Probability as notion of state in quantum and classical theory

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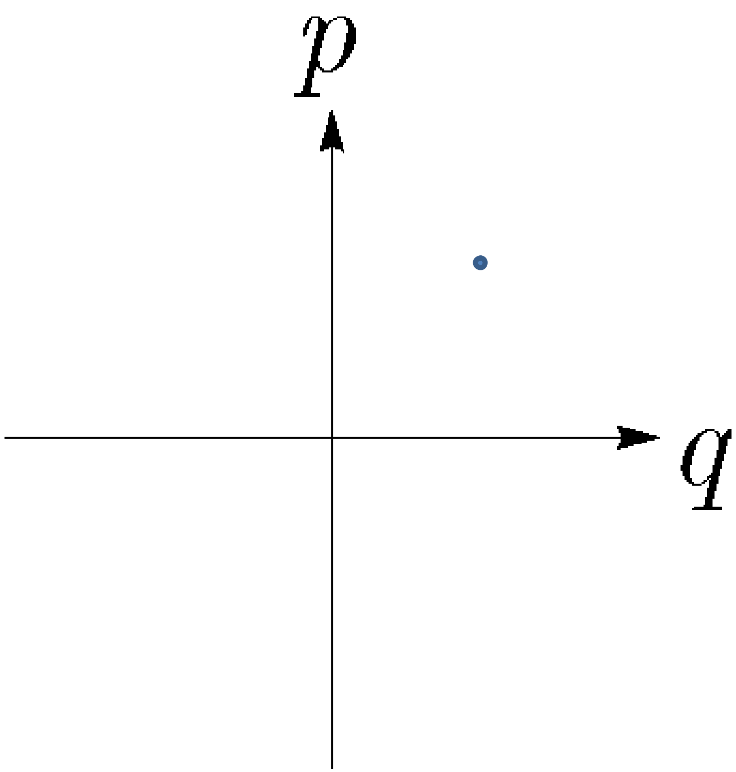
What is Quantum Field Theory? (Sep 14-18, 2011)
CENTRO DE CIECIAS DE BENASQUE PEDRO PASCUAL

M. Asorey, P. Facchi, V. I. Man'ko, G. Marmo, S. Pascazio, and E. G. C. Sudarshan. Radon transform on the cylinder and tomography of a particle on the circle. *Phys. Rev. A* **76**, 012117 (2007)

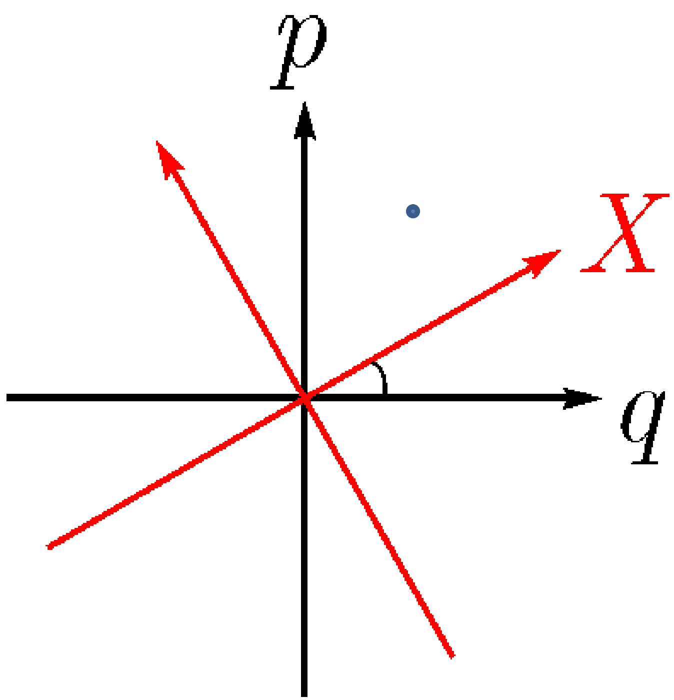
M. Asorey, P. Facchi, V. I. Man'ko, G. Marmo, S. Pascazio, and E. G. C. Sudarshan. Generalized tomographic maps. *Phys. Rev. A* **77**, 042115 (2008)

