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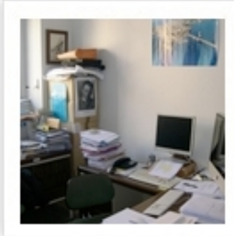
What is Quantum Field Theory?

2011, Sep 14 -- Sep 18



New Lattice Gauge Theories from Quantum Computation

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  UNIVERSITY OF TEXAS A&M, ETH ZURICH
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  ETH ZURICH (SWITZERLAND)

REFERENCES

- "Tricolored Lattice Gauge Theory with Randomness: Fault-Tolerance in Topological Color Codes"
  R. ANDRIST, H. KATZGRABER, H. BOMBIN, M. A. MARTIN-DELGADO
  NEW J. OF PHYS. (2011)

SUPPLEMENTARY MATERIAL

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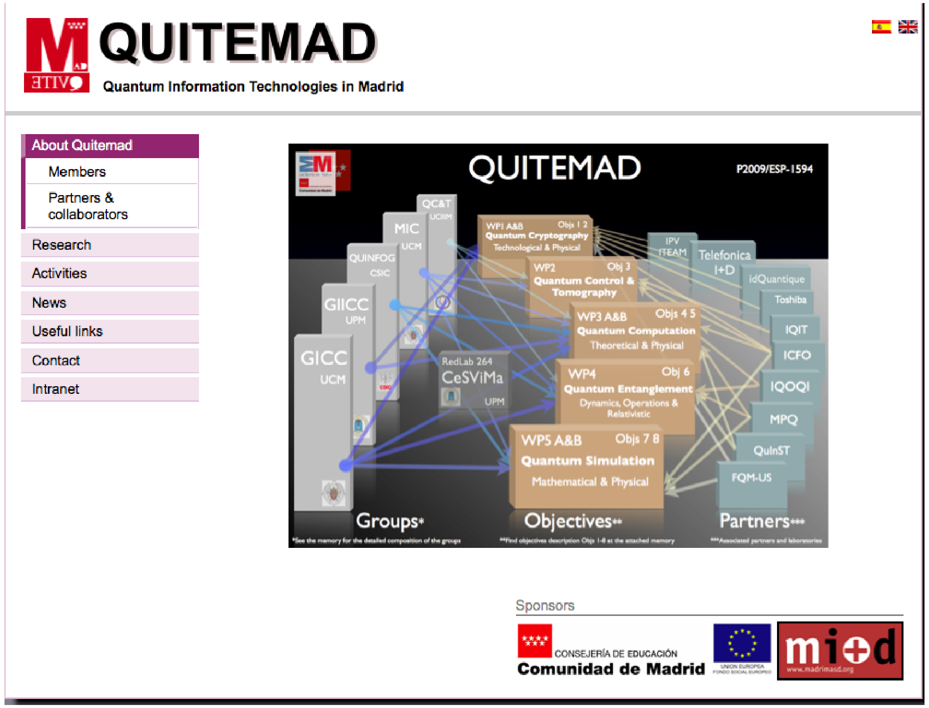
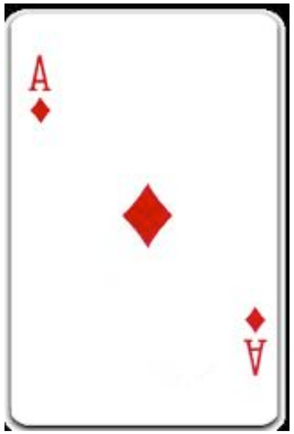


FELICIDADES MANUEL !

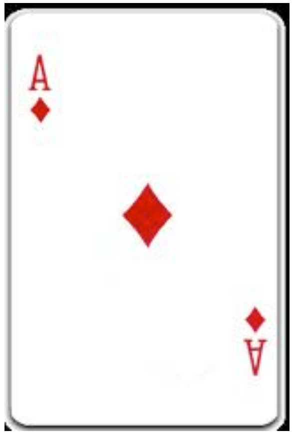
Coat of Arms?



As o rey

FELICIDADES MANUEL !

Coat of Arms?



As o rey

Siempre llevas buenas cartas

PLAY
“JOTA” from MANUEL DE FALLA
7 Canciones Populares

Interpreter: Daniel Shafran (cello) with piano accompaniment

Manuel A. Carballeira

Monte do Gozo

Manuel A. Carballeira

Monte do Gozo

Benasque and mountains

Outline of the TALK

I) The Quest for a Quantum Computer

II) Externally Protected Quantum Computer

III) Topological Color Codes

IV) Mapping to Statistical Models

V) Tricolored Lattice Gauge Theory

VI) Simulations and Threshold

I) The Quest for a Quantum Computer

Challenges for New States of Matter…

Challenge for NEW PHYSICS:

The QUEST for a QUANTUM COMPUTER

Challenges for New States of Matter…

Challenge for NEW PHYSICS:

The QUEST for a QUANTUM COMPUTER

TYPES OF QUANTUM COMPUTERS

According to their Protection

Challenges for New States of Matter…

Challenge for NEW PHYSICS:

The QUEST for a QUANTUM COMPUTER

INTERNALLY PROTECTED QC

EXTERNALLY PROTECTED QC

BARE QUANTUM COMPUTER

Challenges for New States of Matter…

Challenge for NEW PHYSICS:

The QUEST for a QUANTUM COMPUTER

INTERNALLY PROTECTED QC

↑

EXTERNALLY PROTECTED QC

↑

BARE QUANTUM COMPUTER

II) Externally Protected Quantum Computer

Quantum Error Correction

A Critical Ghost

All papers on quantum computing should carry a footnote: *"This proposal, like all proposals for quantum computation, relies on speculative technology, does not in its current form take into account all possible sources of noise, unreliability and manufacturing error, and probably will not work."*

Rolf Landauer
IBM

Get around Landauer's objections by finding systems which naturally enact quantum error correction.

Nature abhors a quantum computer?

Quantum Error Correction

Classical Error Correction

Noisy Cell Phone

Communication over a noisy CHANNEL can be overcome via

ENCODING

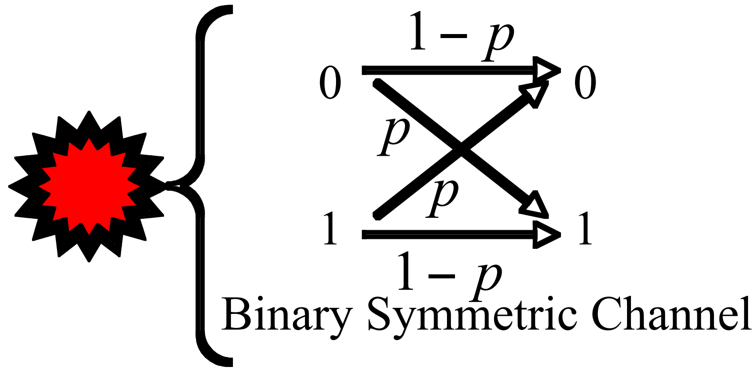
"Hello?" = "Hello? Hello? Hello? Hello?" [using redundancy to encode "Hello"]

Quantum Error Correction

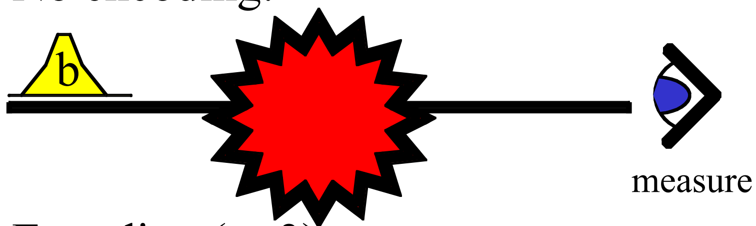
Simple Repetition Code

Encode: $0 \rightarrow 000\ldots000$

$1 \rightarrow \underbrace{111\ldots111}_{\text{n copies}}$

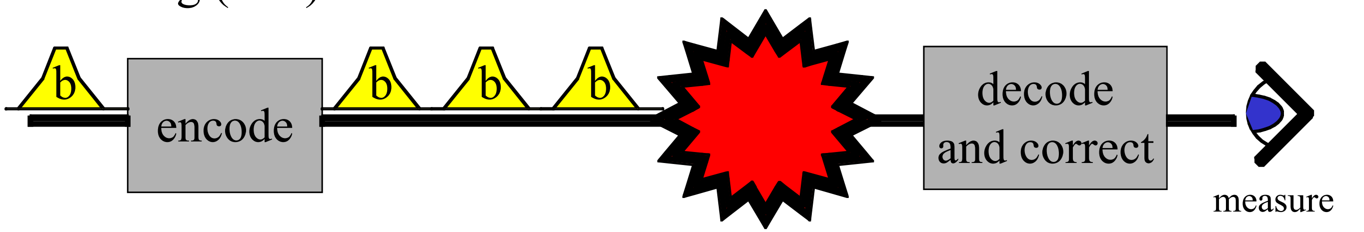


No encoding:



Probability of error = p

Encoding (n=3):



Probability of error $= 3p^2(1-p) + p^3 = 3p^2 - 2p^3 < p, p < \frac{1}{2}$

Quantum Error Correction

Quantum Error Correction must face **major problems**:

• **Quantum Redundacy** is not possible due to No-Cloning Theorem (and even if we overcome this→)

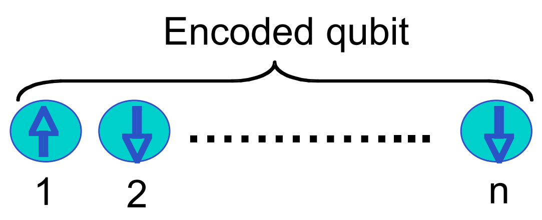
•There exist Quantum Errors with **no classical analog**:

1) Bit-Flip Errors → Classical counterpart

2) **Phase Errors**→ Really Quantum

• We are **not allowed to read the state of the qubit**, which leads to decoherence. The location and the type of error must be acquired without acquiring the knowledge of the qubit state.

Encoding one logical qubit into several physical qubits



Challenges for New States of Matter…

EXTERNALLY PROTECTED QUANTUM COMPUTER

⟶ **Yet to produce NEW PHYSICS**

ACTIVE ERROR CORRECTION = MEDICINE

Rethink in more Familiar Terms

Error Correction ⟷ Medicine

Errors ⟷ Disease

Error Detection (syndrome) ⟷ Diagnose (syndrome)

Correction ⟷ Medicines

Challenges for New States of Matter…

Problem: we cannot touch the quantum state to do Error Detection

➡ We need more than Medicine ……

➡ We need Magic ….

but Quantum Theory is magical … and tricky

We have tricks at our disposal ➡ Ancilla Qubits

Challenges for New States of Matter…

Problem: we cannot touch the quantum state to do Error Detection

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but Quantum Theory is magical … and tricky

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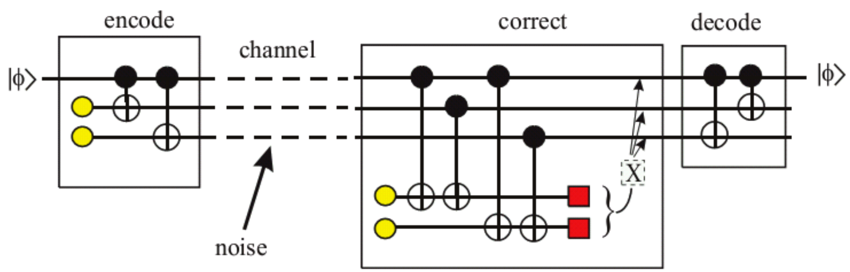
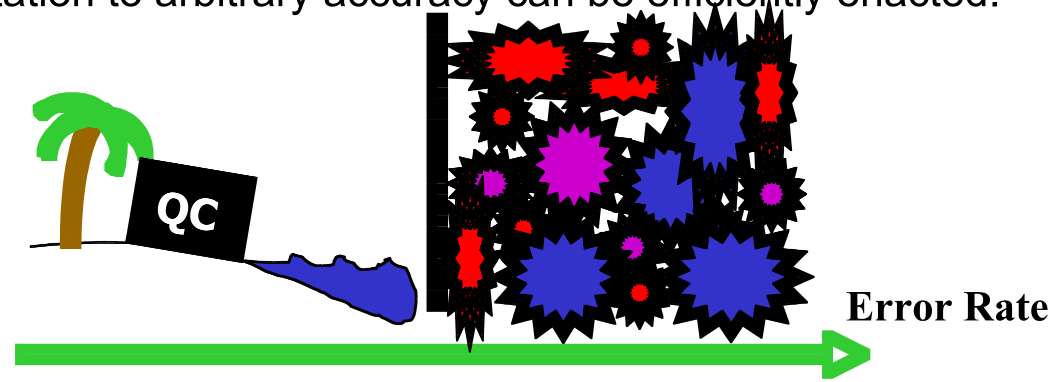


Threshold Theorem

Fault Tolerant Quantum Computing is Possible!

Threshold Theorem:
If the error rate (typical number of operations per decoherence time/ accuracy in quantum evolution / measurement error rate) is smaller than a certain value (the threshold), then quantum computation to arbitrary accuracy can be efficiently enacted.



Quantum Computation is NOT Analog Computation

Challenges for New States of Matter…

- **Good News: Fault-Tolerant Qomputation is possible**

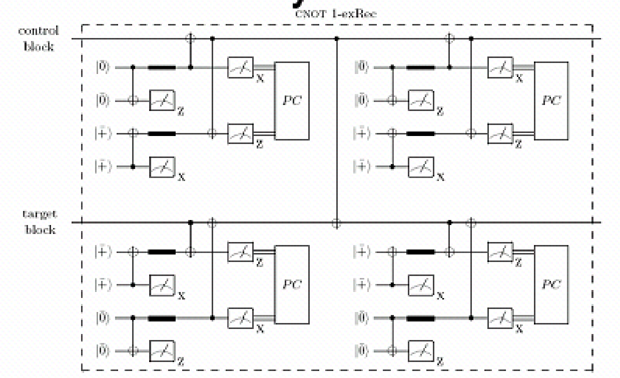
- **Bad News: the threshold is very small**

Caution: the proof is constructive, there could be better thresholds

Challenges for New States of Matter…

Lower bound on the accuracy threshold

A ***good*** gadget (one with sparse faults) is ***correct*** (simulates the ideal gate accurately).



For each of the level-1 extended Rectangles in a universal set, e.g. for the [[7,1,3]] (Steane) code, we can count the number of pairs of malignant locations; the CNOT 1-exRec dominates the threshold estimate. We find a rigorous lower bound on the accuracy threshold for *adversarial independent stochastic noise:*

$$\varepsilon_0 > 2.73 \times 10^{-5}$$

(assuming parallelism, fresh ancillas, nonlocal gates, fast measurements, fast and accurate classical processing, no leakage).

Challenges for New States of Matter…

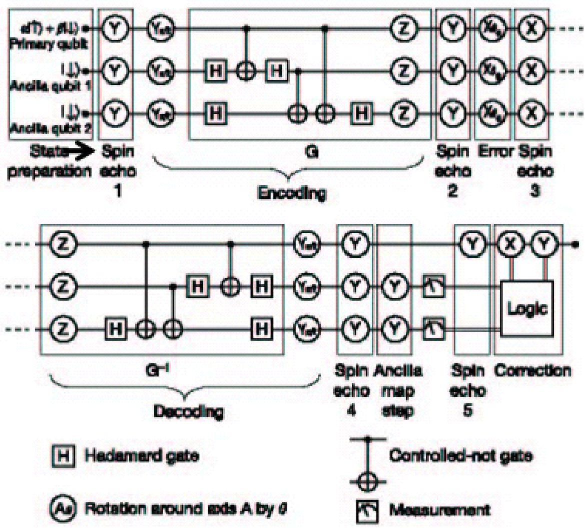
A realization of quantum error correction

J. Chiaverini *et al.*, [*Nature* **432**, 602-605 (2004)] implemented a three-qubit quantum repetition code using trapped ions. They prepared the encoded $|\overline{\psi}\rangle = a|\overline{0}\rangle + b|\overline{1}\rangle$ state, simulated noise that flips each qubit with probability ε, measured the error syndrome, and corrected the error.

The probability P of an encoded error was found to be

$$P = c + 2.6\,|ab|^2\,\varepsilon^2$$

… i.e., quadratic in ε.



Realization of quantum error correction

J. Chiaverini[1], D. Leibfried[1], T. Schaetz[1*], M. D. Barrett[1*], R. B. Blakestad[1], J. Britton[1], W. M. Itano[1], J. D. Jost[1], E. Knill[2], C. Langer R. Ozeri[1] & D. J. Wineland[1]

[1]*Time and Frequency Division*, [2]*Mathematical and Computational Sciences Division, NIST, Boulder, Colorado 80305, USA*

* Present addresses: Max Planck Institut für Quantenoptik, Garching, Germany (T.S.); Physics Department, University of Otago, Dunedin, New Zealand (M.D.B.)

Challenges for New States of Matter…

EXTERNALLY PROTECTED QUANTUM COMPUTER

→ **Good Candidate: TOPOLOGICAL QUANTUM COMPUTER**

ERROR CORRECTION: RANDOM LGT (Lattice Gauge Theory)

PHASE DIAGRAM: NISHIMORI LINE

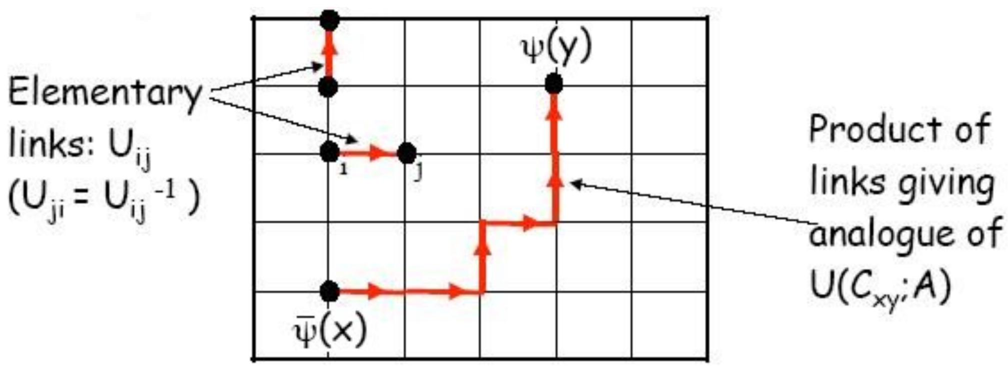
OBSERVABLES: AREA LAW VS. PERIMETER LAW

Challenges for New States of Matter…

EXTERNALLY PROTECTED QUANTUM COMPUTER

→ **Good Candidate: TOPOLOGICAL QUANTUM COMPUTER**

RANDOM LGT (Lattice Gauge Theory)
Gauge group depends on the Topological Code



Challenges for New States of Matter…

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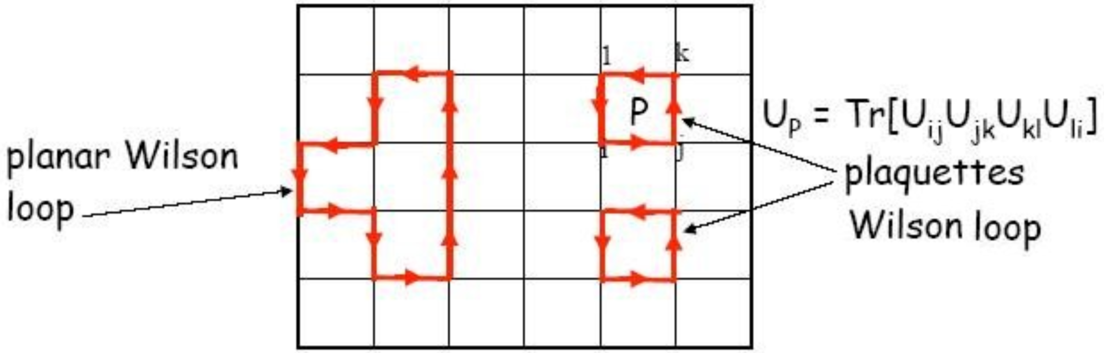

Challenges for New States of Matter…

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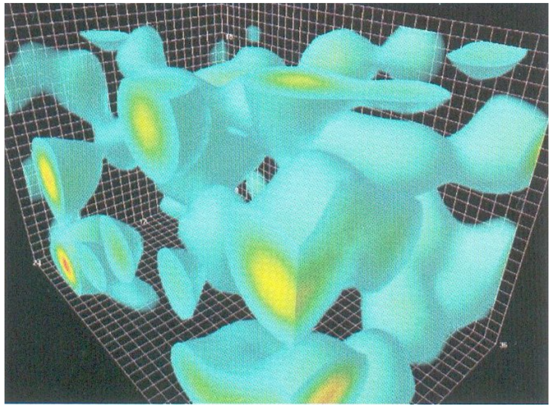
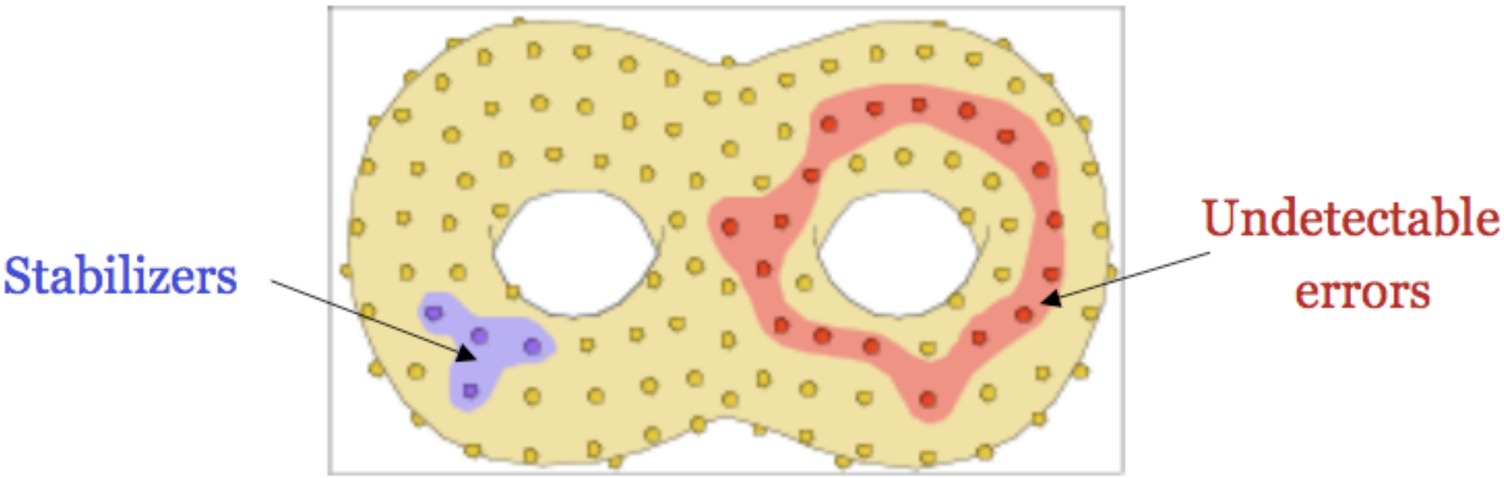


III) Topological Color Codes

Topological Stabilizer Codes

- In order to introduce the idea of a topological stabilizer code (TSC), we must consider a topological space in which our physical qubits are to be placed, for example a surface.
- A TSC is a stabilizer code in which the generators of the stabilizer are **local** and undetectable errors (or encoded operators) are **topologically nontrivial**.



II.Stabilizer Codes

- A **stabilizer code**[1] C of length n is a subspace of the Hilbert space of a set of n qubits. It is defined by a stabilizer group S of Pauli operators, i.e., tensor products of Pauli matrices.

- It is enough to give the **generators** of S. For example:

$$|\psi\rangle \in \mathcal{C} \quad \Longleftrightarrow \quad \forall\, s \in \mathcal{S} \quad s|\psi\rangle = |\psi\rangle$$

- Operators O that belong to the **normalizer** of S

$$\{ZXXZI, IZXXZ, ZIZXX, XZIZX\}$$

leave invariant the code space C. If they do not belong to the stabilizer, then they act non-trivially in the code subspace.

$$O \in \mathbf{N}(\mathcal{S}) \quad \Longleftrightarrow \quad O\mathcal{S} = \mathcal{S}O$$

[1] D. Gottesman 95

II.Stabilizer Codes

- A encoded state can be subject to **errors**.
- To correct them, we measure a set of generators of *S*. The results of the measurement compose the **syndrome** of the error. Errors can be corrected as long as the syndrome lets us distinguish among the possible errors.
- Since correctable errors always form a vector space, it is enough to consider Pauli operators, which form a basis.
- We say that a Pauli error *e* is **undetectable** if it belongs to **N(*S*)-*S***. In such a case, the syndrome says nothing:

$$\forall s \in \mathcal{S} \qquad s\, e|\psi\rangle = e\, s|\psi\rangle = e|\psi\rangle$$

- A set of Pauli errors *E* is correctable iff:

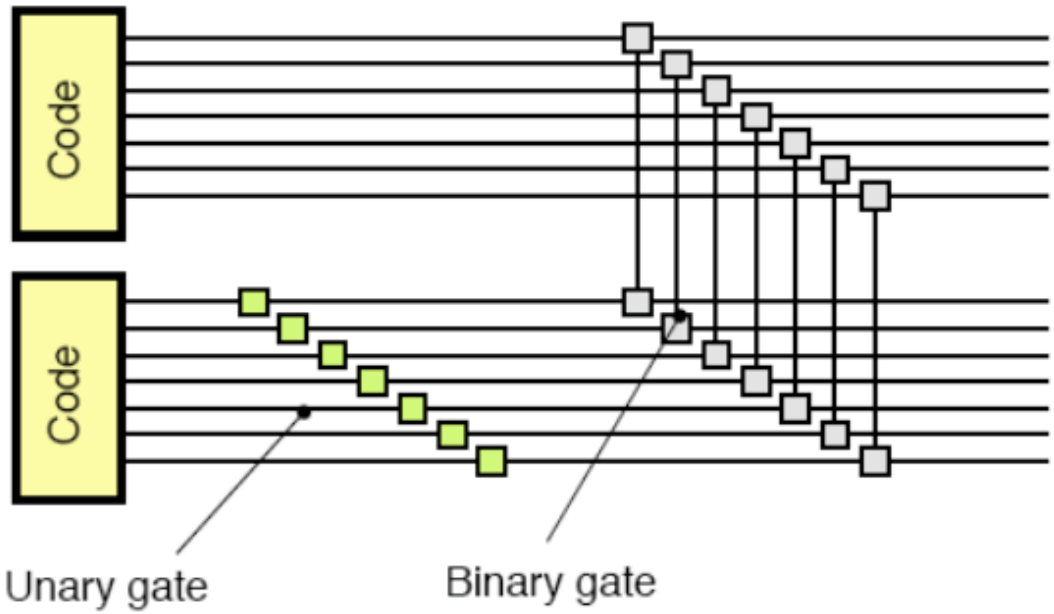
$$E^{\dagger}E \cap \mathcal{N}(\mathcal{S}) \in \mathcal{S}.$$

Stabilizer Codes

- A **stabilizer code**[1] C of length n is a subspace of the Hilbert space of a set of n qubits. It is defined by a stabilizer group S of Pauli operators, i.e., tensor products of Pauli matrices.

$$|\psi\rangle \in \mathcal{C} \quad \Longleftrightarrow \quad \forall\, s \in \mathcal{S} \quad s|\psi\rangle = |\psi\rangle$$

- Some stabilizer codes are specialy suitable for quantum computation. They allow to perform operations in a **transversal** and **uniform** way:



[1] D. Gottesman 95

Stabilizer Codes

Gate Sets

- Several codes allow the transversal implementation of

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \qquad K = \begin{pmatrix}1 & 0\\0 & i\end{pmatrix} \qquad \Lambda = \begin{pmatrix}I_2 & 0\\0 & X\end{pmatrix}$$

  which generate the **Clifford group**. This is useful for quantum information tasks such as teleportation or **entanglement distillation**.
- Quantum **Reed-Muller** codes[1] are very special. They allow **universal computation** through transversal gates

$$K^{1/2} = \begin{pmatrix}1 & 0\\0 & i^{1/2}\end{pmatrix} \qquad \Lambda = \begin{pmatrix}I_2 & 0\\0 & X\end{pmatrix}$$

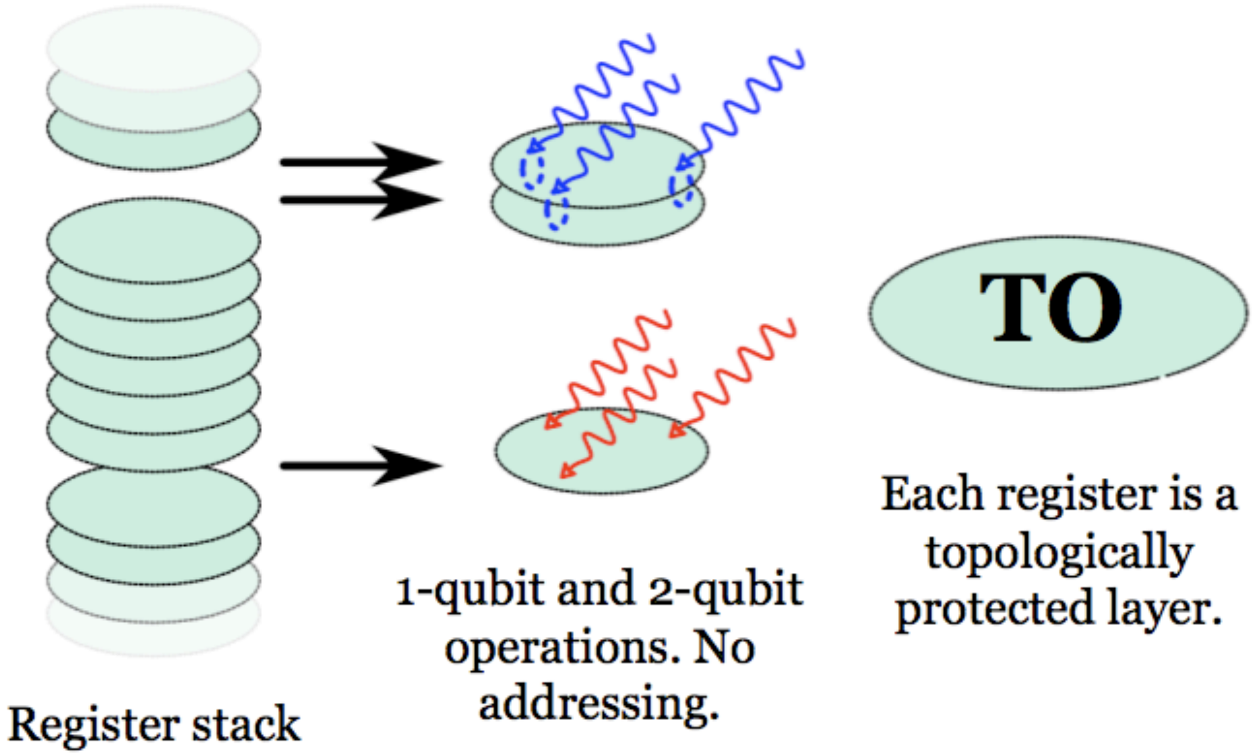
  and transversal measurements of X and Z.
- We will see how both sets of operations can be transversally implemented in 2D and 3D topological color codes:

Color Codes = Transversality + Topology

[1] E. Knill *et al.*

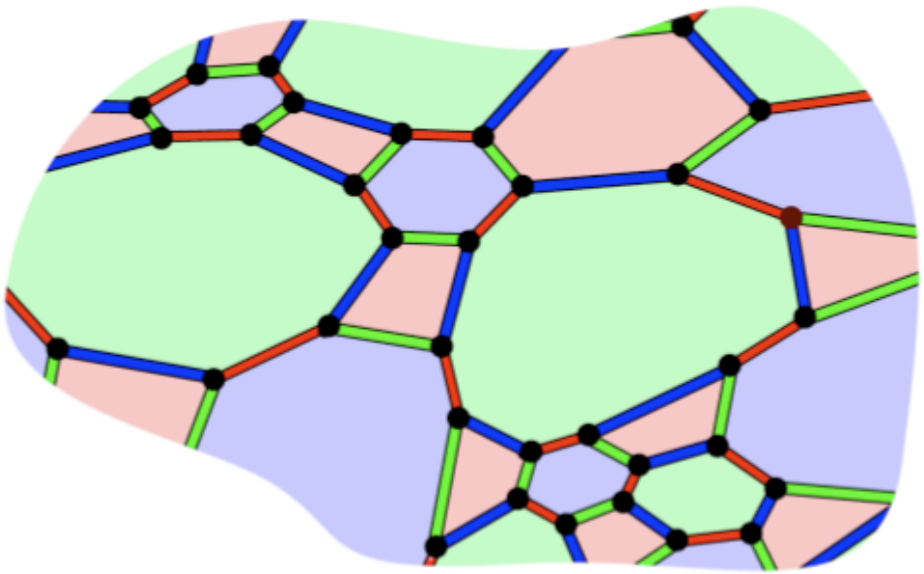
2-Colexes

- **Goal**: 2-dimensional layers as quantum registers, protected by TO. Operations on encoded qubits without selective addressing of physical qubits.



2-Colexes

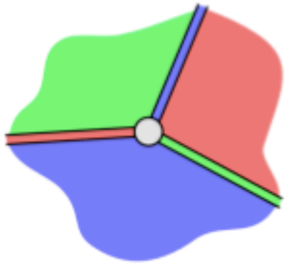
- A 2-colex is a **trivalent** 2-D lattice with **3-colored** **faces**.

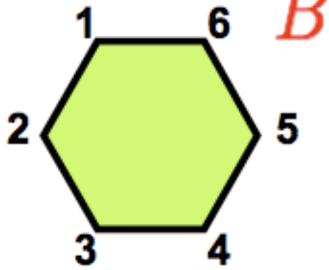


- **Edges can be 3-colored accordingly**. Blue edges connect blue faces, and so on.
- The name 'colex' is for 'color complex'. *D*-colexes of arbitrary dimension can be defined. Their key feature is that the whole structure of the complex is contained in the 1-skeleton and the coloring of the edges.

2-Colexes

- To construct a **color code** from a 2-colex, we place 1 qubit at each **vertex** of the lattice. The generators of S are **face operators**:





$$B_f^X = X_1X_2X_3X_4X_5X_6$$

$$B_f^Z = Z_1Z_2Z_3Z_4Z_5Z_6$$

- Transversal Clifford gates should belong to $\mathbf{N}(S)$. We have:

$$\hat{H} B_f^X \hat{H}^\dagger = B_f^Z \qquad \hat{K} B_f^X \hat{K}^\dagger = (-)^{\frac{v}{2}} B_f^X B_f^Z$$

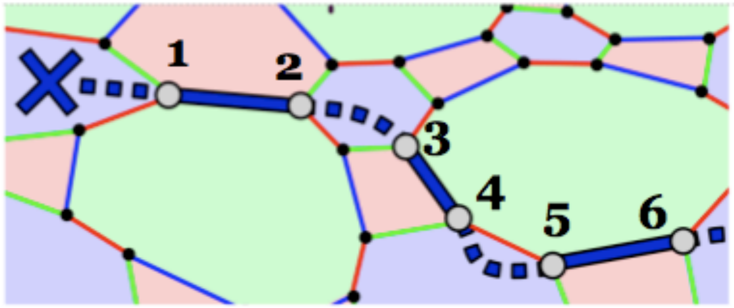
$$\hat{H} B_f^Z \hat{H}^\dagger = B_f^X \qquad \hat{K} B_f^Z \hat{K}^\dagger = B_f^Z$$

- Here v is the number of vertices in the face. If it is a multiple of 4 for every face, then K is in $\mathbf{N}(S)$. H always is.
- As for the CNot gate, it is clearly in $\mathbf{N}(S)$ (it is a CSS code).

2-Colexes

- In order to understand 2-D color codes, we have to introduce **string operators** in the picture. As in surface codes, we play with Z_2 homology. However, there is a **new ingredient**, **color**.
- A **blue string** is a collection of **blue links**

Endpoint



String operators

$S^X = X_1 X_2 X_3 X_4 X_5 X_6 \dots$

$S^Z = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 \dots$

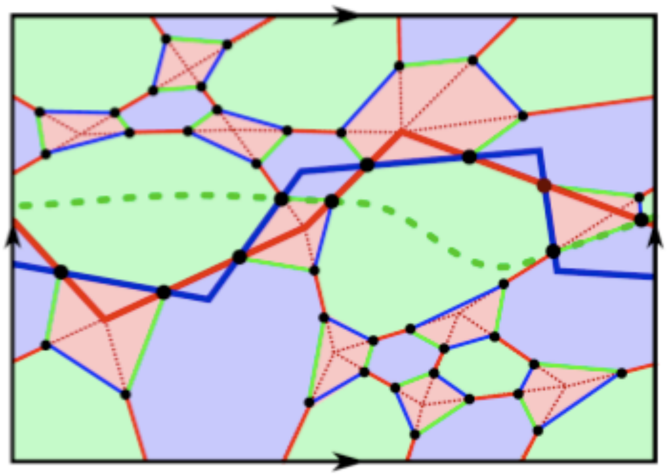
(hexagonal bishop with flavor X or Z):

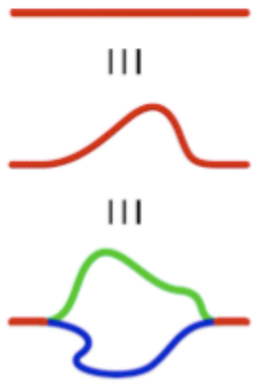
- Strings can have endpoints, located at faces of the same color. However, in that case the corresponding string and face operators will not commute. Therefore, a string operator belongs to $\mathbf{N}(S)$ iff the string has no endpoints.

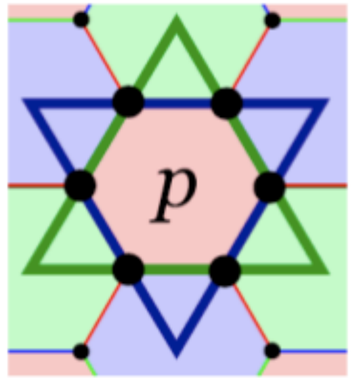
2-Colexes

Continous Visualization of Color Strings

- For each color we can form a **shrunk graph**. The red one is:







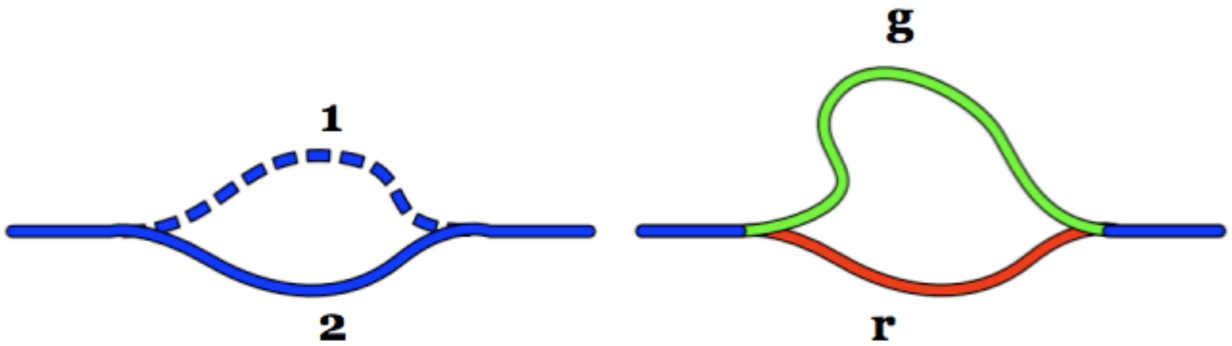
Red faces → vertices
Red edges → edges
Blue and green faces → faces, faces

A red face is also blue or green string

- Thus for each color homology works as in surface codes. The new feature is the possibility to **combine** homologous blue and red string operators of the same kind to get a green one.

2-Colexes

- Strings can be **deformed** and colors **branched**:



$$S_1^\sigma \sim S_2^\sigma \qquad S_r^\sigma S_g^\sigma \sim S_b^\sigma$$

Equivalent strings act equally on the Ground State.

2-Colexes

- Since there are two independent colors, the number of encoded qubits should double that of a surface code. Lets check this for a surface **without boundary** using the Euler characteristic for any *shrunk* lattice. $\chi = V + F - E$
- Face operators are subject to the **conditions**

$$\prod_{f \in \bullet} B_f^\sigma = \prod_{f \in \bullet} B_f^\sigma = \prod_{f \in \bullet} B_f^\sigma \,,$$

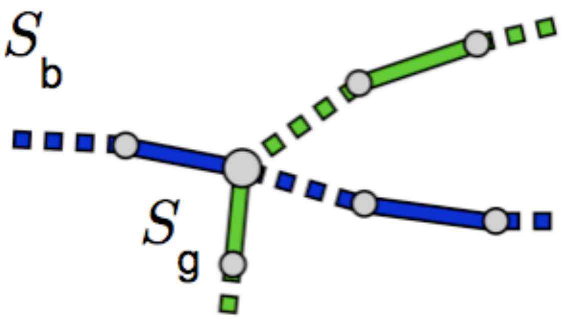
so that the total number of generators is $g = 2(F + V - 2)$

- The number of physical qubits is $n = 2E$. Therefore the number of encoded qubits q is twice the first Betti number of the manifold:

$[[n, k, d]]$ $\qquad k = n - g = 4 - 2\chi = 2h_1$

2-Colexes

- In order to form a **Pauli basis** for the operators acting on encoded qubits, we can use as in surface codes those string operators (SO) that are not homologous to zero.
- To this end, we need the commutation rules for SO.
- Clearly SO of the same type (*X* or *Z*) always commute.
- A string is made up of edges with two vertices each. Therefore, two SO of the same color have an even number of qubits in common an they commute.
- SO of **different colors** can **anticommute**, but only if they **cross** an odd number of times:

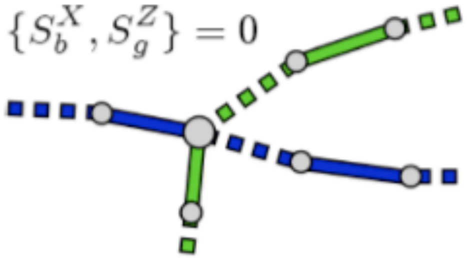


$$\{S_b^X, S_g^Z\} = 0$$

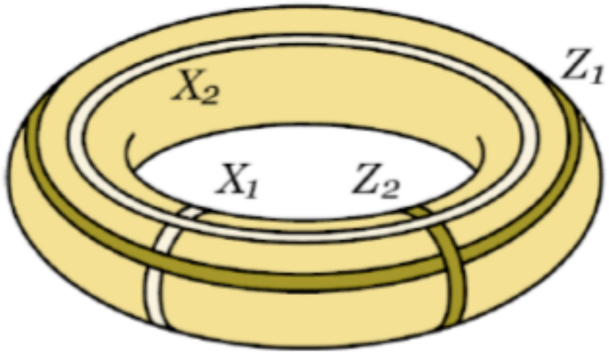
2-Colexes

String Operators

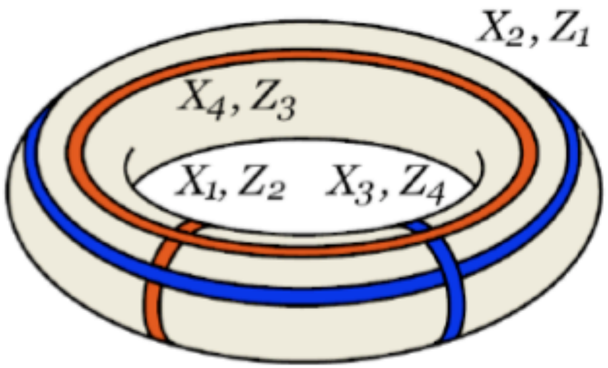
- For each colored string S, there are a **pair** of **string operators**, S^X and S^Z, products of Xs or Zs along S.
- String operators either commute or anticommute.
- Two string operators **anticommute** when they have **different color and type** and **cross** an odd number of times.



- As in surface codes, encoded X and Z operators can be chosen from closed string operators which are not boundaries.
- The number of **encoded qubits** is **twice** as in a surface code:



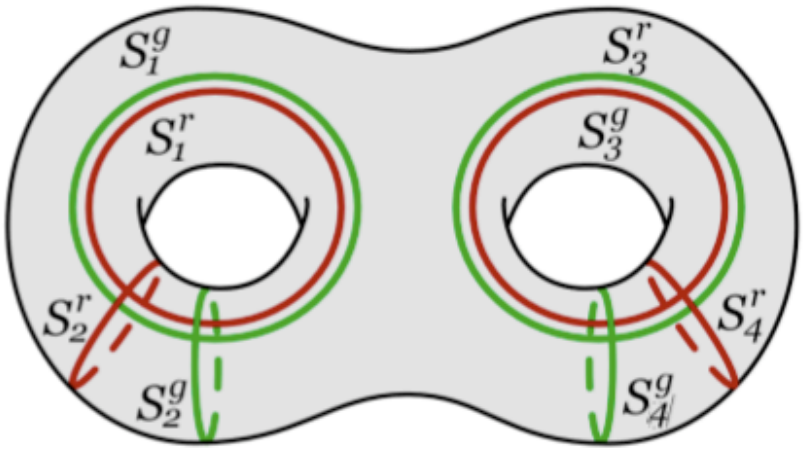
Surface code: 2 qubits



Color code: 4 qubits

2-Colexes

- Now we can construct the desired operator basis for the encoded qubits. In a 2-torus a possible choice is:



$$\begin{array}{ll} S_1^{gX} \leftrightarrow X_1 & S_2^{rZ} \leftrightarrow Z_1 \\ S_2^{rX} \leftrightarrow X_2 & S_1^{gZ} \leftrightarrow Z_2 \\ S_2^{gX} \leftrightarrow X_3 & S_1^{rZ} \leftrightarrow Z_3 \\ \vdots & \vdots \end{array}$$

$$X_i Z_j = (-1)^{\delta_{i,j}} Z_j X_i$$

Encoded qubits = $2h_1$

h_1 = first Betti number

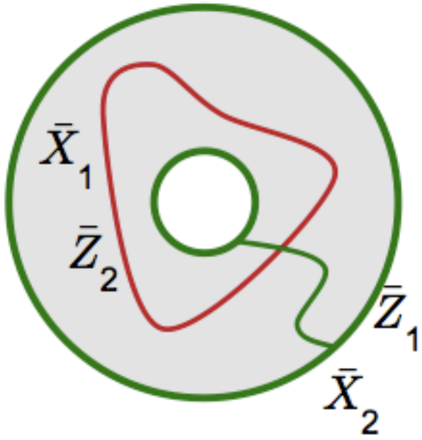
- **However**, if we apply the transversal H gate to such a code the resulting encoded gate is not H. The underlying reason is that for a string S we **never** have

$$\{S_b^X, S_g^Z\} = 0$$

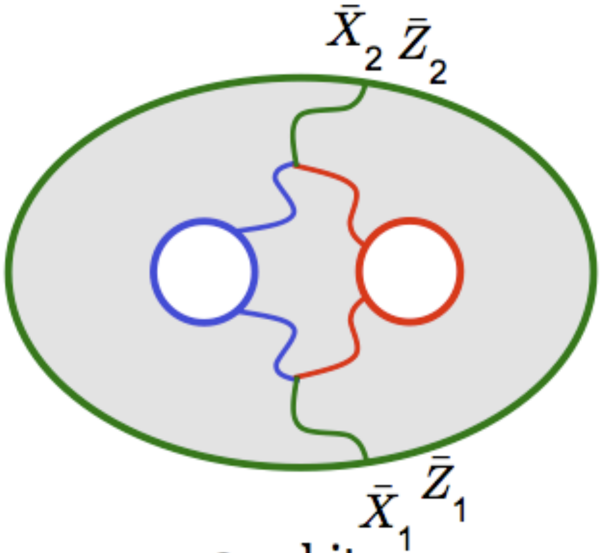
2-Colexes

Way out:

- But we can consider surfaces with **boundary**. To this end, we take a sphere, which encodes no qubit, and **remove faces**.
- When a face is removed, the resulting boundary must have its color, and only strings of that color can end at the boundary.



2 qubits



2 qubits

Toy Baryon or String-Net[1]

$$\{T^{\mathsf{x}}, T^{\mathsf{z}}\} = 0$$

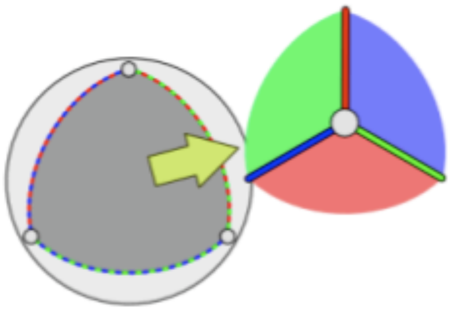
As desired!

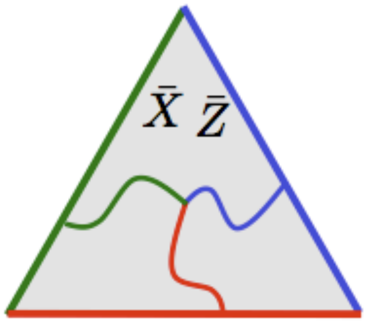
[1]Wen et al.