M. Asorey, P. Facchi, G. Florio, V.I. Man'ko, G. Marmo, S. Pascazio, E.C.G. Sudarshan. Robustness of raw quantum tomography. *Phys. Lett. A* **375**, 861 (2011)

Classical Picture



$$\int f(q,p)\,dq\,dp = 1$$



$$X = q\cos\theta + p\sin\theta$$

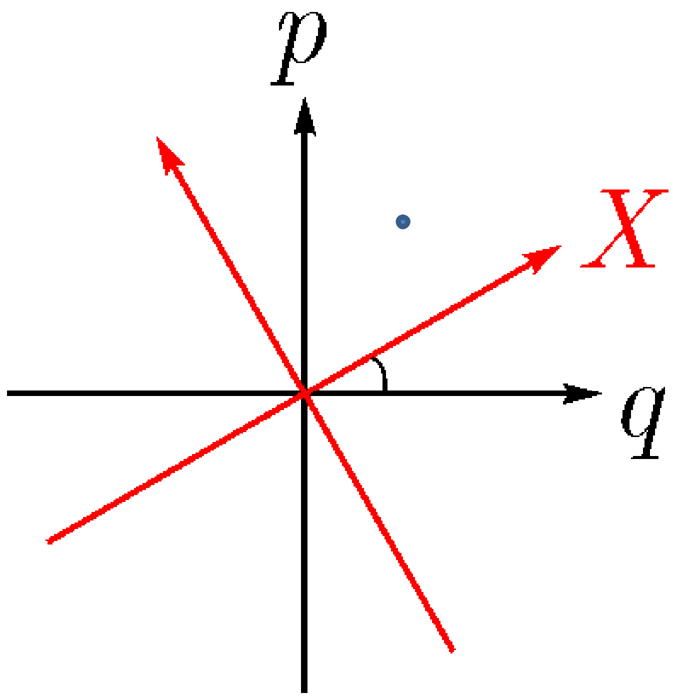
$$w(X,\theta) = \langle \delta\left(X - q\cos\theta - p\sin\theta\right)\rangle$$
$$= \int f(q,p)\delta\left(X - q\cos\theta - p\sin\theta\right)\, dq\, dp$$

$$
\begin{aligned}
M(X,\mu,\nu) &= \langle \delta\left(X-\mu q-\nu p\right)\rangle \\
&= \int f(q,p)\delta\left(X-\mu q-\nu p\right)\, dq\, dp
\end{aligned}
$$

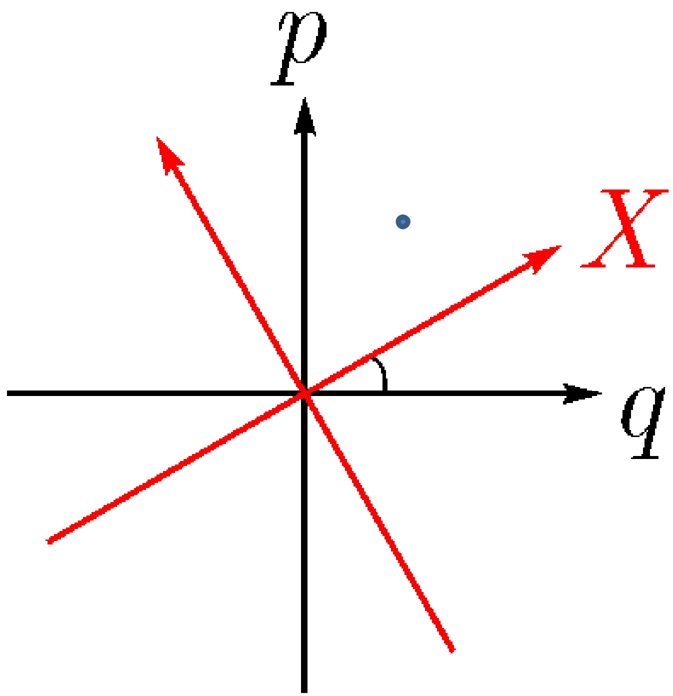
$$
\begin{aligned}
\mu &= s\cos\theta \\
\nu &= s^{-1}\sin\theta
\end{aligned}
$$