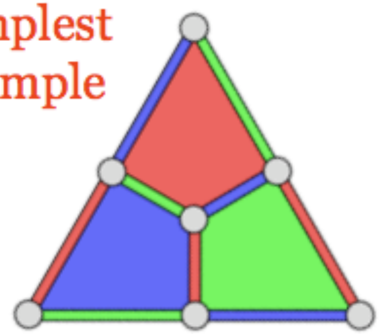
2-Colexes

Look for 2-colexes with string-nets:

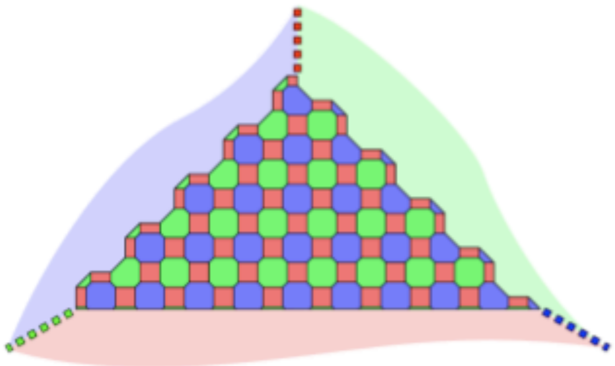
- We can even encode a single qubit an remove the need for holes. If we remove a site and neighboring links and faces from a 2-colex in a **sphere**, we get a **triangular** code:







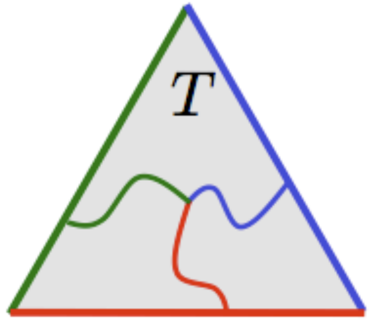
- We can construct triangular codes of arbitrary sizes. The vertices per face can be 4 and 8 so that K is in $\mathbf{N}(S)$.



2-Colexes

- The transversal H clearly amounts to an encoded H:

$$H: \begin{array}{l} X \longrightarrow Z \\ Z \longrightarrow X \end{array} \qquad \hat{H}: \begin{array}{l} T\mathsf{x} \longrightarrow T\mathsf{z} \\ T\mathsf{z} \longrightarrow T\mathsf{x} \end{array}$$



- This is also true for K. The anticommutation properties of T imply that its support consists of an odd number of qubits:

$$K: \begin{array}{l} X \longrightarrow iXZ \\ Z \longrightarrow Z \end{array} \qquad \hat{K}: \begin{array}{l} T\mathsf{x} \longrightarrow \pm iT\mathsf{x}\, T\mathsf{z} \\ T\mathsf{z} \longrightarrow T\mathsf{z} \end{array}$$

- Therefore, the **Clifford group** can be implemented transversally in triangular codes.

2-Colexes

Triangular Codes

- Encoded ***X* and *Z*** operators:

$$\hat{X} = X^{\otimes n} \quad \hat{Z} = Z^{\otimes n} \qquad \{\hat{Z}, \hat{X}\} = 0$$

n = # physical qubits.

$$[\hat{X}, B_f^Z] = 0, \quad [\hat{Z}, B_f^X] = 0$$

- The **Clifford group** is implemented with global operators:

$$\hat{H} = H^{\otimes n} \qquad \hat{K} = K^{\otimes n} \qquad \hat{\Lambda} = \Lambda^{\otimes n}$$

$$\hat{H}\hat{X}\hat{H}^\dagger = \hat{Z} \qquad \hat{K}\hat{X}\hat{K}^\dagger = \pm i\hat{X}\hat{Z} \qquad \hat{\Lambda}\widehat{IX}\hat{\Lambda}^\dagger = \widehat{IX}, \quad \hat{\Lambda}\widehat{XI}\hat{\Lambda}^\dagger = \widehat{XX}$$

$$\hat{H}\hat{Z}\hat{H}^\dagger = \hat{X} \qquad \hat{K}\hat{Z}\hat{K}^\dagger = \hat{Z} \qquad \hat{\Lambda}\widehat{IZ}\hat{\Lambda}^\dagger = \widehat{ZZ}, \quad \hat{\Lambda}\widehat{ZI}\hat{\Lambda}^\dagger = \widehat{ZI}$$

$$\hat{H}B_f^X\hat{H}^\dagger = B_f^Z \qquad \hat{K}B_f^X\hat{K}^\dagger = B_f^X B_f^Z \qquad \hat{\Lambda}IB_f^X\hat{\Lambda}^\dagger = IB_f^X, \quad \hat{\Lambda}B_f^X I\hat{\Lambda}^\dagger = B_f^X B_f^X$$

$$\hat{H}B_f^Z\hat{H}^\dagger = B_f^X \qquad \hat{K}B_f^Z\hat{K}^\dagger = B_f^Z \qquad \hat{\Lambda}IB_f^Z\hat{\Lambda}^\dagger = B_f^Z B_f^Z, \quad \hat{\Lambda}B_f^Z I\hat{\Lambda}^\dagger = B_f^Z I$$

2-Colexes

- **Ground State GS** can be described by applying string-net operators to the GS:

We can give an expression for the states of the logical qubit
$\{|\bar{0}\rangle, |\bar{1}\rangle\}$

:

$$|\bar{0}\rangle = \prod_{\mathrm{b}} (1 + B_b^X) \prod_{\mathrm{p}} (1 + B_p^X) |0\rangle^{\otimes n}$$

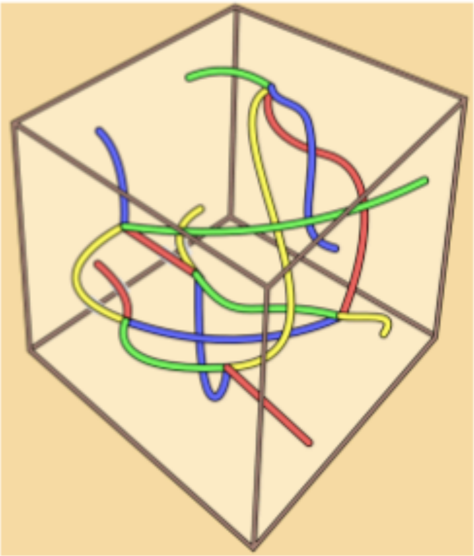
and

$$|\bar{1}\rangle := \hat{X} |\bar{0}\rangle$$

$$\hat{Z} |\bar{l}\rangle = (-1)^l |\bar{l}\rangle \qquad l = 0, 1$$

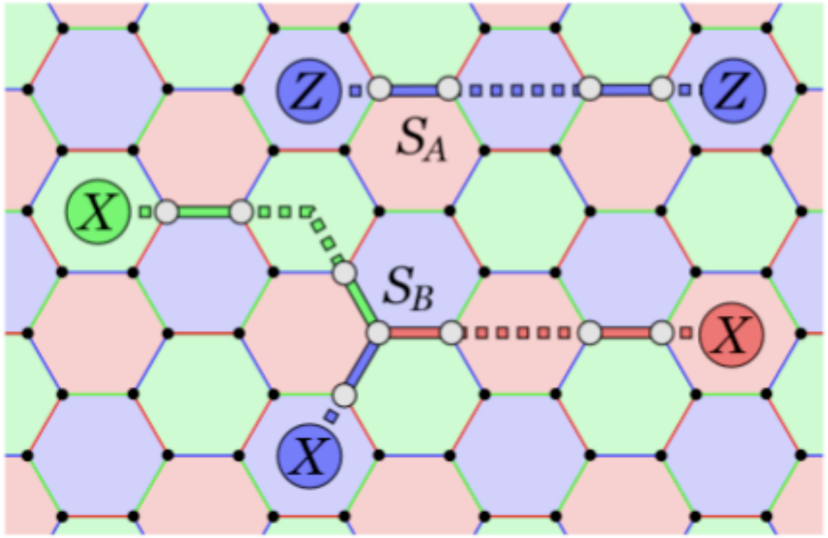
.

$$|\bar{0}\rangle = \sum_{\mathrm{string-nets}} B_s^X |0\rangle^{\otimes n}$$



2-Colexes

- **Excitations** can be created applying string operators to the GS:



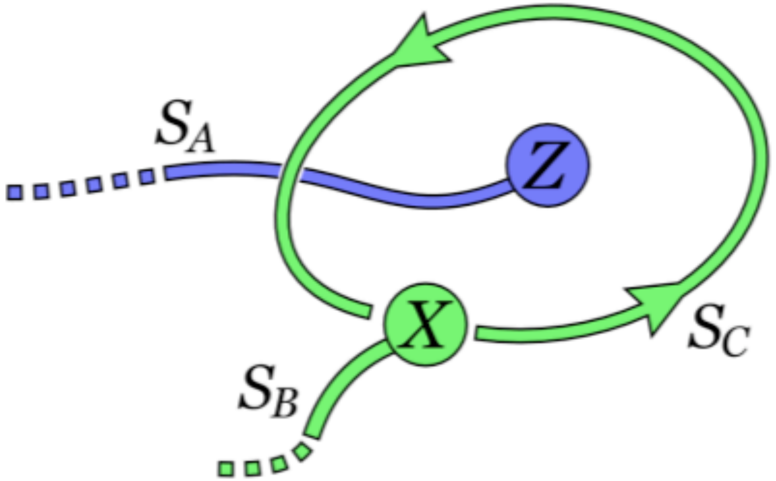
$S^z_A\, S^x_B\, |\mathrm{GS}\rangle$

Each endpoint is a quasiparticle,
a violation of a face condition.

Anyons

2-Colexes

- The quasiparticles that populate the system are **abelian anyons**.
- When, for example, a green *X* excitation loops around a blue *Z* excitation, the system gets a global **minus sign**:



$$\{S^{z}_{A}, S^{x}_{C}\} = 0$$

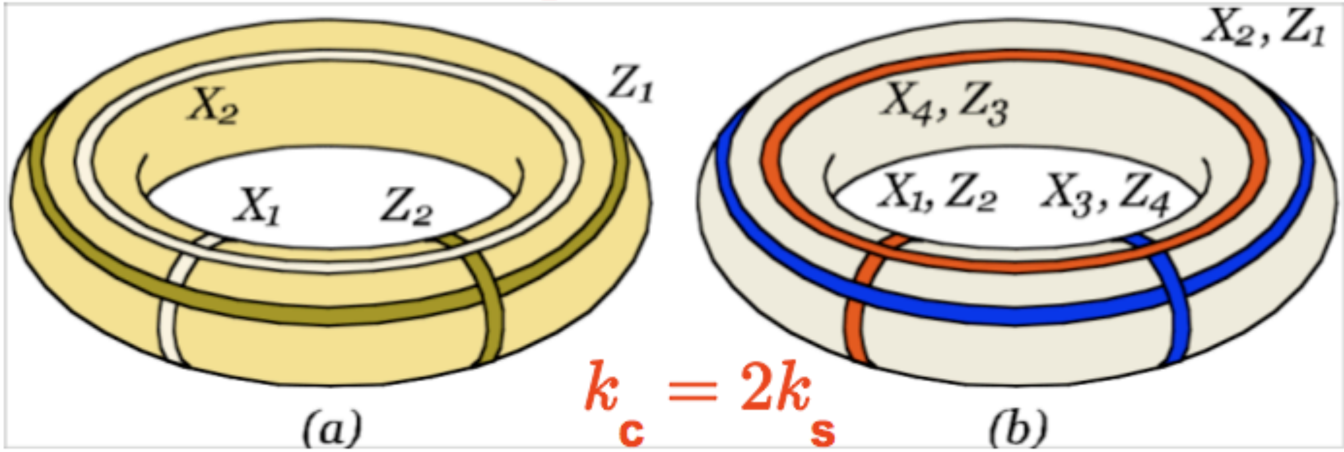
$$S^{x}_{C}\,(S^{z}_{A}\,S^{x}_{B}\,|\mathrm{GS}\rangle) = -S^{z}_{A}\,S^{x}_{B}\,|\mathrm{GS}\rangle$$

- Note that excitations, or their braiding, **play no role** in our computational model. All the operations are carried out in the ground state of the system.

2-Colexes

Topological 2D Stabilizer Codes: Comparative Study

Pauli operator bases in the torus



$k_c = 2k_s$

- A color code encodes twice as much logical qubits as a surface code does

•We compute the topological error correcting rate $C := n/d^2$
for surface codes C_s

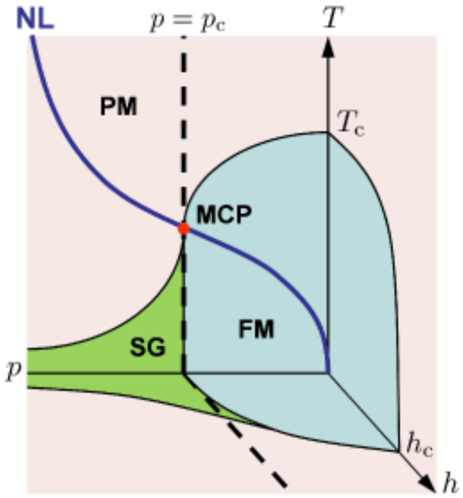
and color codes C_c
in several instances.

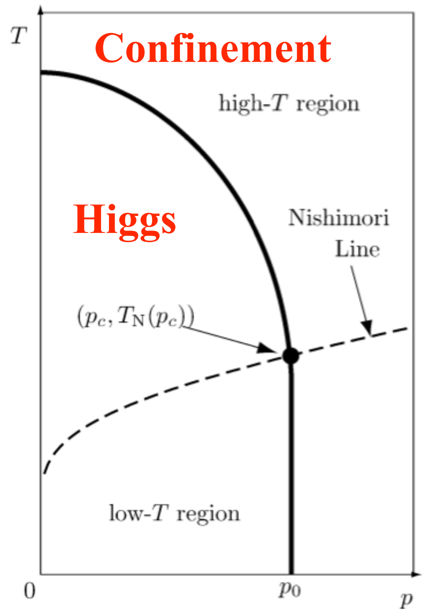
Ⅳ) Mapping to Statistical Models

Challenges for New States of Matter…

EXTERNALLY PROTECTED QUANTUM COMPUTER

→ **Good Candidate: TOPOLOGICAL QUANTUM COMPUTER**

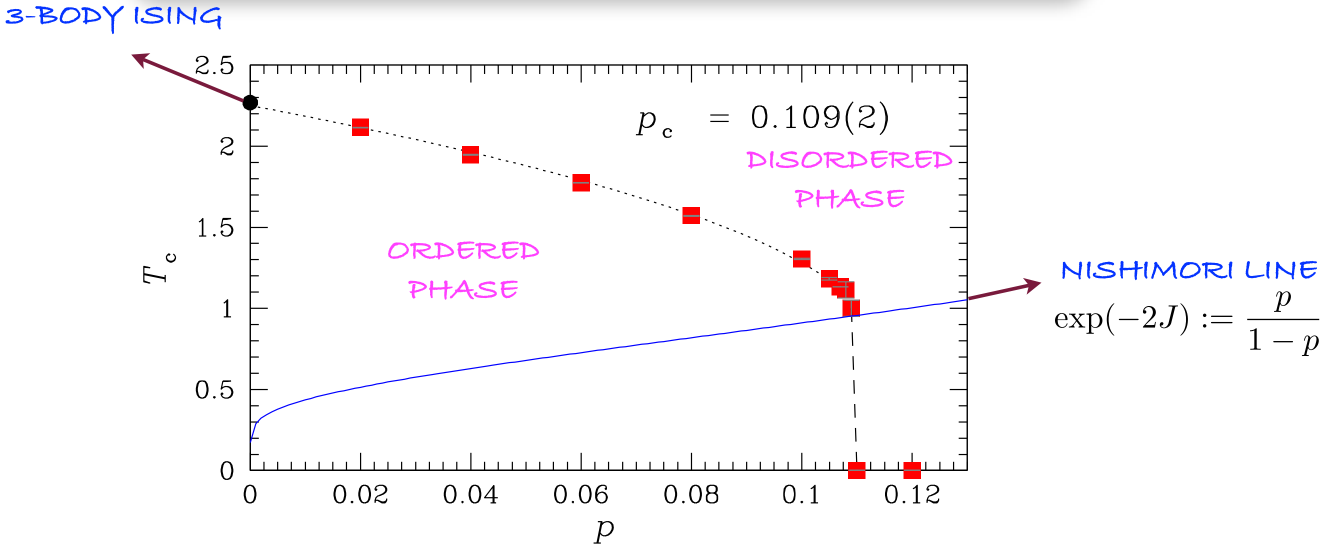




FUNDAMENTAL RESULT

- PHASE DIAGRAM: RANDOM 3-BODY ISING MODEL

$$Z[J,\tau] := \sum_{\sigma} e^{J \sum_{\langle ijk \rangle} \tau_{ijk}\sigma_i\sigma_j\sigma_k}$$



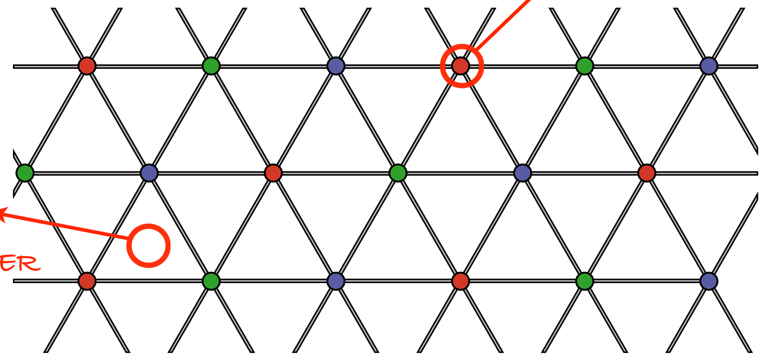
TOPOLOGICAL COLOR CODES

-

ENHANCING QUANTUM CAPABILITIES WITHOUT LOWERING RESISTANCE TO NOISE

FUNDAMENTAL RESULT

CLASSICAL SPIN σ_i



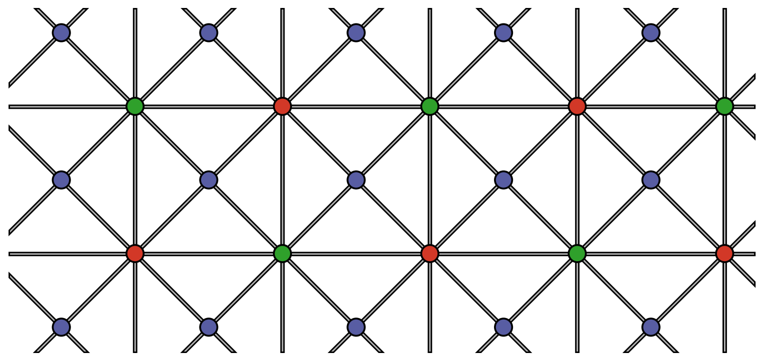
TRIANGULAR LATTICE

$$\tau_\triangle = \tau_{ijk}$$

QUENCHED DISORDER

$$P(\tau_{ijk}) := p\ \delta(\tau_{ijk}+1) + (1-p)\ \delta(\tau_{ijk}-1)$$

(a)

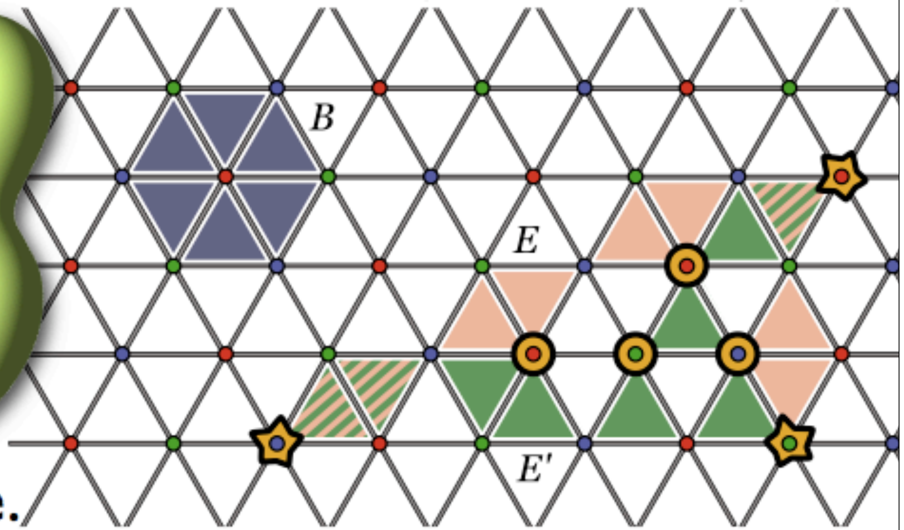


UNION-JACK LATTICE

(b)

Topological color codes

- Start from a 2D 3-colorable triangular lattice.
- Embed the lattice in a nontrivial compact surface with $g \geq 1$.



- A qubit is placed on each triangle.
- The stabilizer group is given by the following vertex operators acting on nearby triangles: $X_v := \bigotimes_{\triangle : v \in \triangle} X_\triangle \qquad Z_v := \bigotimes_{\triangle : v \in \triangle} Z_\triangle$

  the vertex operators pairwise commute and square to unity.
- The code is defined on the subspace with $X_v = Z_v = 1 \quad \forall\, v$.
- The resulting collection of ±1 eigenvalues is the error syndrome.
- The stabilizers do not mix X (bit-flip) and Z (phase) operators.

Topological color codes

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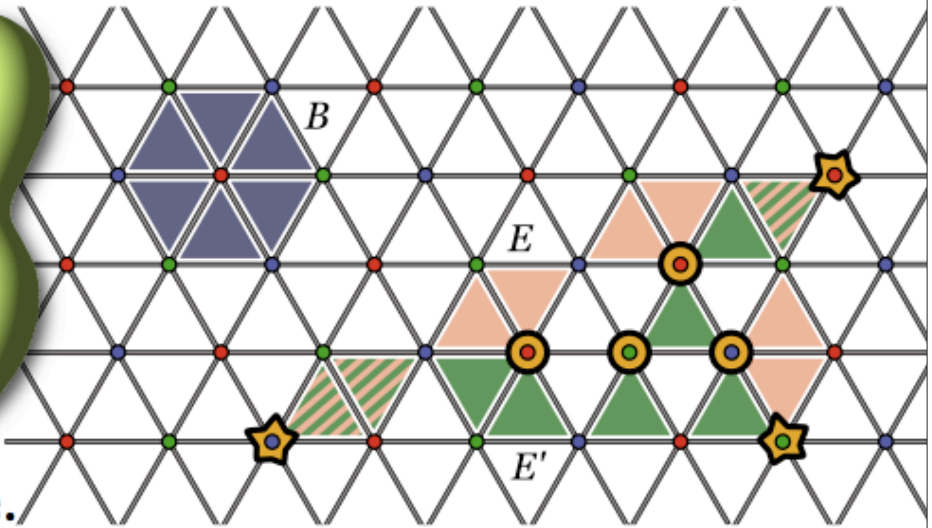
- Embed the lattice in a nontrivial compact surface with $g \geq 1$.



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  the vertex operators pairwise commute and square to unity.
- The code is defined on the subspace with $X_v = Z_v = 1 \;\; \forall\, v$.
- The resulting collection of ±1 eigenvalues is the error syndrome.

We can treat bit-flip and phase errors separately.

Threshold: map to a statistical model

- Error correction is achievable if in the limit of infinite size:

$$\sum_E P(E)P(\overline{E}|\partial E) \to 1$$

Dennis *et al.*, J Math Phys (0

  - $P(E) \propto [p/(1-p)]^{|E|}$, with E a bit-flip error with probability p.
  - $P(\overline{E}|\partial E)$ is the probability that a syndrome ∂E was caused by an error in the homology class $\overline{E}$.

- Mapping:
  - Set $\exp(-2J) = p/(1-p)$ for the Nishimori line. It follows that $P(E) \propto \exp(\sum_\triangle \tau_\triangle)$.
  - $\tau_\triangle = \pm 1$ negative when $\triangle \in E$.
  - Insert classical spin variables $\sigma_i = \pm 1$ at the vertices. We obtain

$$P(\bar{E}) \propto Z[J,\tau] := \sum_\sigma e^{J \sum_{\langle ijk \rangle} \tau_{ijk} \sigma_i \sigma_j \sigma_k}$$

Katzgraber *et al.*, PRL submitted (09

Random 3-body Ising model

- Hamiltonian:

$$\mathcal{H} = J \sum_{\langle ijk \rangle} \tau_{ijk} \sigma_i \sigma_j \sigma_k$$

- Details:
  - Ising spins are placed on the vertices of a triangular lattice in 2D.
  - A bit-flip error corresponds to $\tau_{ijk} = -1$ with probability p.
  - $p > 0$: glassy Ising model with 3-body interactions without spin-reversal symmetry.

- Error threshold:
  - Compute the p–T_c phase diagram of the model.
  - p_c corresponds to the critical p along the Nishimori line where *ferromagnetic* order ($p = 0$) is *lost*.

Dennis *et al.*, J Math Phys (02)

Probing criticality: correlation length

- Study the finite-size two-point correlation function.
- *k*-space susceptibility...

$$\chi(\mathbf{k}) = \frac{1}{N}\sum_{ij}\langle\sigma_i\sigma_j\rangle_T e^{i\mathbf{k}(\mathbf{R}_i-\mathbf{R}_j)}$$

- Perform an Ornstein-Zernicke approximation...

$$[\chi(k)/\chi(0)]^{-1} = 1 + \xi^2 k^2 + \mathcal{O}[(\xi k)^4]$$

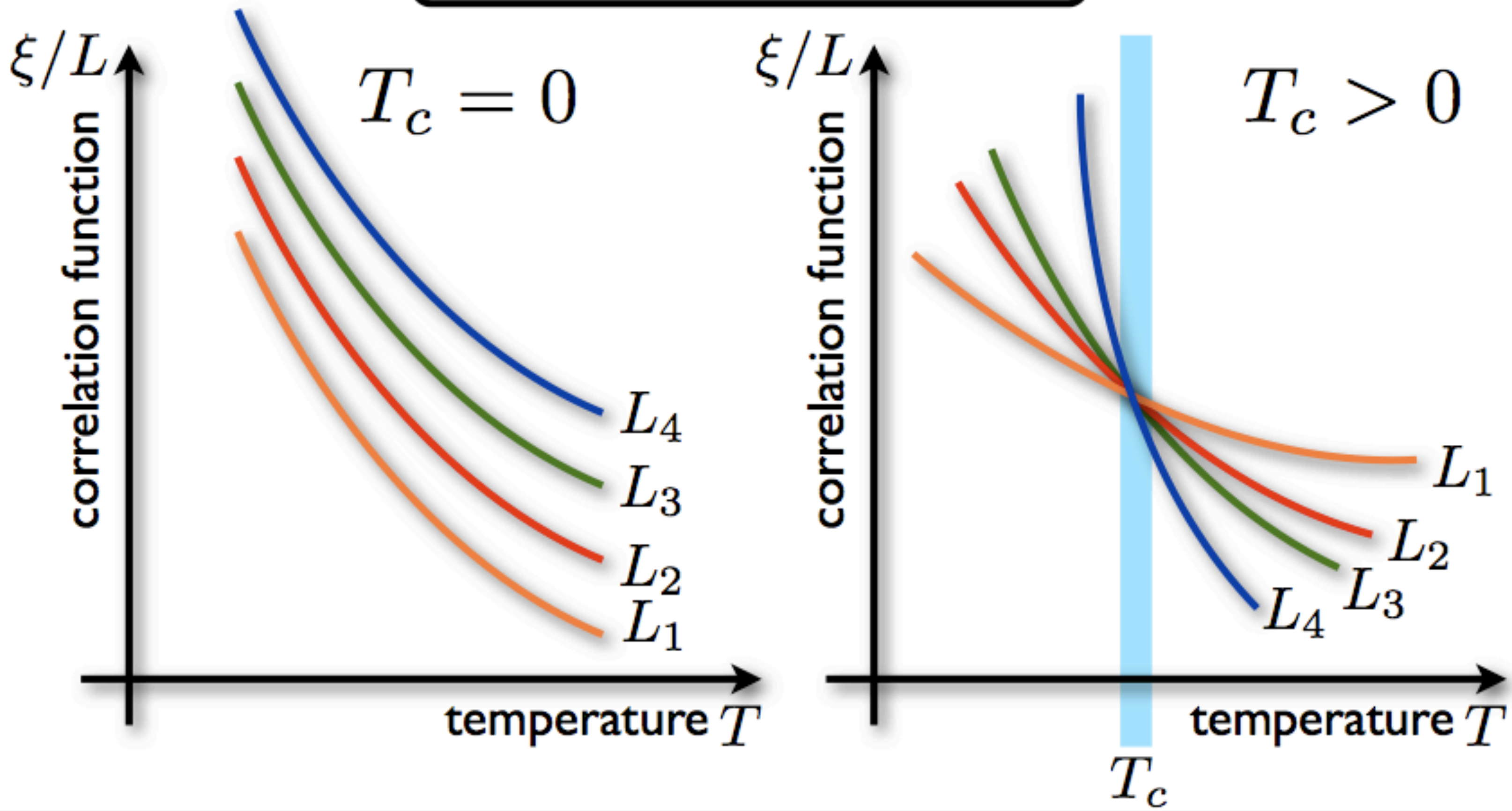
- Compute the two-point correlation function:

$$\xi = \frac{1}{2\sin(k_{\rm min}/2)}\sqrt{\frac{[\chi(0)]_{\rm av}}{[\chi(k_{\rm min})]_{\rm av}} - 1}$$

Probing criticality: correlation length

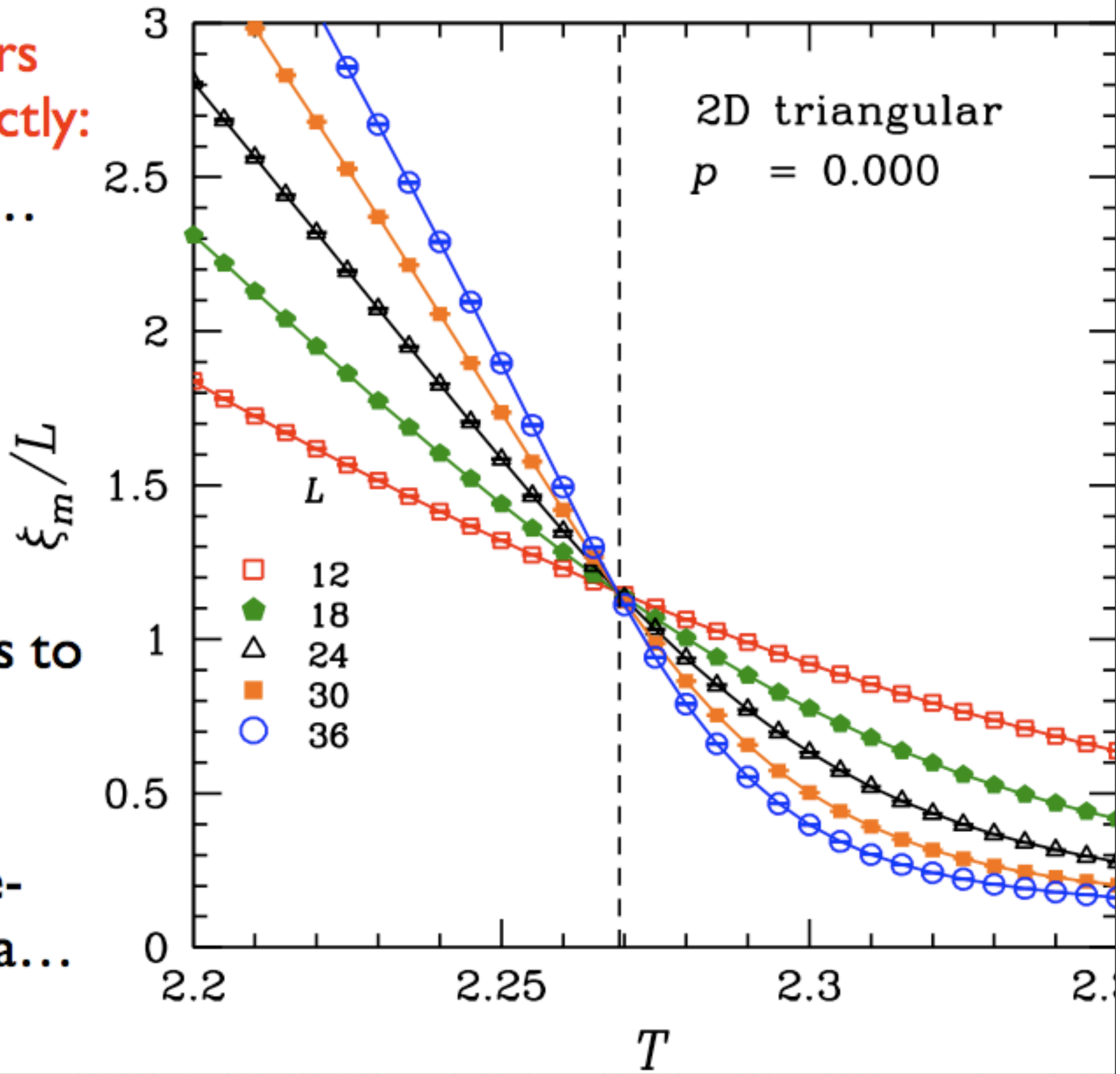
- Study the finite-size two-point correlation function
- Scaling behavior:

$$\frac{\xi}{L} = \tilde{X}\left(L^{1/\nu}[T - T_c]\right)$$



Benchmark case: p = 0

- The critical parameters can be computed exactly:
  - $T_c = T_c^{\text{ising}} = 2.269\ldots$
  - $\nu = \alpha = 2/3$
- Agreement with exact results.
- No visible corrections to scaling.
- Next: Perform a finite-size scaling of the data…



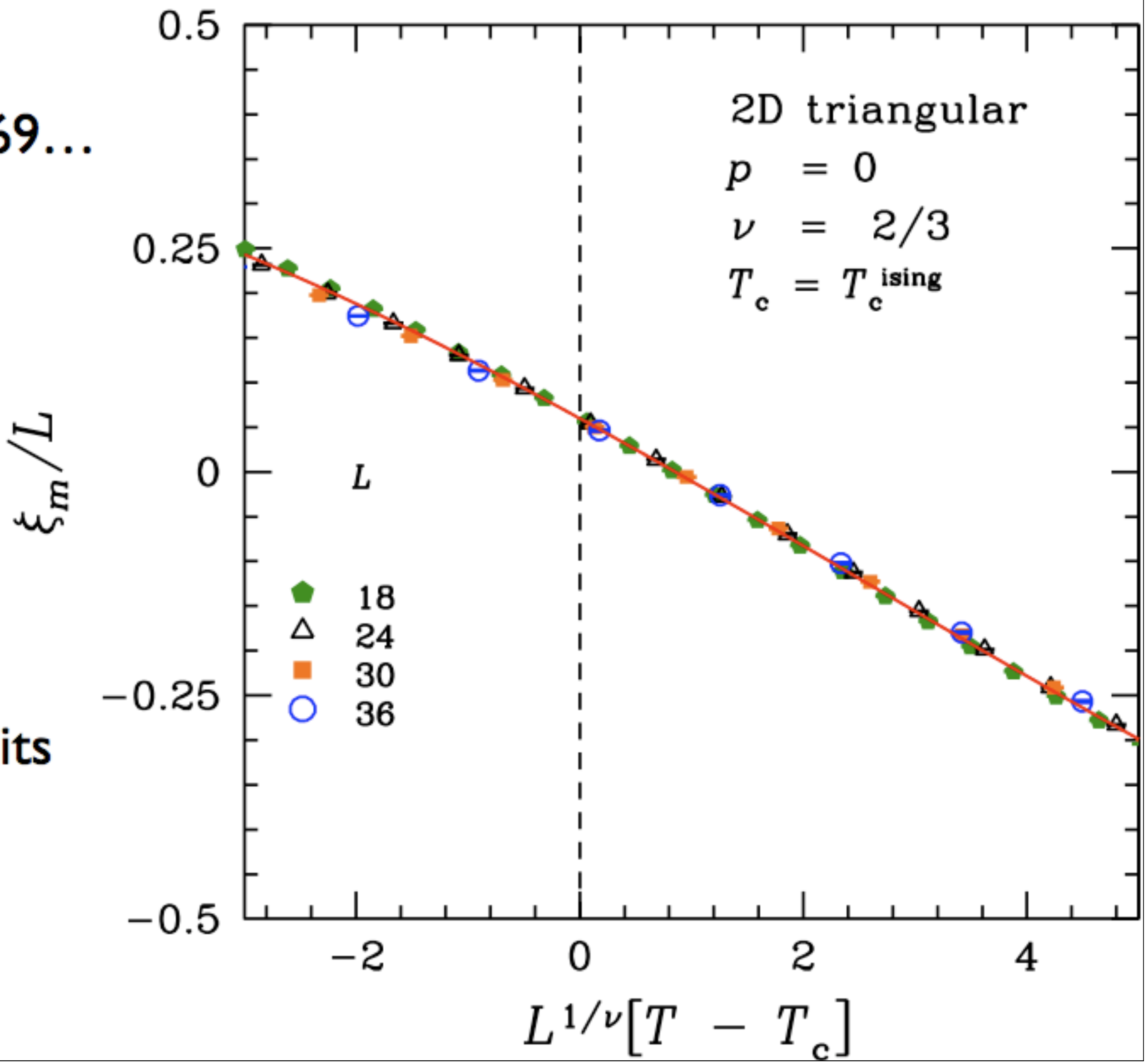
Scaling with known exponents (p = 0)

- Fixed parameters:
  - $T_c = T_c^{\text{ising}} = 2.269\ldots$
  - $\nu = 2/3$

- The finite-size scaling is perfect.
- The code works!

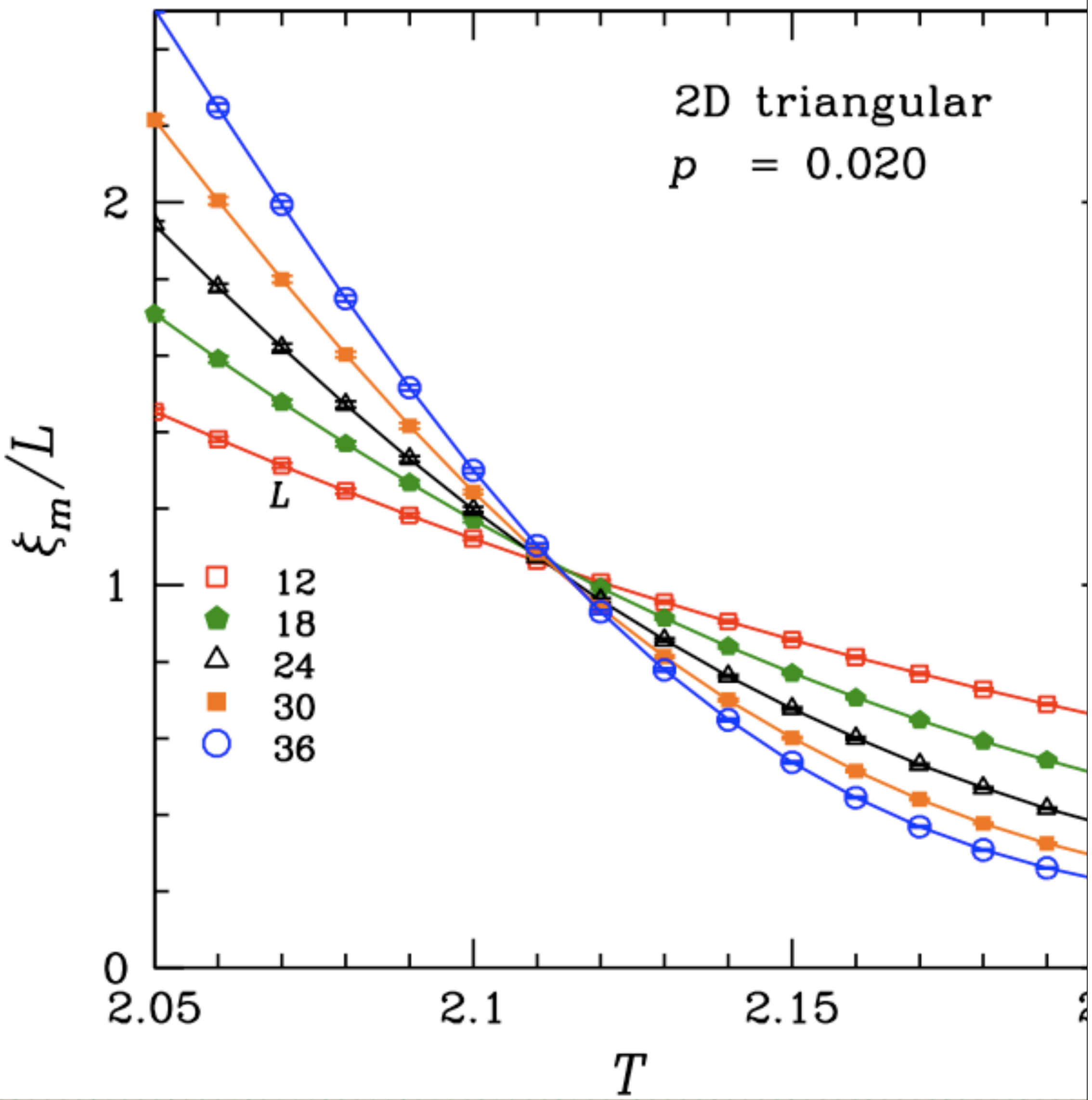
- Next: introduce errors by flipping bits

$$\tau_{ijk} \longrightarrow -\tau_{ijk}$$



Introduce qubit errors: $p > 0$

- For each value of p compute T_c.
- Corrections around $p \sim 0.108$.



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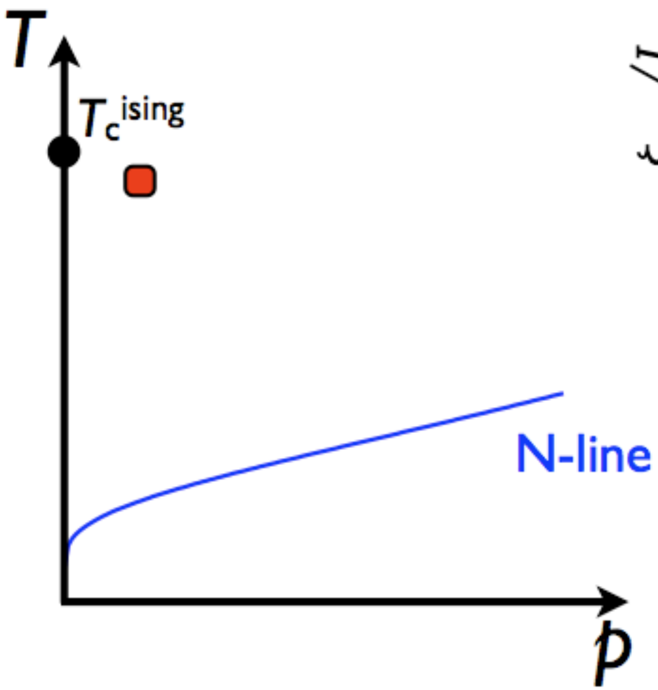

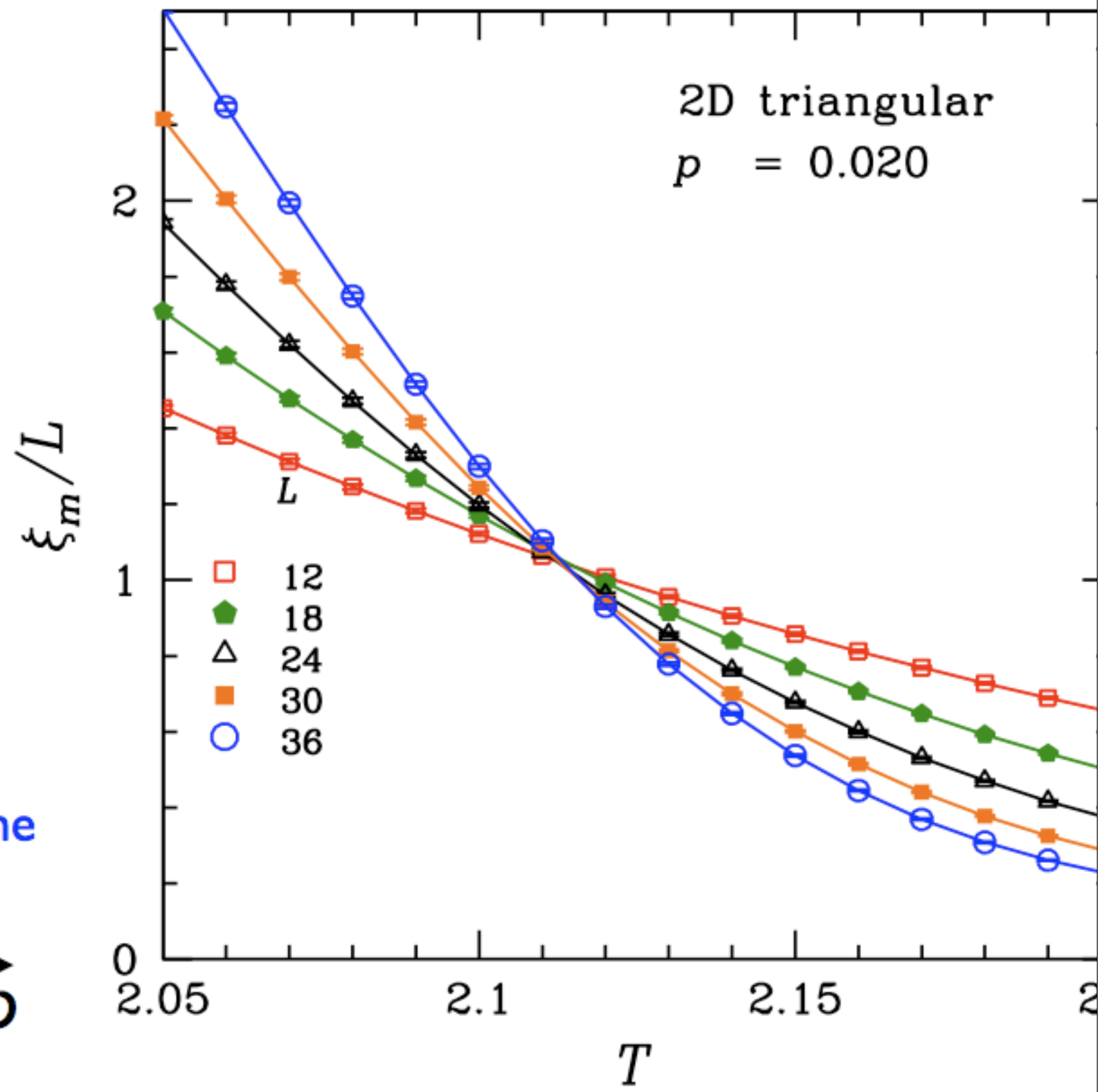

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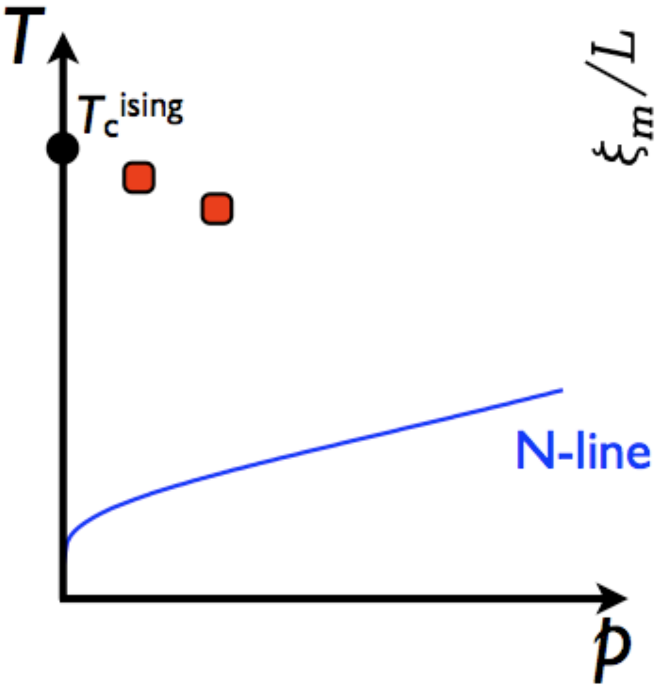

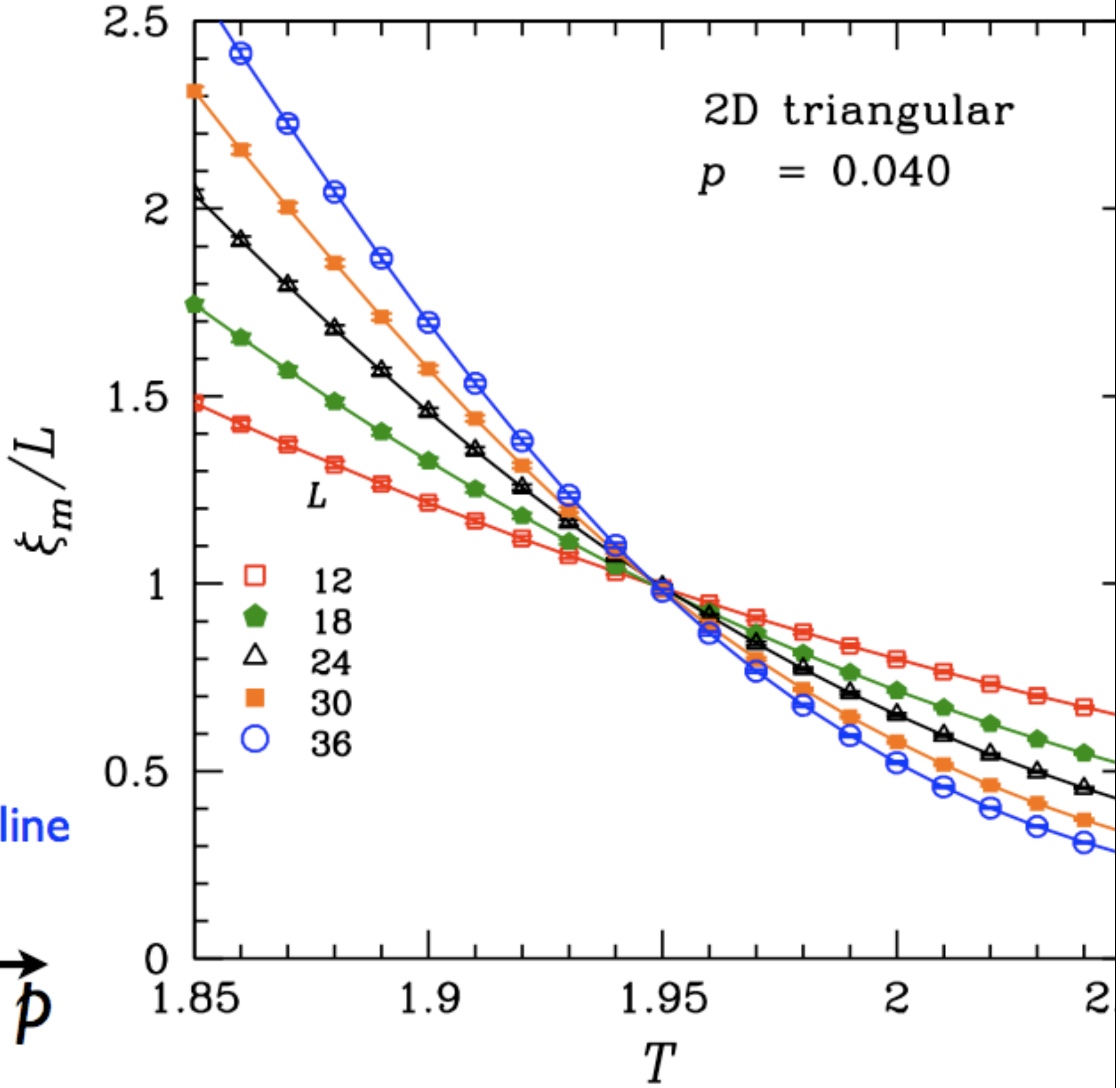

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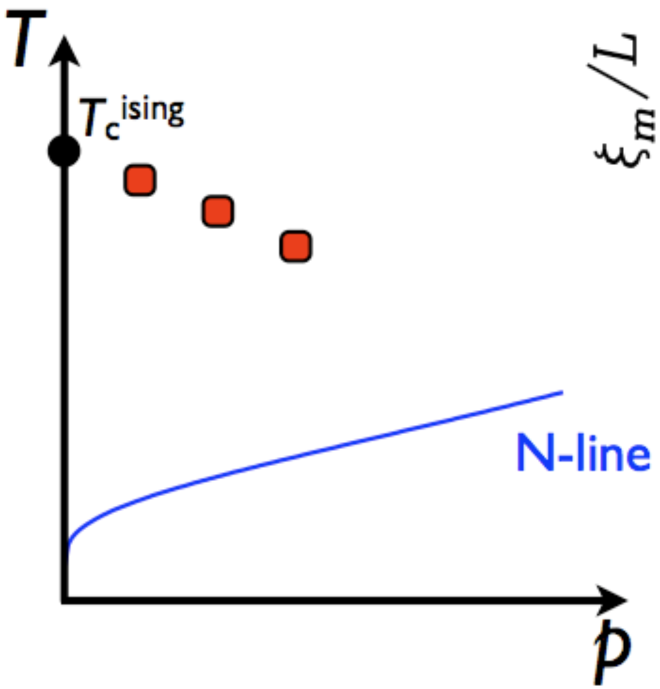