Man'ko, O. V., Man'ko, V. I., J. Russ. Laser Res. 18, 407--444 (1997)

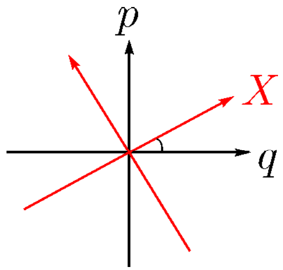
$$
\begin{aligned}
w(X,\theta) &= M(X, \cos\theta, \sin\theta) \\
M(X,\mu,\nu) &= \frac{1}{\sqrt{\mu^2+\nu^2}}\, w\left(\frac{X}{\sqrt{\mu^2+\nu^2}}, \tan^{-1}\frac{\nu}{\mu}\right)
\end{aligned}
$$



$$f(q,p) = \frac{1}{4\,\pi^2} \int M(X,\mu,\nu) e^{i(X-\mu q-\nu p)} dX\, d\mu\, d\nu \geq 0$$



$$\langle q^n \rangle = \int M(X,1,0)X^n\,dX,$$
$$\langle p^n \rangle = \int M(X,0,1)X^n\,dX$$



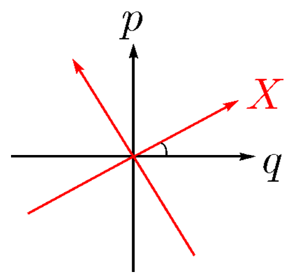
Quantum Picture

$$w(X,\theta) = \langle \delta \left(X - \hat{q} \cos\theta - \hat{p} \sin\theta \right) \rangle$$

$$M(X,\mu,\nu) = \langle \delta \left(X - \mu\hat{q} - \nu\hat{p} \right) \rangle$$

$$\int M(X,\mu,\nu)\, dX = 1$$

Mancini, S., Man'ko, V. I., Tombesi, P, Found. Phys. 27, 801-824 (1997).