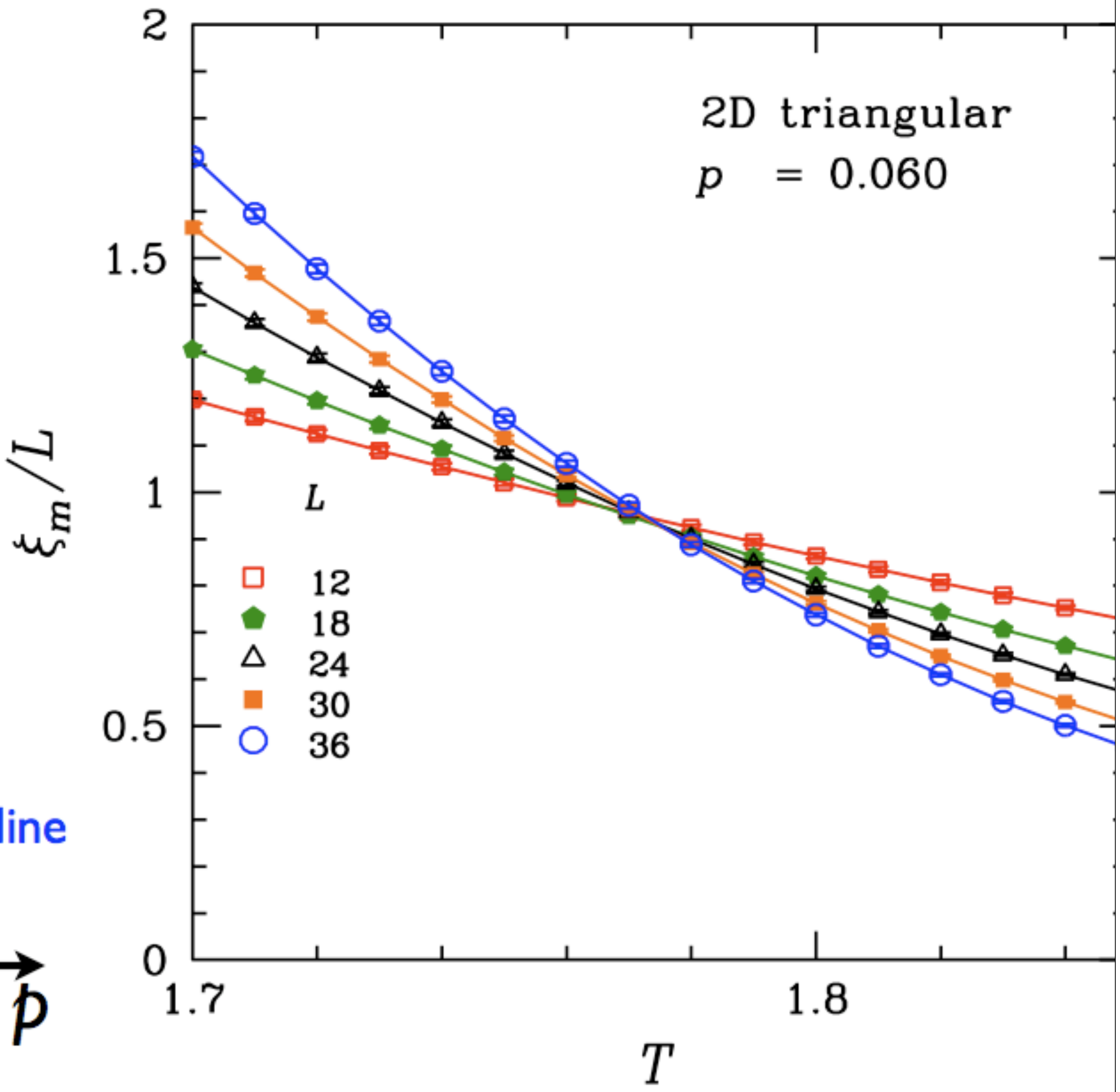

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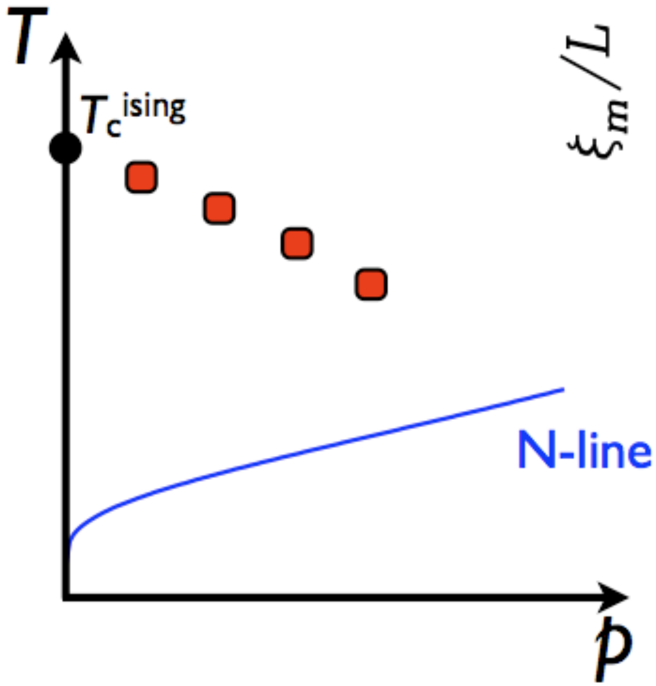

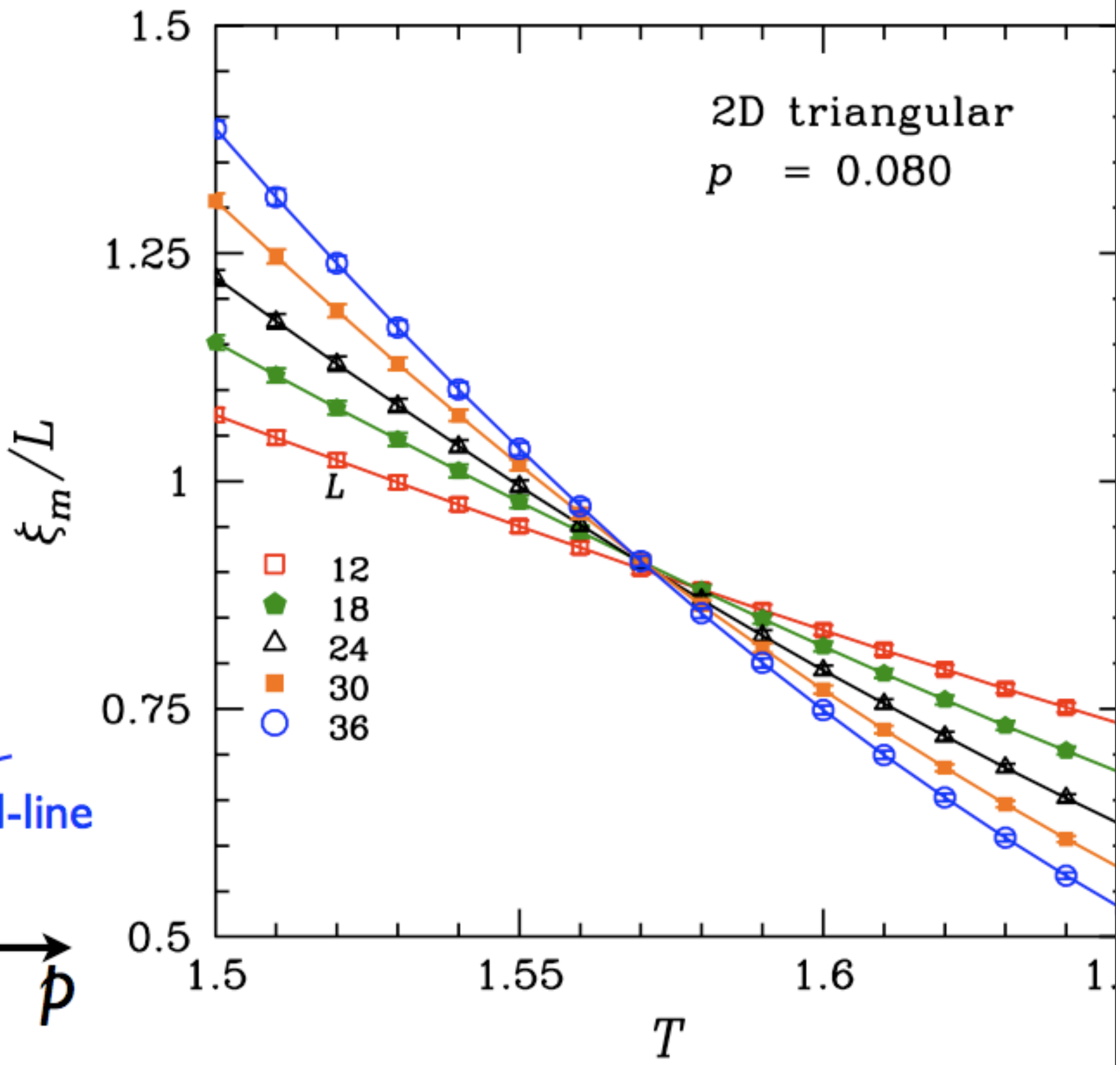

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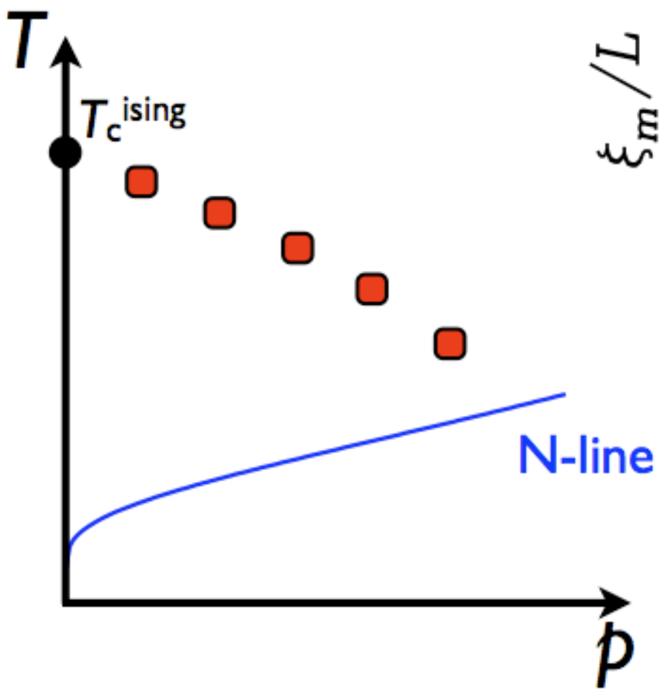

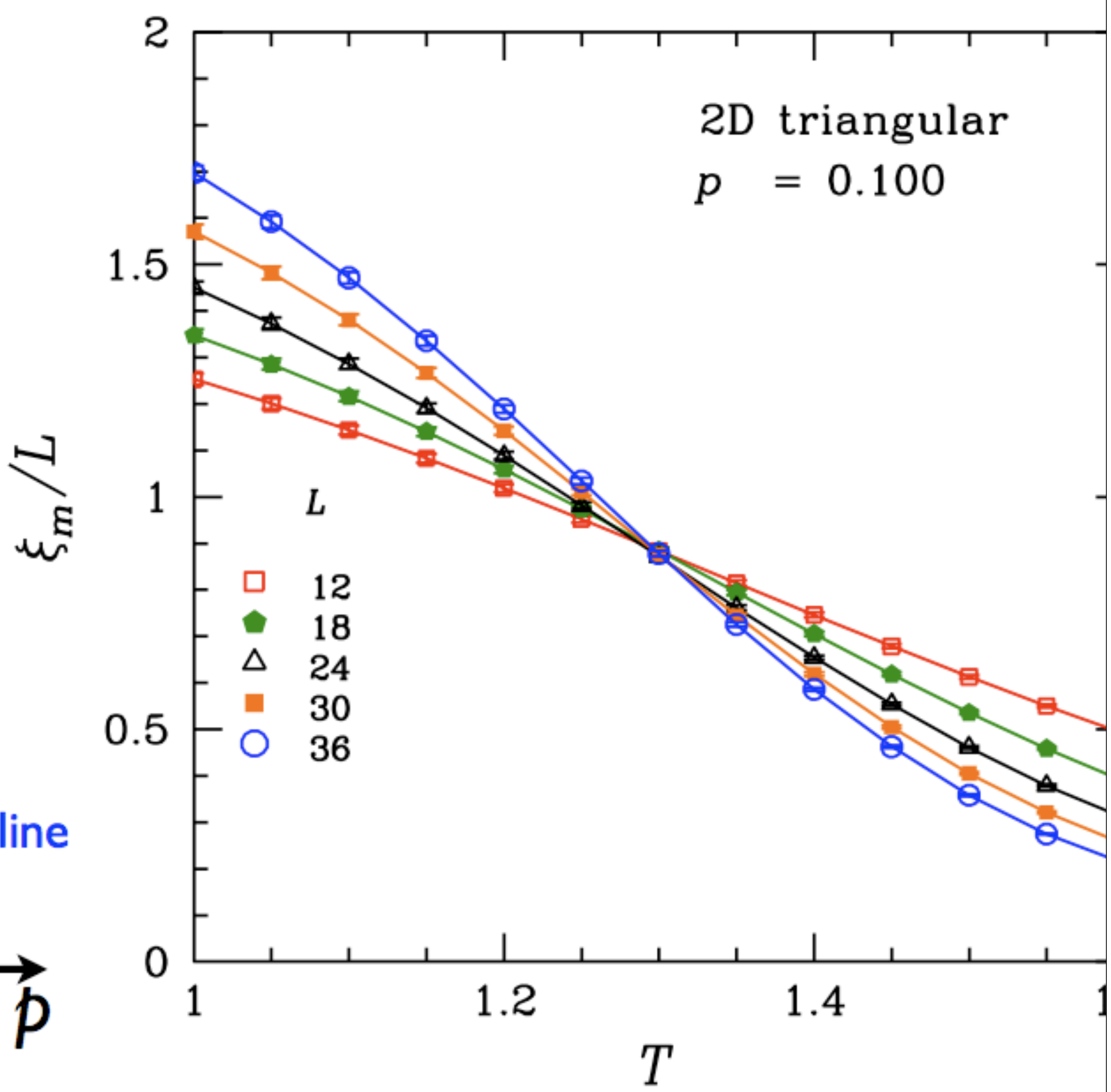

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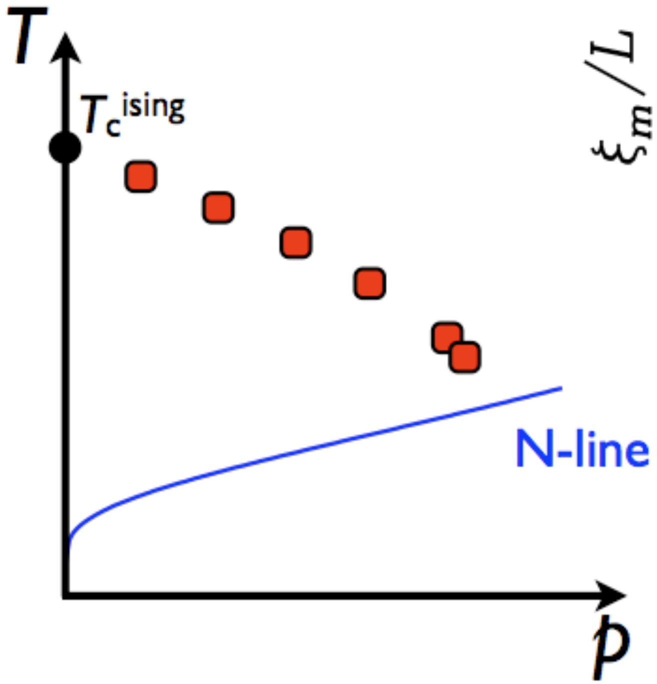

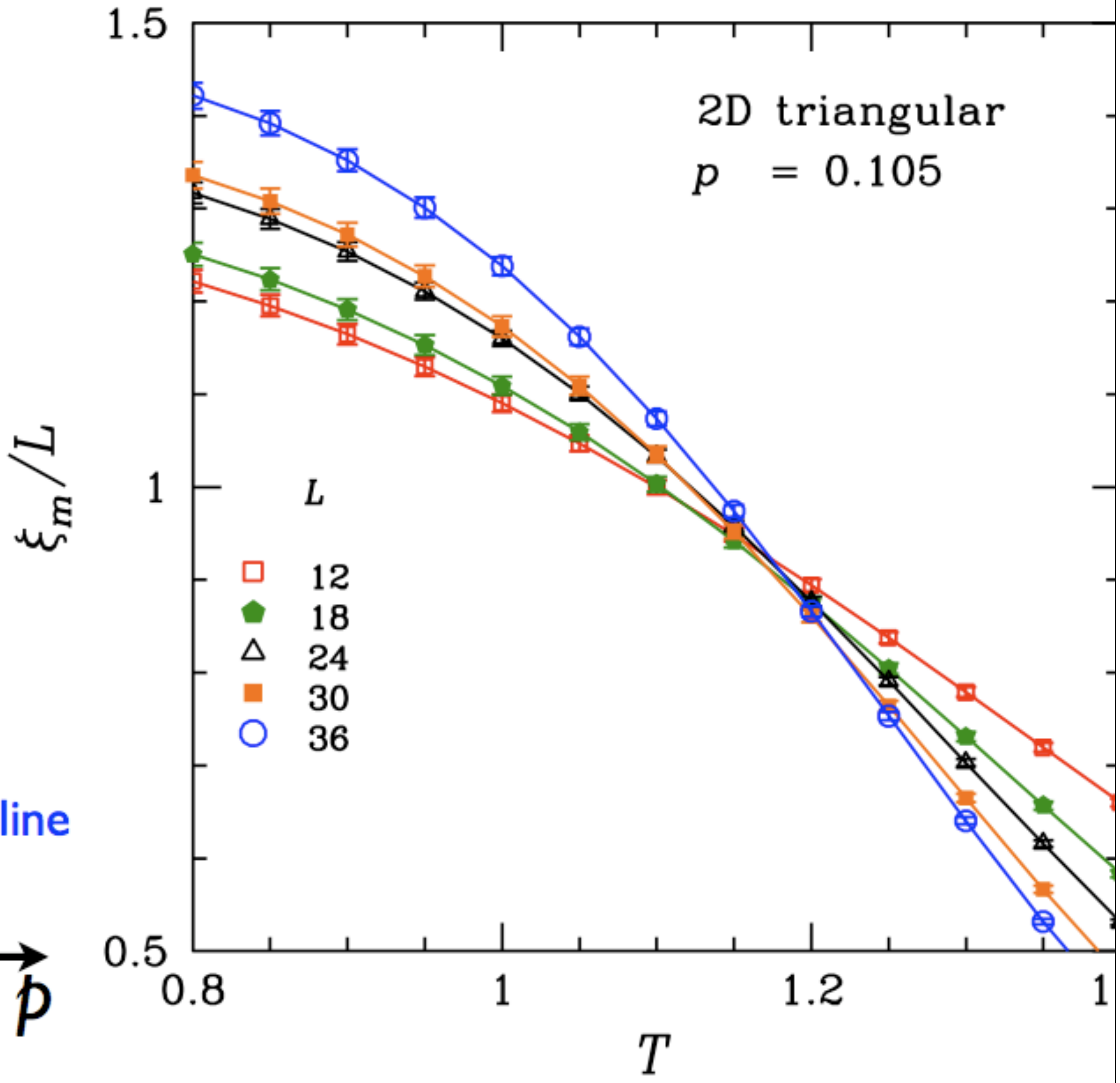

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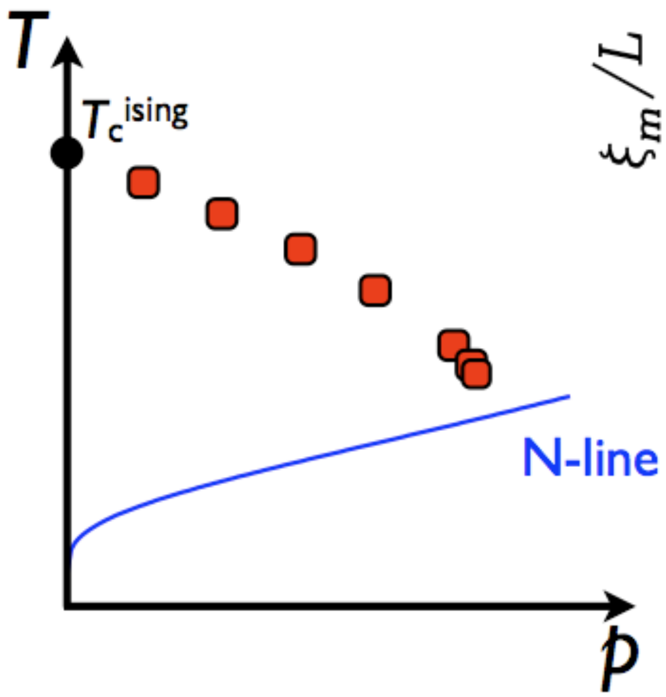

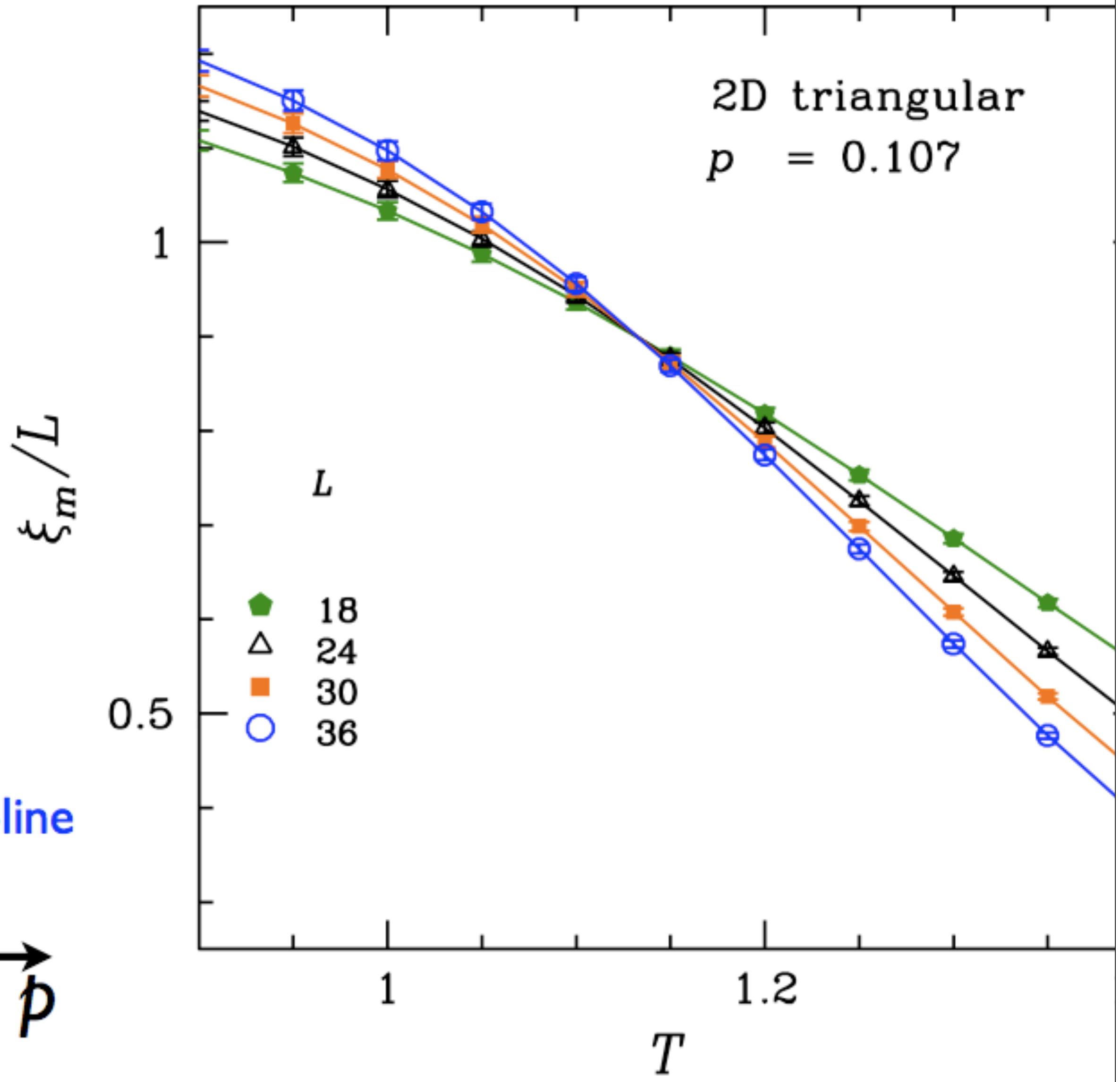

Introduce qubit errors: $p > 0$

- For each value of p compute T_c.
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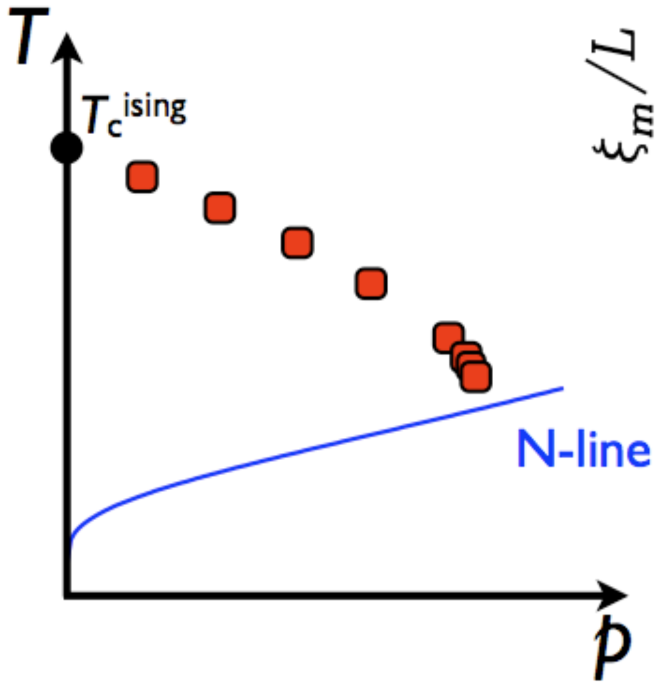

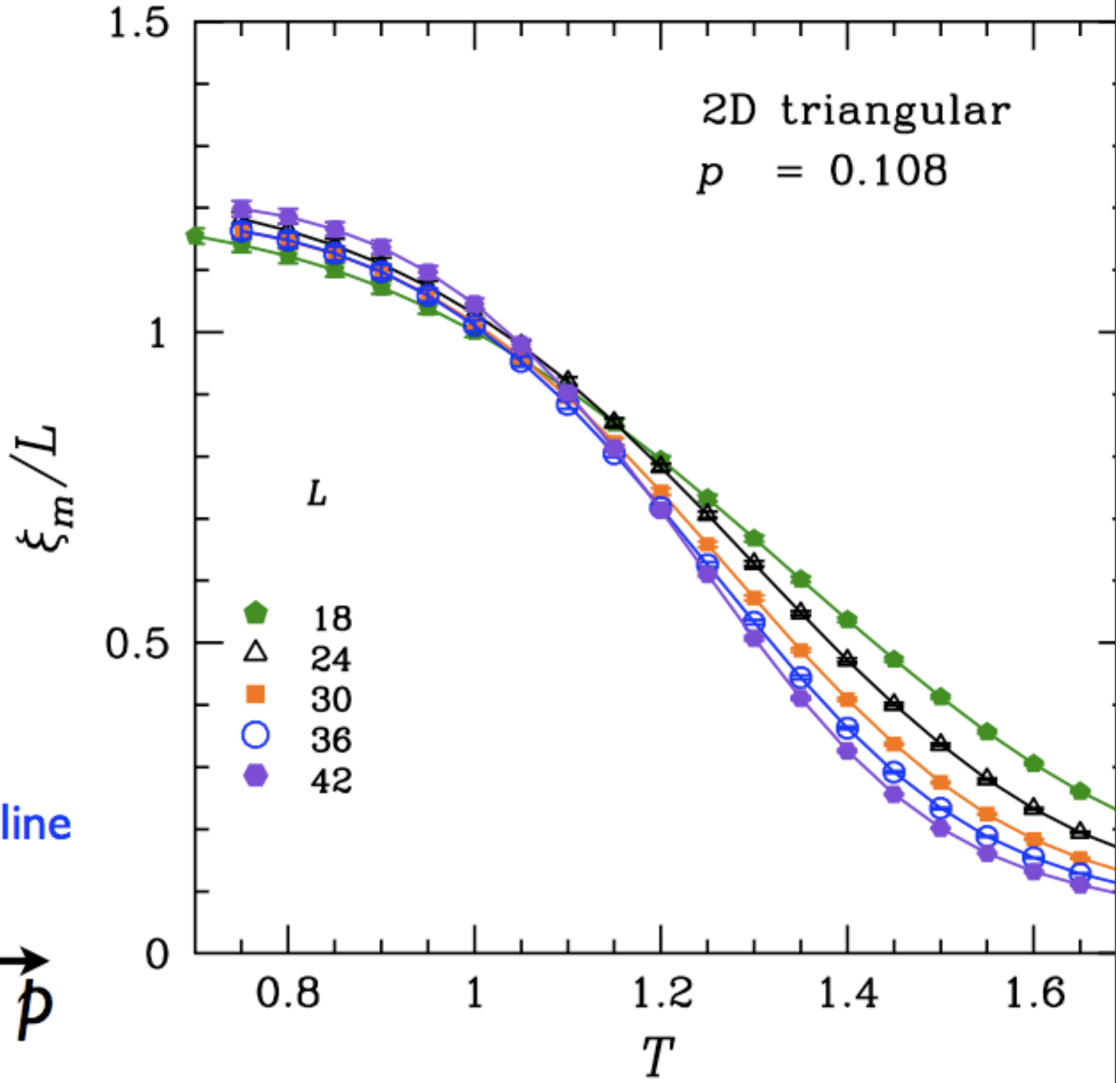

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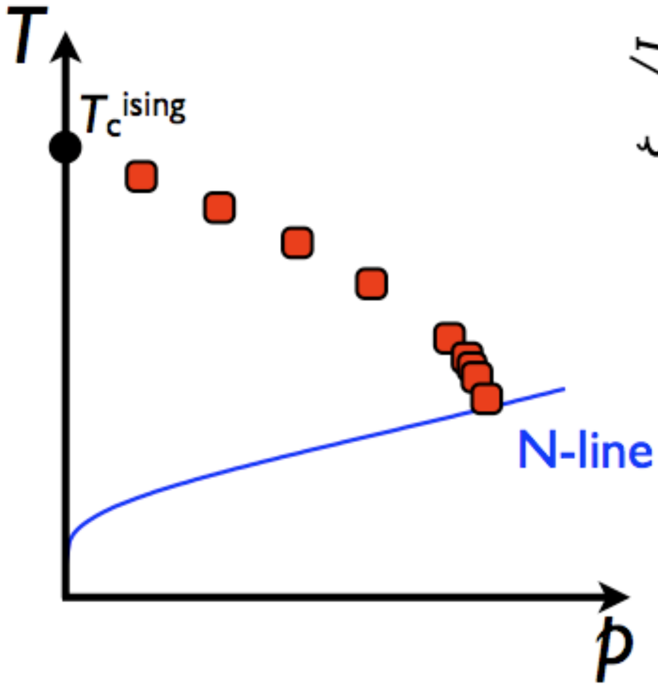

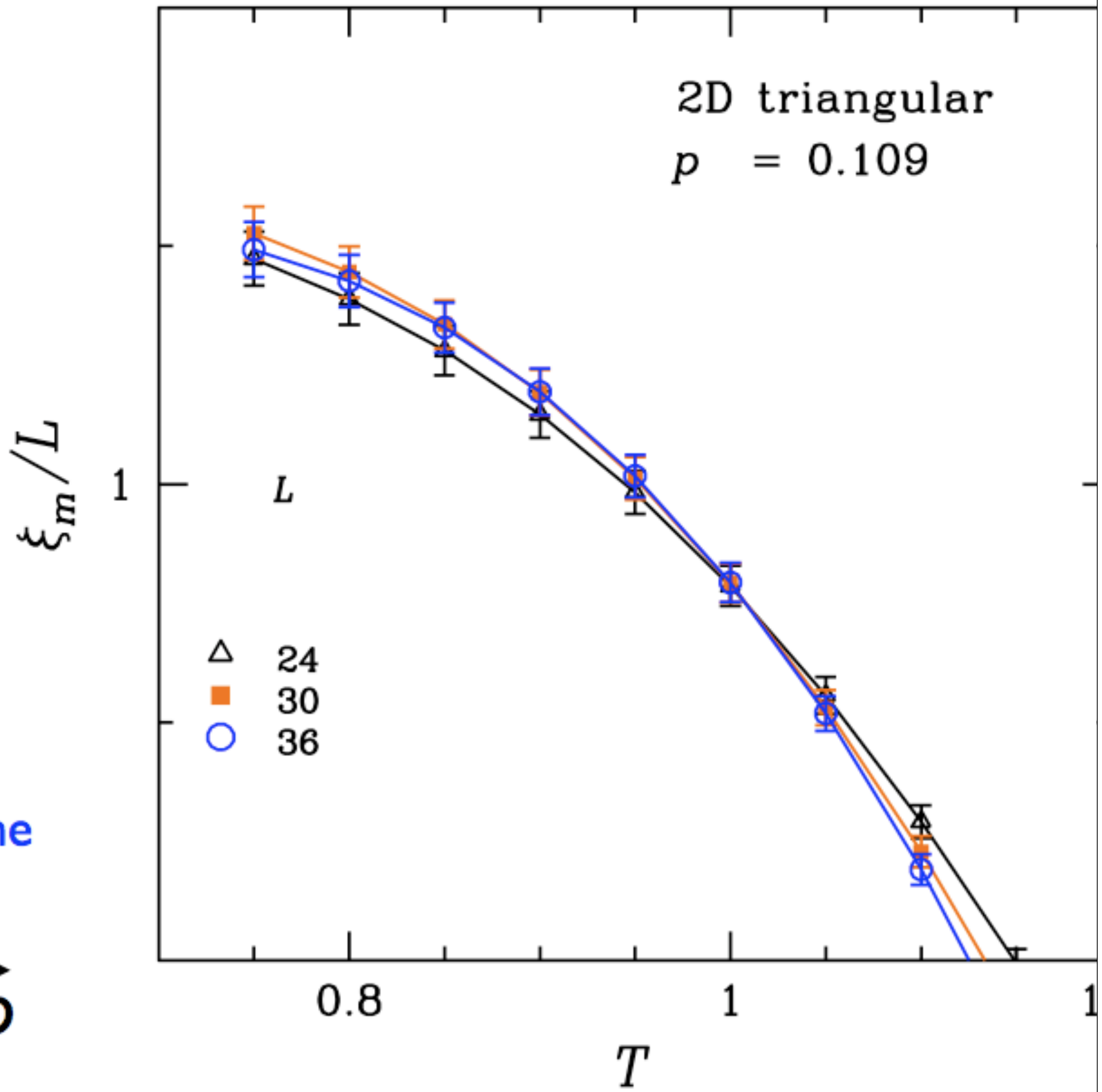

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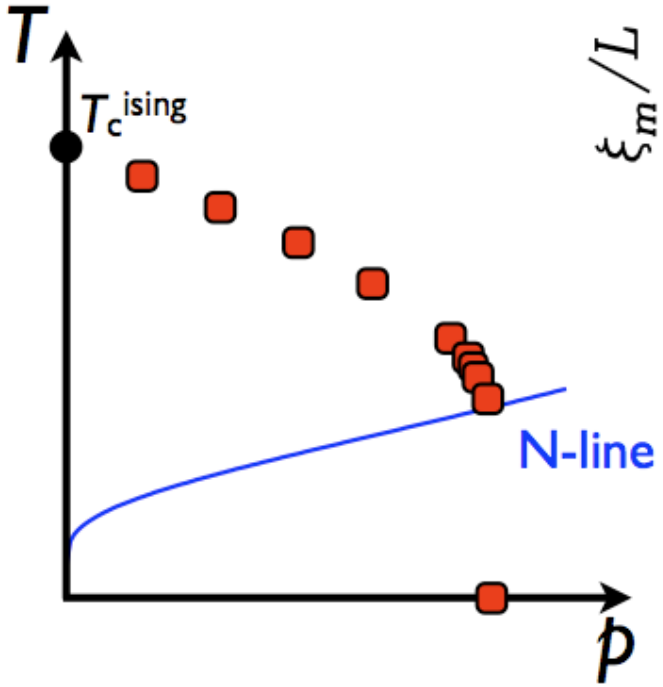

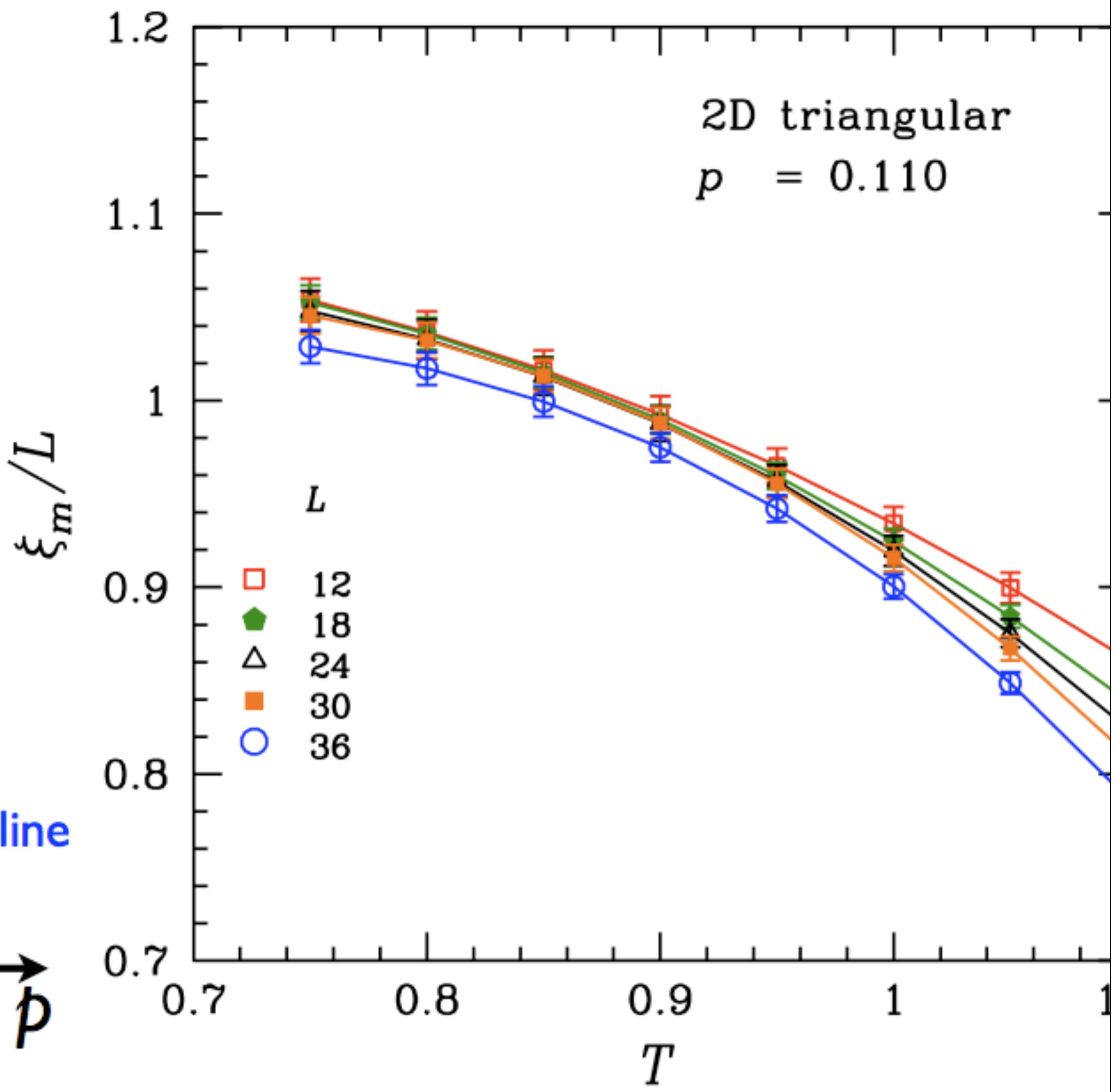

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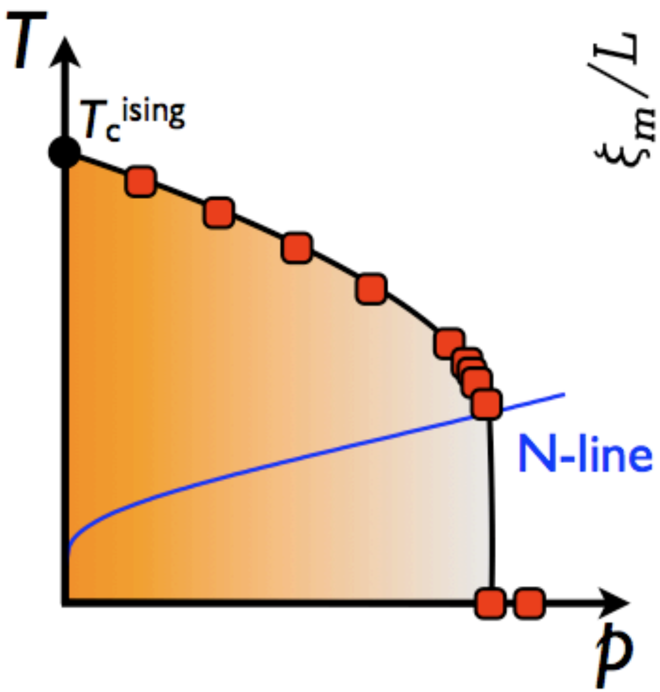
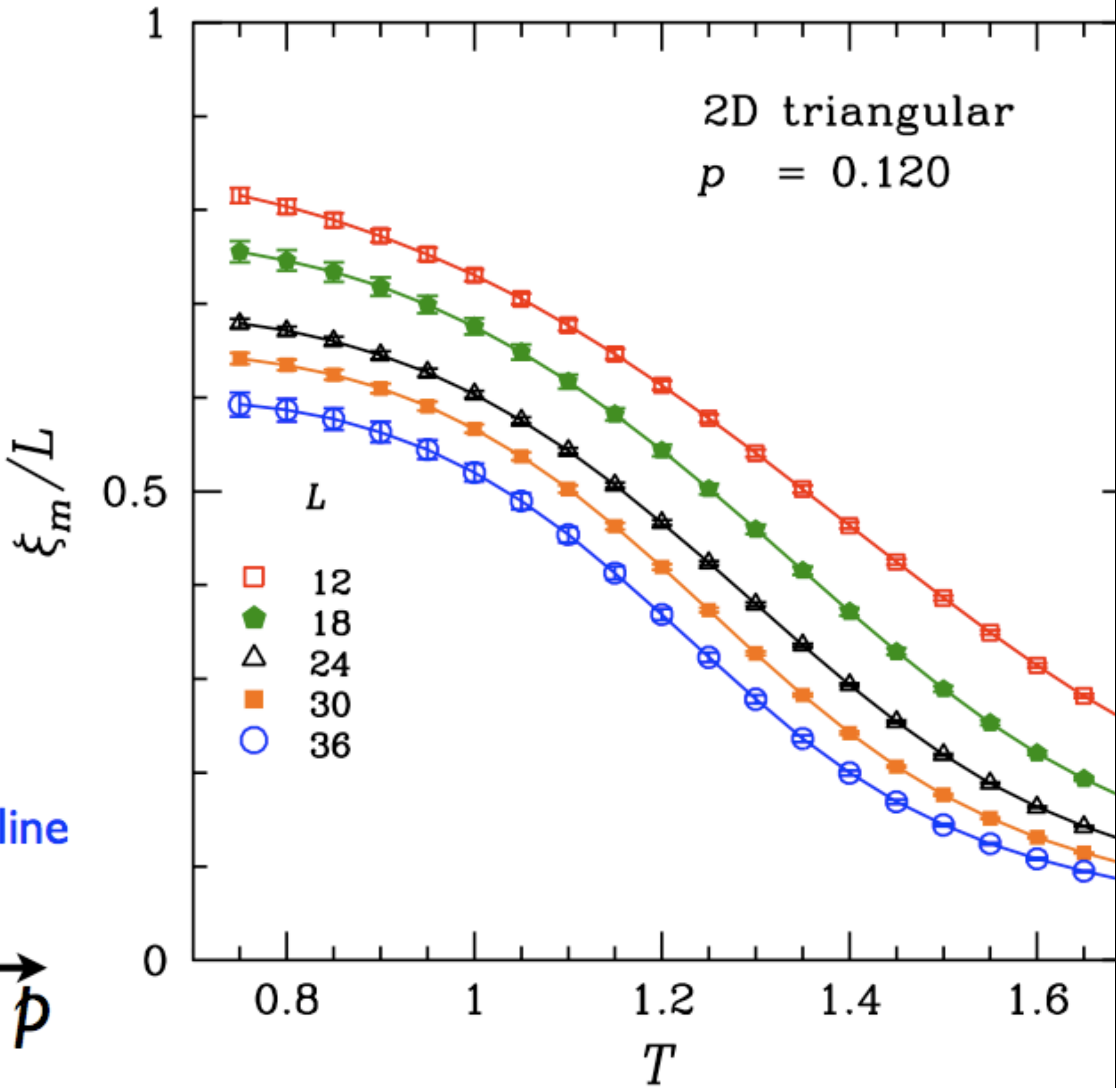


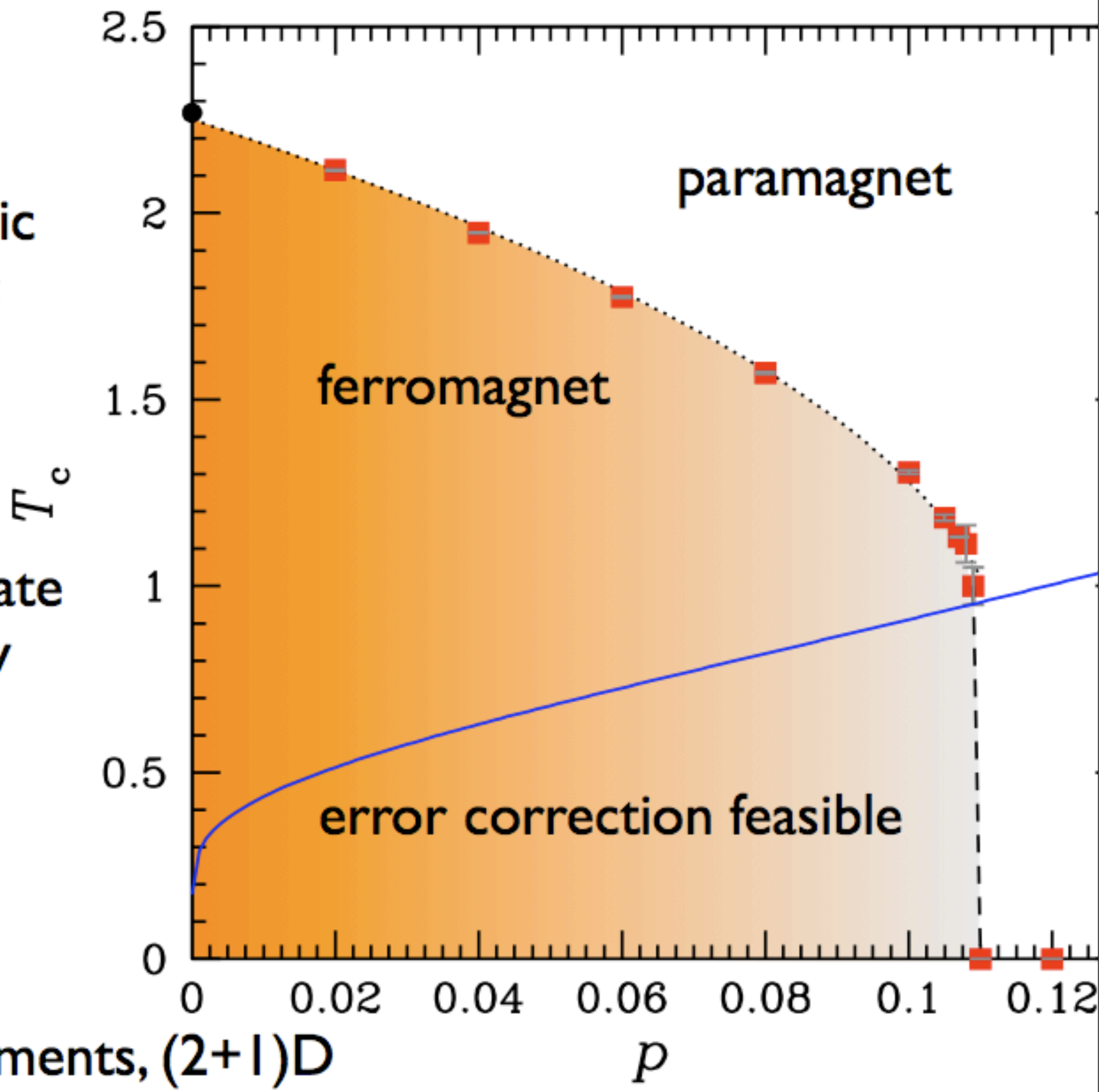


p–T Phase diagram & concluding remarks

- 50 CPU years later…
- The error threshold is comparable to the toric code value. We obtain:

$$p_c = 0.109(2)$$

- Note: p_c does not violate the Gilbert-Varshamov bound $p \sim 0.110027$.