$$\psi(y)$$

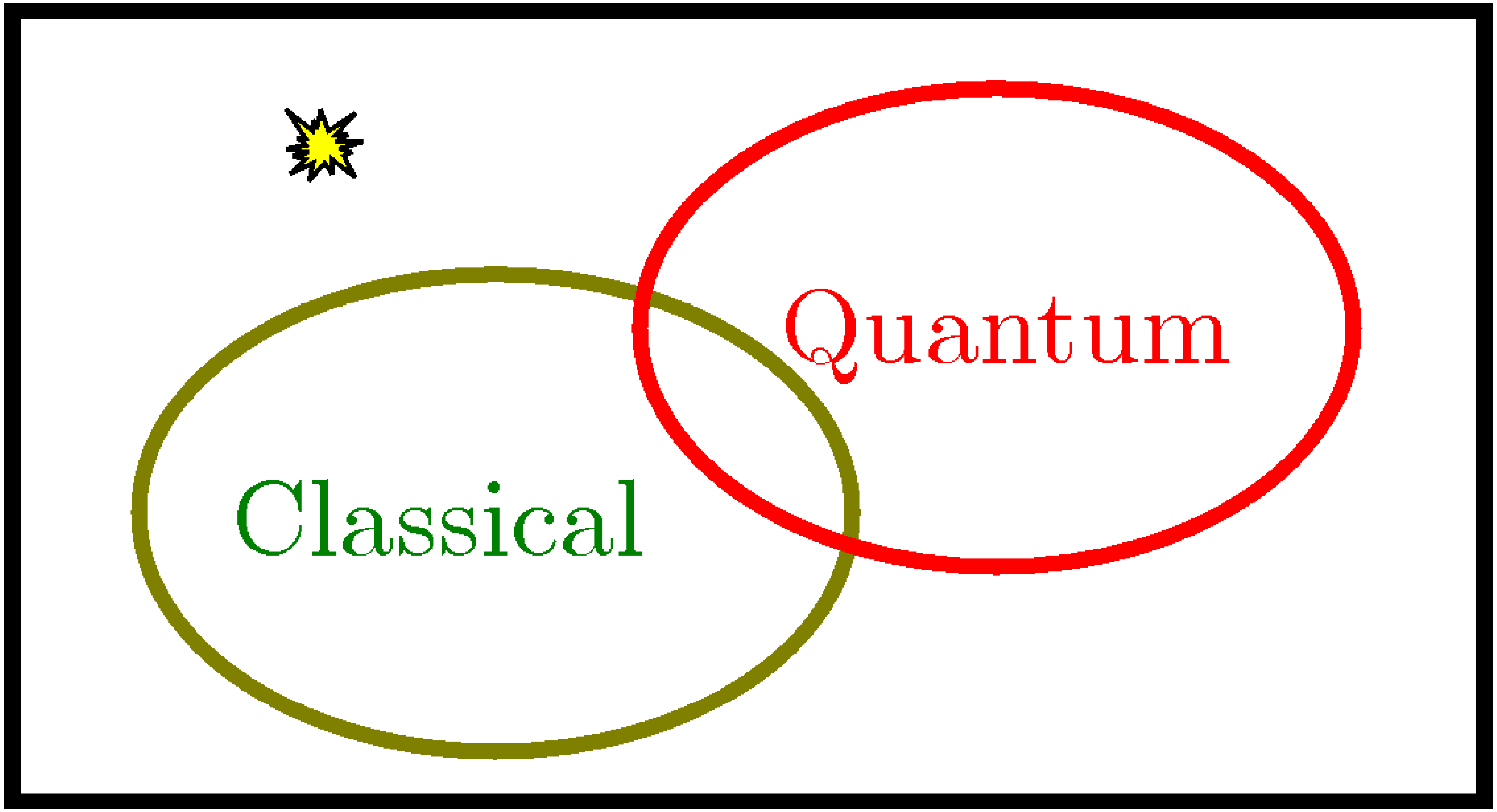
$$M(X,\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left[i\left(\frac{\mu}{2\nu}y^2 - \frac{Xy}{\nu}\right)\right]dy\right|^2$$

Man'ko V. I., Mendes, R. V, Phys. Lett. A 263, 53--56 (1999) [ArXiv Physica/9712022]

$$\widehat{\rho} = \frac{1}{2\pi} \int M(X, \mu, \nu) e^{i(X - \mu\widehat{q} - \nu\widehat{p})} \, dX \, d\mu \, d\nu$$

See also review:

Ibort, A., Man'ko, V. I., Marmo, G., Simoni, A., Ventriglia F., Phys. Scr. 79, 065013 (2009)



Review “An introduction to the tomographic picture of quantum mechanics”
Ibort, A., Man'ko, V. I., Marmo, G., Simoni, A., Ventriglia F., Phys. Scr. 79, 065013 (2009)

$$\langle X^n \rangle (\mu, \nu) = \int X^n M(X, \mu, \nu)\, dX, \qquad n = 1, 2, \ldots$$

$$\sigma_{PP}\sigma_{QQ} = \left(\int X^2 M(X, 0, 1)\, dX - \left[\int X M(X, 0, 1)\, dX \right]^2 \right) \times \left(\int X^2 M(X, 1, 0)\, dX - \left[\int X M(X, 1, 0)\, dX \right]^2 \right) \geq \frac{1}{4}.$$