- Future: faulty measurements, (2+1)D

p–T Phase diagram & concluding remarks

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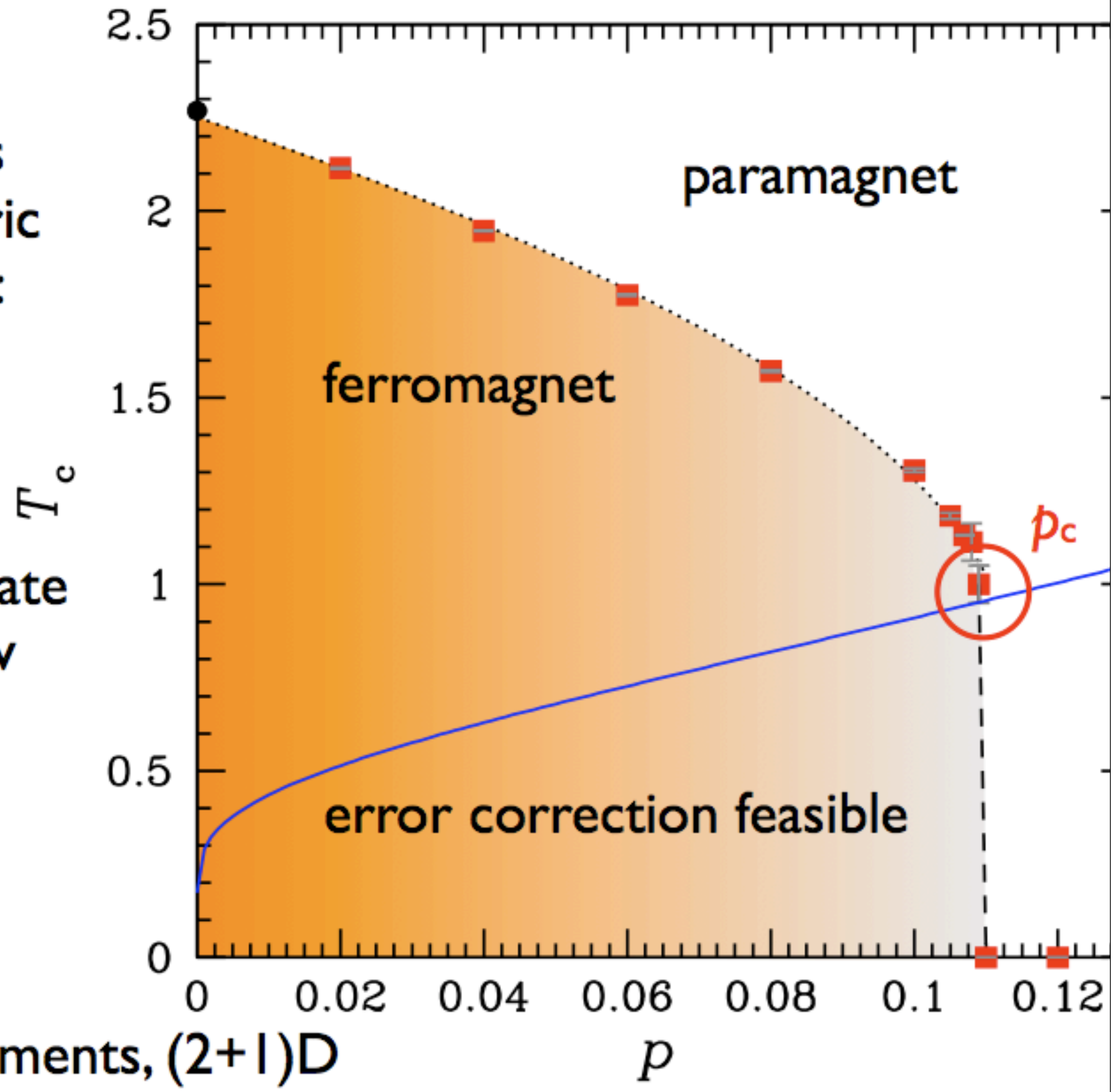

p–T Phase diagram & concl … s

Esto es un ejemplo donde un ordenador clasico ayuda a saber si un cierto tipo de QC es posible en realidad

- 50 CPU years later…
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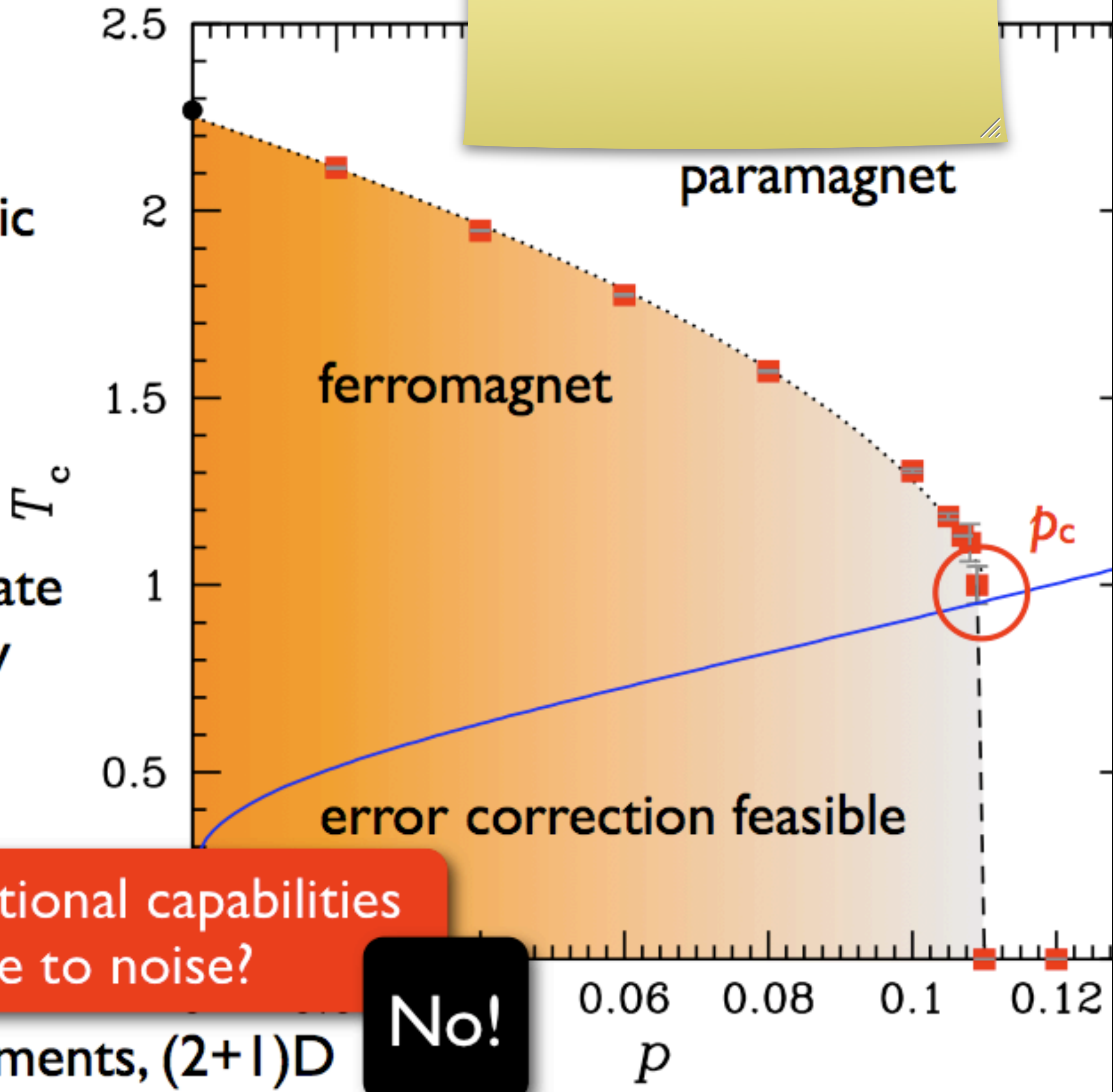


Do the wider computational capabilities imply a lower resistance to noise?

No!

- Future: faulty measurements, (2+1)D

V) Tricolored Lattice Gauge Theory

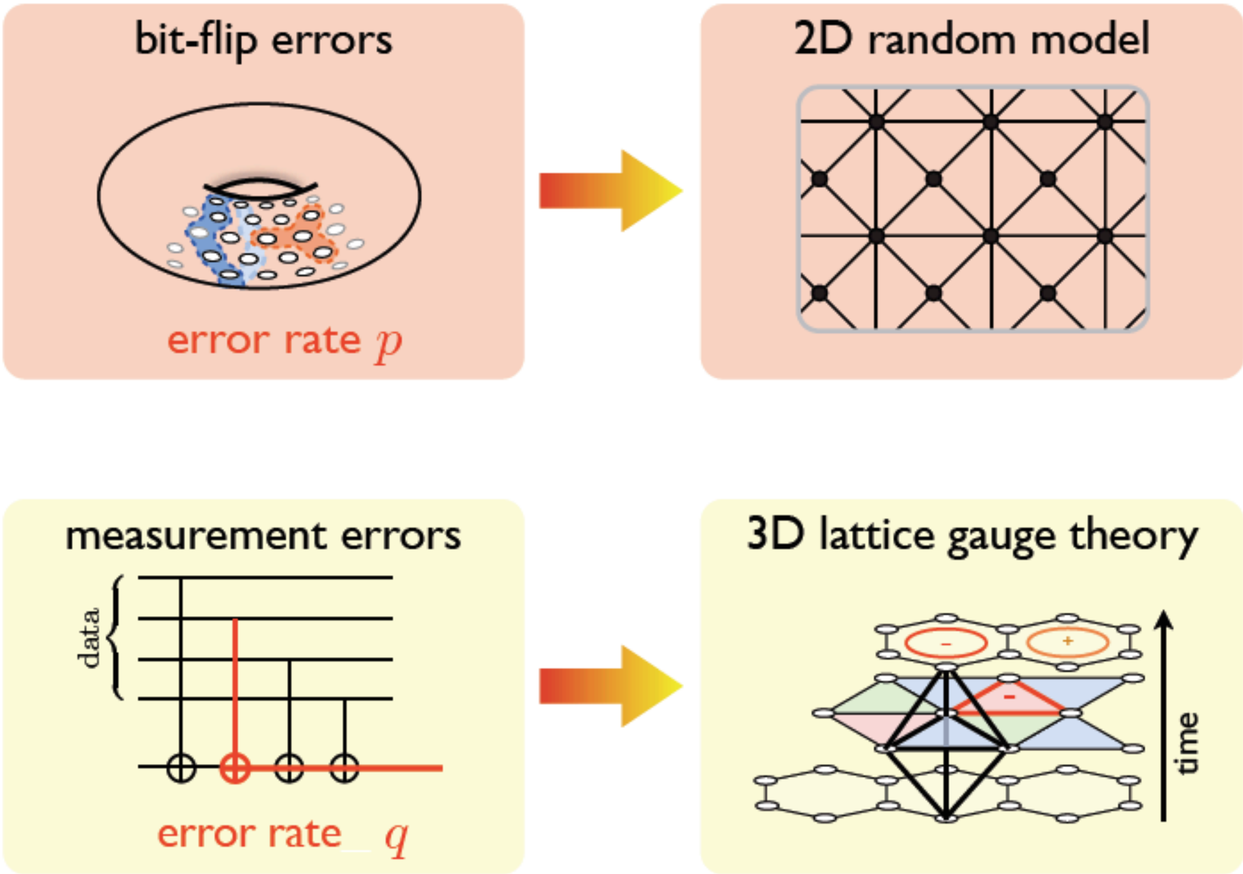
IV) Fault-Tolerant Quantum Computer

Measurement Errors

in addition to

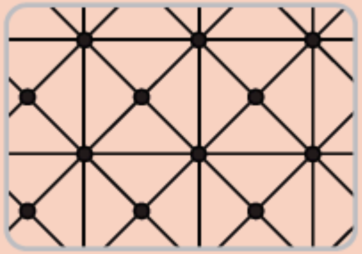
Qubit Errors

Add Measurement Errors ...



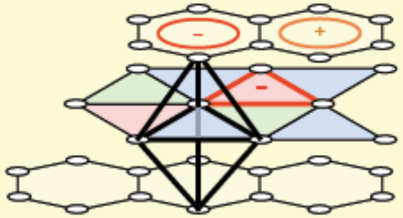
3D Disordered Lattice Gauge Theory

2D random model



$$\mathcal{H} = J \sum_{\langle ijk \rangle} \tau_{ijk} S^i S^j S^k$$

3D lattice gauge theory



$$\mathcal{H} = -\sum J_j [S^j]_5 - \sum K_k [S^k]_6$$

$$J_j, K_k = \begin{cases} -1 & \text{probability } p, q \\ +1 & (1-p), (1-q) \end{cases}$$

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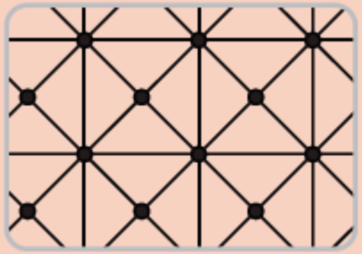
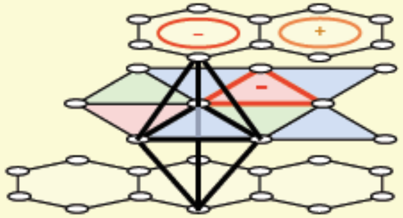


$$\mathcal{H} = J \sum_{\langle ijk \rangle} \tau_{ijk} S^i S^j S^k$$

for simplicity $p = q$...

3D lattice gauge theory



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Mapping

- error probabilities ➜ **quenched couplings**

$$[\,\cdot\,] := \sum_E P(E)\ \cdot$$

- dominant error class in average ➜ **divergent free energy** (large system limit)

$$[\Delta_D(\beta, E)] \to \infty, \qquad D \in \mathcal{Z}(\mathcal{S}) - \langle i\mathbf{1}\rangle\mathcal{S}$$

- error threshold ➜ phase transition

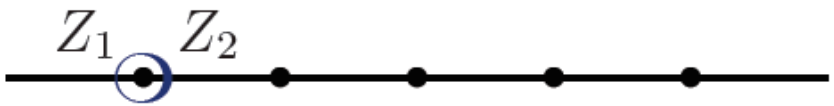
Fault-Tolerance

- topological quantum memory scenario
- two kinds of (effective) errors (➜ interactions)
  - on physical qubits
  - on **measurements** of check operators
- two kinds of error equivalences (➜ spins)
  - between errors at equal times (check operators, as before)
  - related to two **unnoticed, equal and consecutive errors**: they cancel each other!

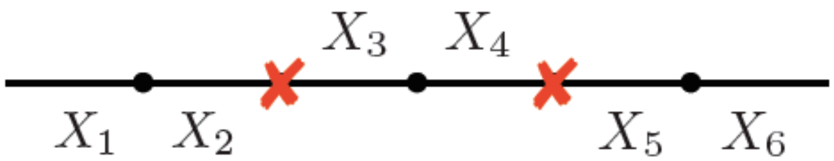
Fault-Tolerance

- e.g. fault-tolerance in **repetition code**
  - corrects bit-flips only
  - qubits: links forming a line
  - check ops: vertices

$$S_i := Z_i Z_{i+1}$$



  - trivial error correction: two errors per syndrome



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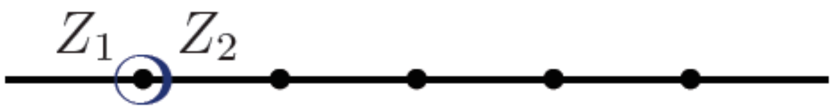

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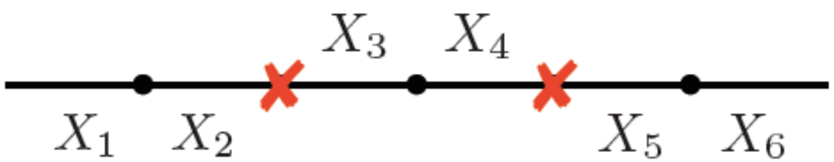
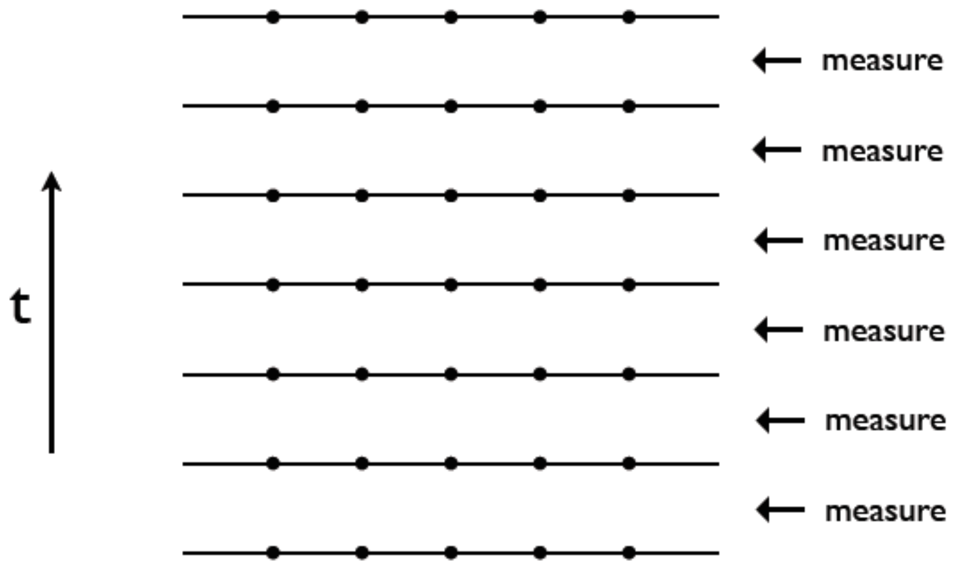


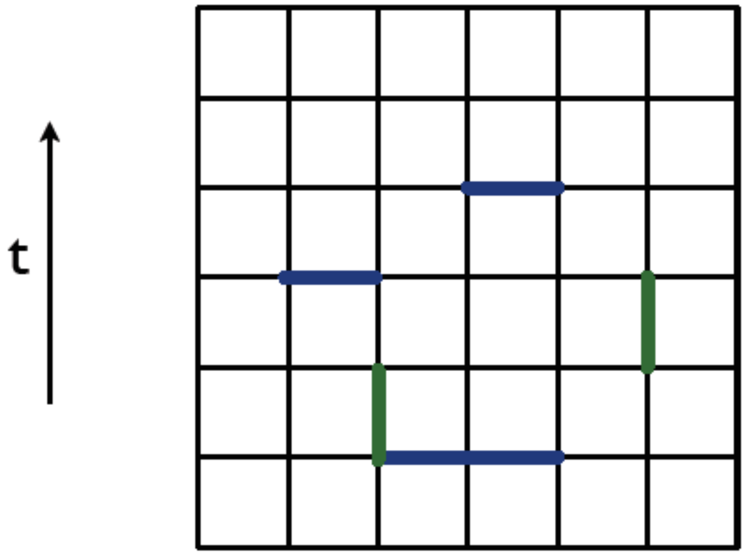
Fault-Tolerance

- fault-tolerance: measure check ops in parallel repeatedly
- add a dimension to represent **time**



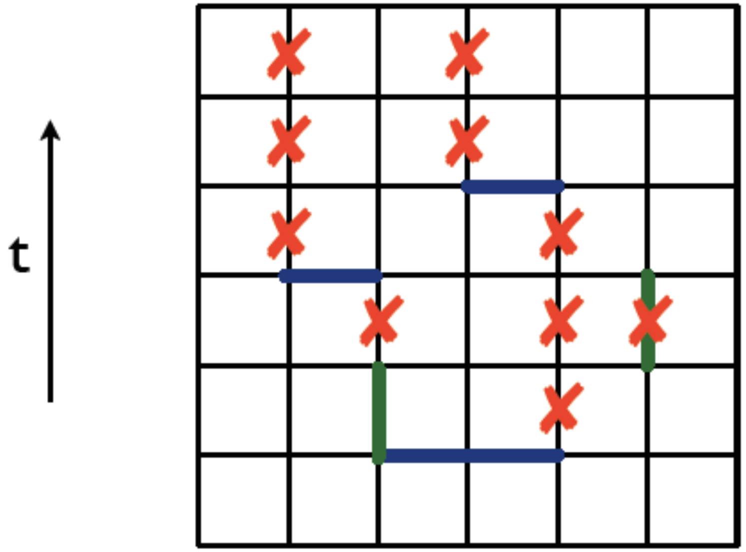
Fault-Tolerance

- errors on physical **qubits** ➜ horizontal links
- errors on **measurements** ➜ vertical links



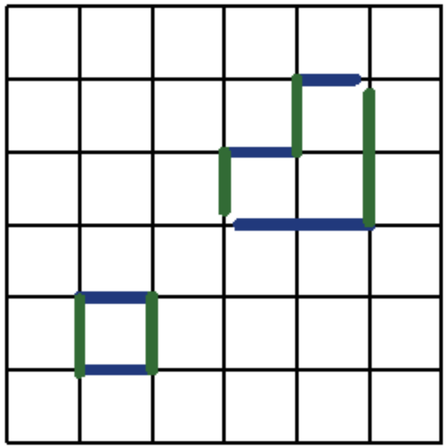
Fault-Tolerance

- **changes** on measurement outcome ➜ endpoints



Fault-Tolerance

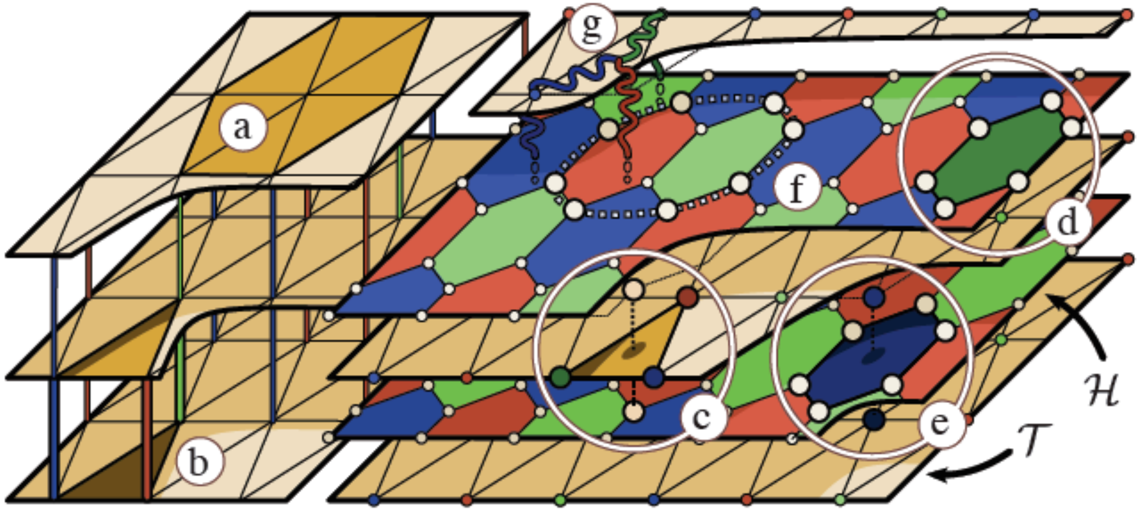
- error equivalence up to **homology**



- same as toric code!