$$\sigma_{QQ}\sigma_{PP} - \sigma_{QP}^2 \geq \frac{1}{4}$$

$$\sigma_{XX}(\mu,\nu) = \mu^2\sigma_{QQ} + \nu^2\sigma_{PP} + 2\,\mu\nu\sigma_{QP}$$

$$\boxed{\sigma_{QP} = \sigma_{XX}\left(\theta = \frac{\pi}{4}\right) - \frac{1}{2}(\sigma_{QQ} + \sigma_{PP})}$$

$$\sigma_{XX}\left(\theta = \frac{\pi}{4}\right) = \langle X^2\rangle\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) - \left[\langle X\rangle\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right]^2$$

$$
\begin{aligned}
F(\theta) \;=\; & \left(\int X^2 w(X,\theta)dX - \left[\int Xw(X,\theta)\,dX\right]^2\right) \\
& \times \left(\int X^2 w\left(X,\theta+\frac{\pi}{2}\right)\,dX - \left[\int Xw\left(X,\theta+\frac{\pi}{2}\right)\,dX\right]^2\right) \\
& - \left\{\int X^2 w\left(X,\theta+\frac{\pi}{4}\right)\,dX - \left[\int Xw\left(X,\theta+\frac{\pi}{4}\right)\,dX\right]^2\right. \\
& - \frac{1}{2}\left[\int X^2 w(X,\theta)dX - \left[\int Xw(X,\theta)dX\right]^2\right. \\
& \left.\left. + \int X^2 w\left(X,\theta+\frac{\pi}{2}\right)\,dX - \left[\int Xw\left(X,\theta+\frac{\pi}{2}\right)\,dX\right]^2\right]\right\}^2 - \frac{1}{4}
\end{aligned}
$$

$$
F(\theta) \geq 0
$$

Man'ko, V. I., Marmo, G., Simoni, A., Ventriglia F., Adv. Sci. Lett. 2, 517-520 (2009)

$$i\frac{\partial\rho(x,x',t)}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right)\rho(x,x',t) + \left(U(x) - U(x')\right)\rho(x,x',t) \tag{10}$$

$$\frac{\partial W(q,p,t)}{\partial t} + p\frac{\partial W(q,p,t)}{\partial q} + \frac{1}{i}\left[U\left(q - \frac{i}{2}\frac{\partial}{\partial p}\right) - \text{c.c.}\right]W(q,p,t) = 0. \tag{11}$$

$$\frac{\partial}{\partial t} w(X,\theta,t) = \left[\cos^2\theta \frac{\partial}{\partial\theta} - \frac{1}{2}\sin 2\theta \left\{1 + X\frac{\partial}{\partial X}\right\}\right] w(X,\theta,t)$$
$$+2\left[\mathrm{Im}\ U\left\{\sin\theta\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial X}\right]^{-1} + X\cos\theta + i\frac{\sin\theta}{2}\frac{\partial}{\partial X}\right\}\right] w(X,\theta,t). \quad (12)$$

Ya.A. Korennoy and V.I. Man'ko, Journal of Russian Laser Research, **32**, 75 (2011)

$$
\begin{aligned}
Ew_E(\vec{X},\vec{\theta}) = &\left[\sum_{\sigma=1}^{n} m_\sigma \omega_\sigma^2 \left\{ \frac{\cos^2\theta_\sigma}{2}\left[\frac{\partial}{\partial X_\sigma}\right]^{-2}\left(\frac{\partial^2}{\partial\theta_\sigma^2}+1\right)\right.\right. \\
&- \frac{X_\sigma}{2}\left[\frac{\partial}{\partial X_\sigma}\right]^{-1}\left(\cos^2\theta_\sigma + \sin 2\theta_\sigma \frac{\partial}{\partial\theta_\sigma}\right) + \frac{X_\sigma^2}{2}\sin^2\theta_\sigma - \frac{1}{8}\cos^2\theta_\sigma \frac{\partial^2}{\partial X_\sigma^2} \Bigg\} \\
&\left. + \operatorname{Re} U \left\{ \sin\theta_\sigma \frac{\partial}{\partial\theta_\sigma}\left[\frac{\partial}{\partial X_\sigma}\right]^{-1} + X_\sigma \cos\theta_\sigma + i\frac{\hbar \sin\theta_\sigma}{2m_\sigma\omega_\sigma}\frac{\partial}{\partial X_\sigma}\right\}\right] w_E(\vec{X},\vec{\theta})
\end{aligned}
$$

$$w(X,\theta) = \frac{e^{-X^2}}{\sqrt{\pi}} \sum_{n,m} \frac{\langle(\hat{a}^\dagger)^n \hat{a}^m\rangle e^{i(n-m)\theta}}{\sqrt{2^{n+m}} n! m!} \; H_{n+m}(X) \tag{17}$$

$$= \int \frac{d\xi}{2\pi} \; e^{\xi^2/4 - i\xi X} \left(\sum_{k,l} \frac{\langle \hat{a}^k (\hat{a}^\dagger)^l \rangle}{k! l!} \left(\frac{i\xi}{\sqrt{2}}\right)^{k+l} e^{i(l-k)\theta} \right). \tag{18}$$

$$w(\mathbf{x}) \equiv w(m,u) = \langle jm|u\hat{\rho}u^{\dagger}|jm\rangle = \mathrm{Tr}\Big(\hat{\rho}u^{\dagger}|jm\rangle\langle jm|u\Big) = \mathrm{Tr}\Big(\hat{\rho}\hat{U}(\mathbf{x})\Big) \tag{29}$$

$$\hat{U}(\mathbf{x}) = u^{\dagger}|jm\rangle\langle jm|u \tag{30}$$

$$\sum_{m=-j}^{j} w(m, u) = 1 \tag{31}$$

$$\frac{2j+1}{8\pi^2} \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \sin\beta \mathrm{d}\beta \int_0^{2\pi} \mathrm{d}\gamma \; w(m, \alpha, \beta, \gamma) = 1 \tag{32}$$

$$\hat{\rho} = \int w(\mathbf{x}) \hat{D}(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{33}$$

$$\int \mathrm{d}\mathbf{x} = \sum_{m=-j}^{j} \frac{1}{8\pi^2} \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \sin\beta \mathrm{d}\beta \int_0^{2\pi} \mathrm{d}\gamma \tag{34}$$

$$\hat{U}(\mathbf{x}) = \sum_{L=0}^{2j} \sum_{M=-L}^{L} (-1)^{j-m+M} \langle jm; j-m | L0 \rangle \, D_{0-M}^{(L)}(\alpha, \beta, \gamma) \, \hat{T}_{LM}^{(j)},$$

$$\hat{D}(\mathbf{x}) = \sum_{L=0}^{2j} (2L+1) \sum_{M=-L}^{L} (-1)^{j-m+M} \langle jm; j-m | L0 \rangle \, D_{0-M}^{(L)}(\alpha, \beta, \gamma) \, \hat{T}_{LM}^{(j)}$$

S.N. Filippov and V.I. Man'ko, Journal of Russian Laser Research **30**, 129 (2009),
E-print `arXiv:0904.1124v1 [quant-ph]`

Conclusions

- We reviewed the *probability representation of quantum mechanics*, where the quantum states are described by probability distributions as an alternative to density operators.
- The probability representation is discussed for continuous variables.
- The essence of this work is the *consideration of experiments to check quantum mechanics*, or better to say, to study the accuracy with which one can check experimentally the uncertainty relations for photon quadratures.
- It is worth pointing out that in the classical domain one does not have these bounds on $F(\theta)$.

Congratulations to Manolo with Birth Day