Fault-Tolerance

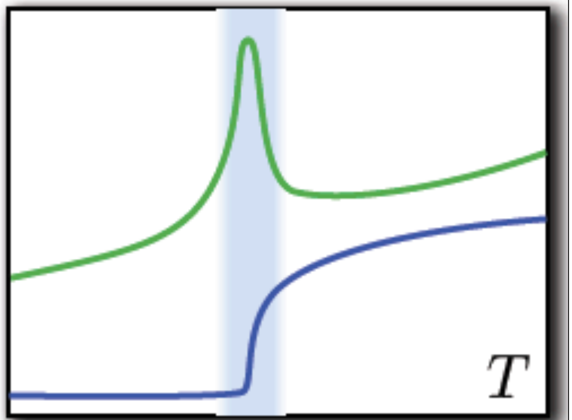
- in general, fault-tolerance **adds one dimension**
- toric code + bit-flip + f-t ➜ random Z_2 gauge model
- color code + bit-flip + f-t ➜ random $Z_2 x Z_2$ "tricolored" gauge model (Andrist *et al.* '10)



VI) Simulations and Threshold

Order parameter

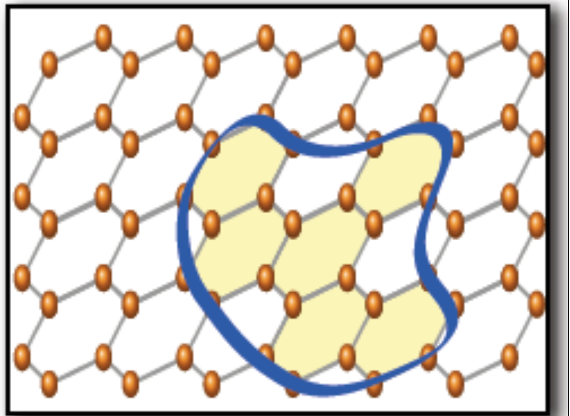
- Problems:
  - Local order parameters (magnetization) do not work for LGTs.
  - The transition is *first order*.
  - Both specific heat and energy imprecise.



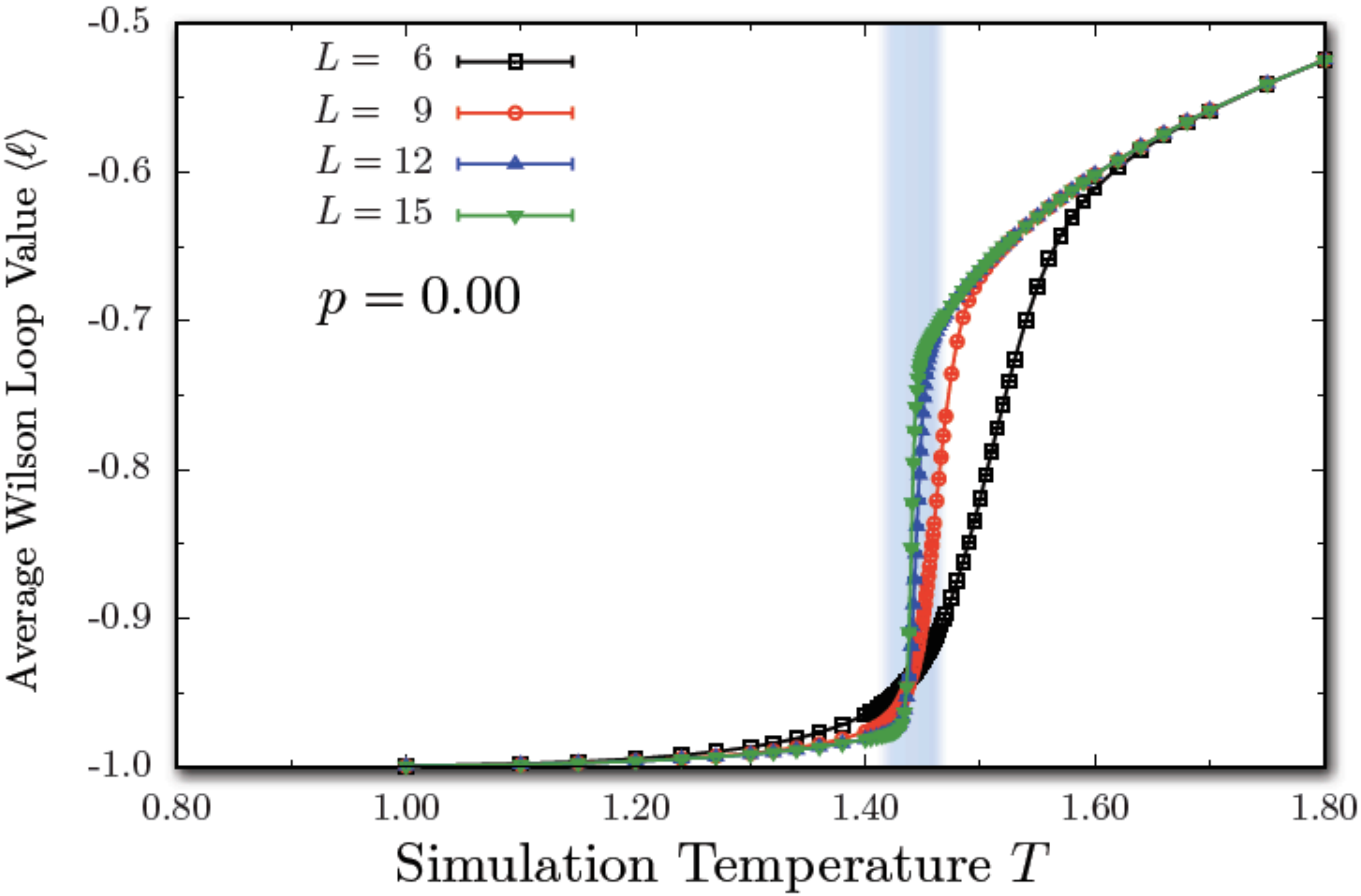
- Solution:
  - Wilson loops in the hexagon plane

$$\ell = \frac{1}{N_{\mathrm{loops}}} \sum_{\mathrm{loops}} \prod_{j \in \mathrm{loop}} S_j$$

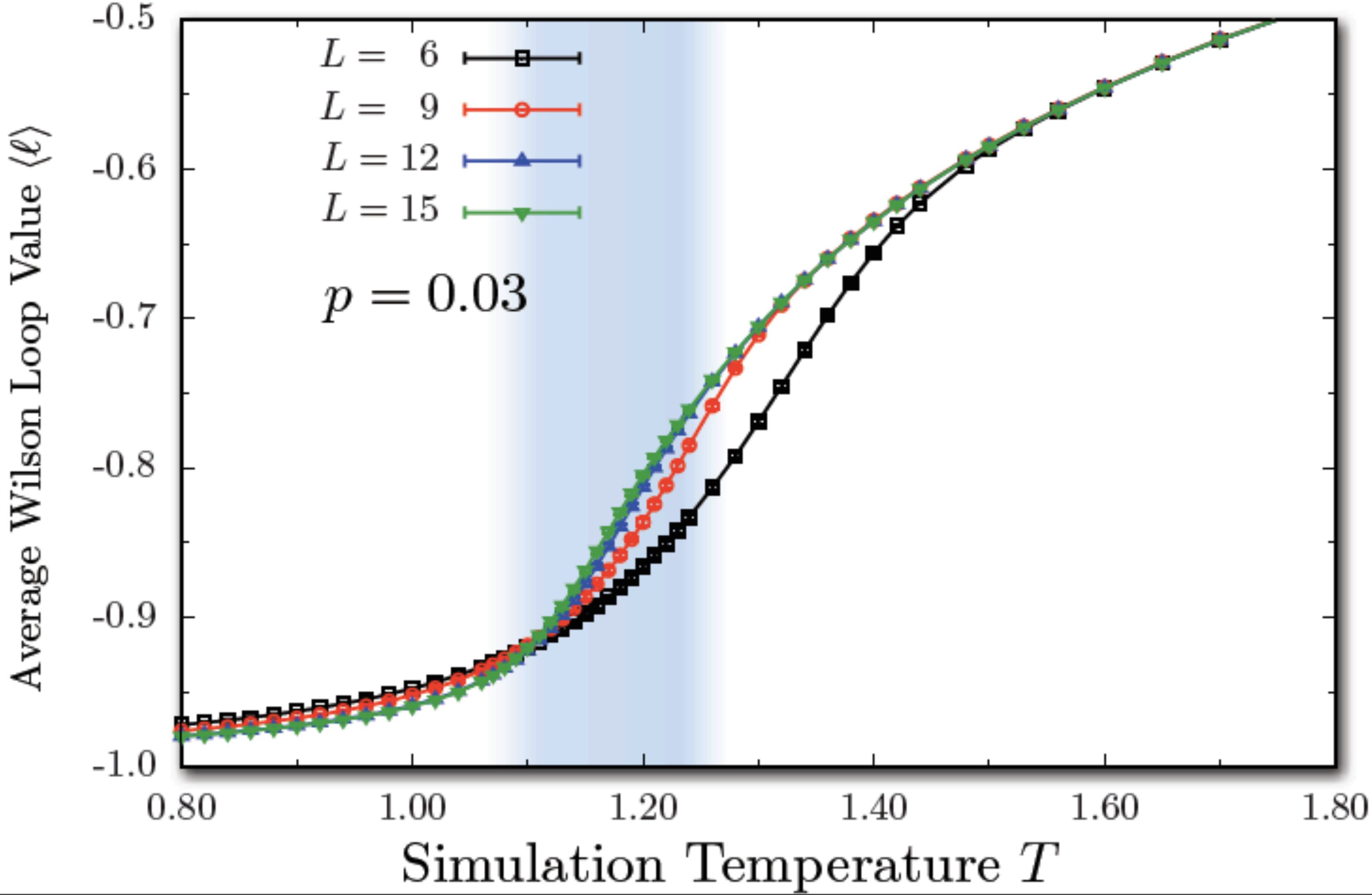
  - Note: here we use minimal loops over one plaquette to reduce corrections.



Average Wilson loop value (no errors)



Average Wilson loop value (p = 3%)



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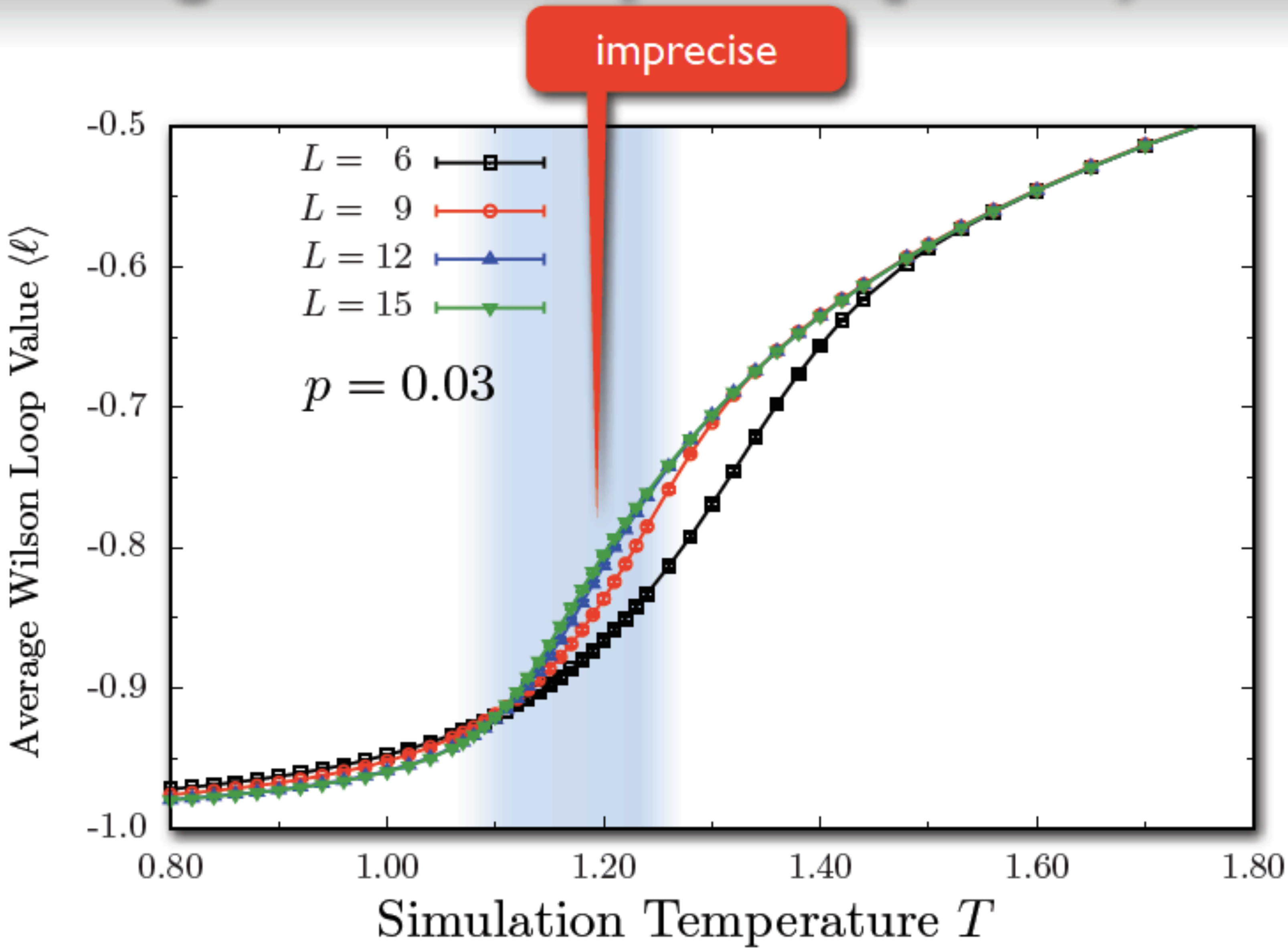
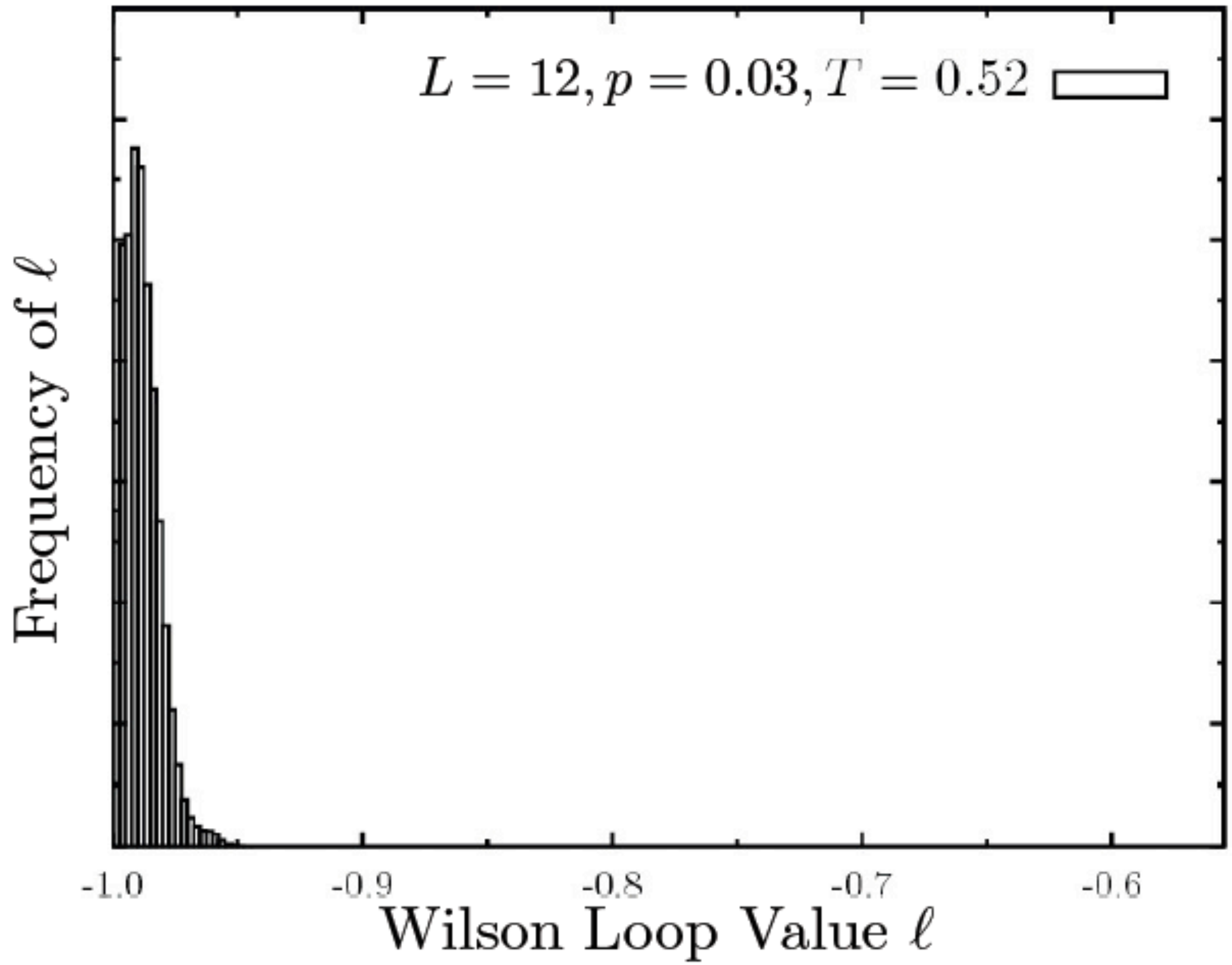


Wilson loop distribution (p = 3%)



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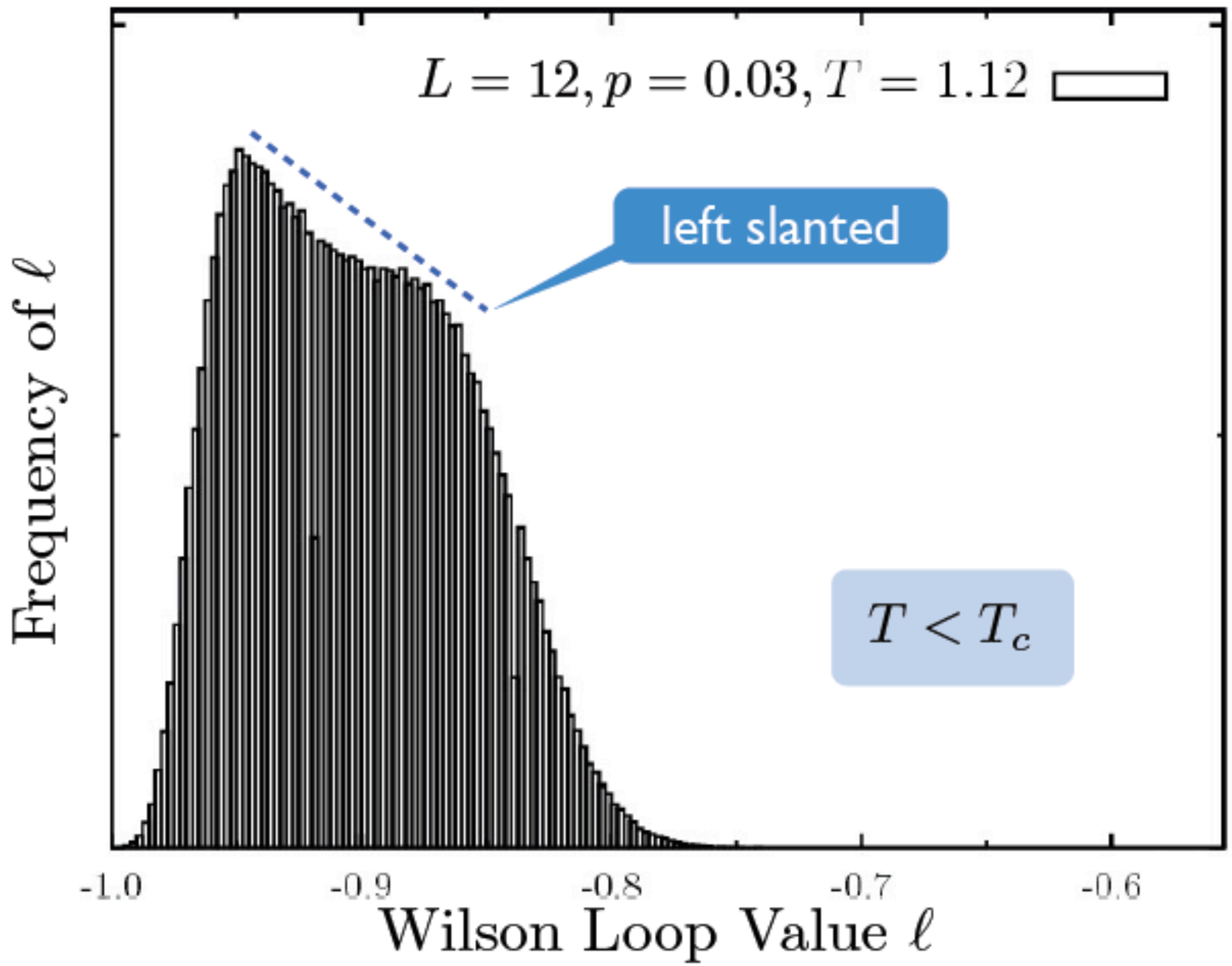

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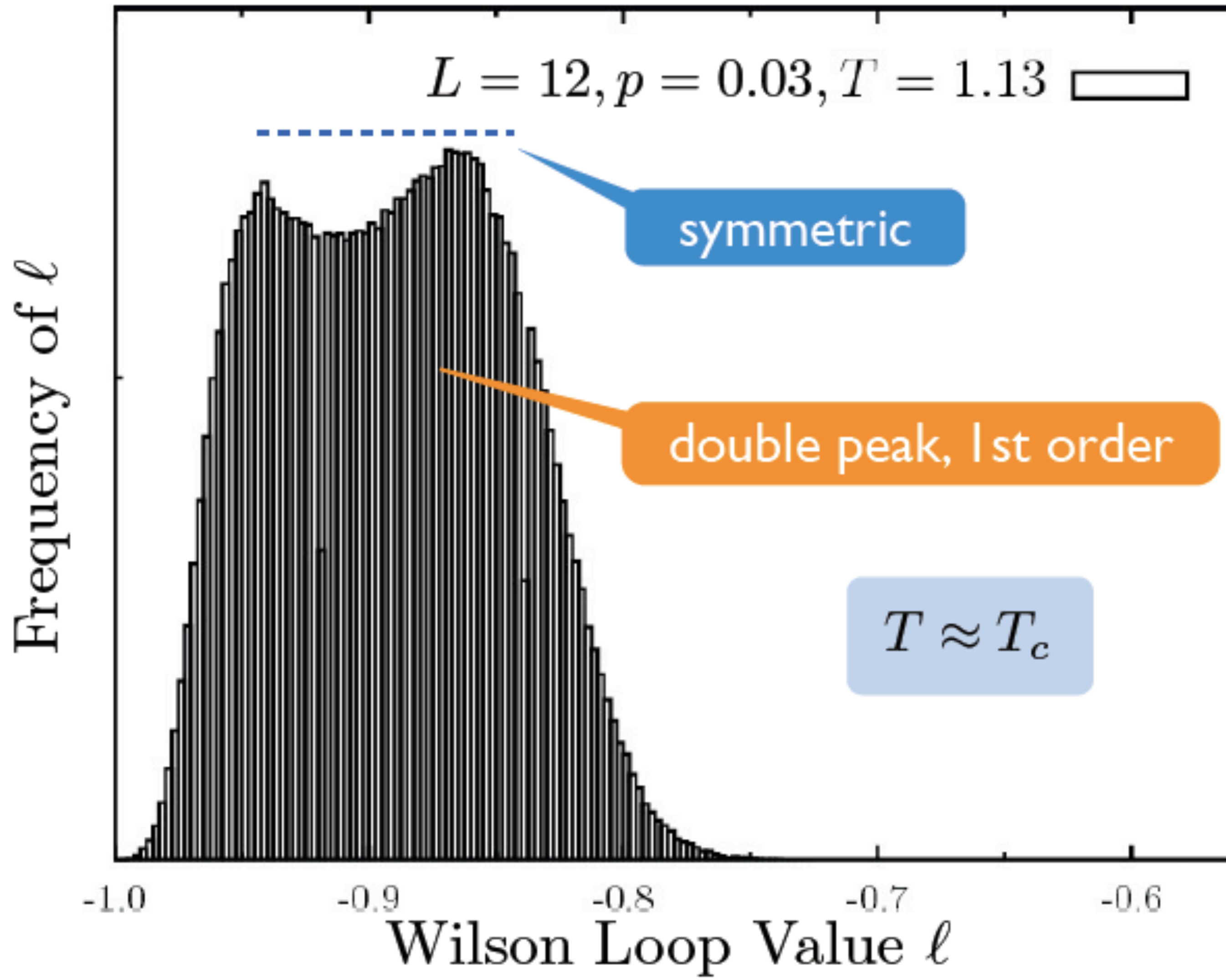

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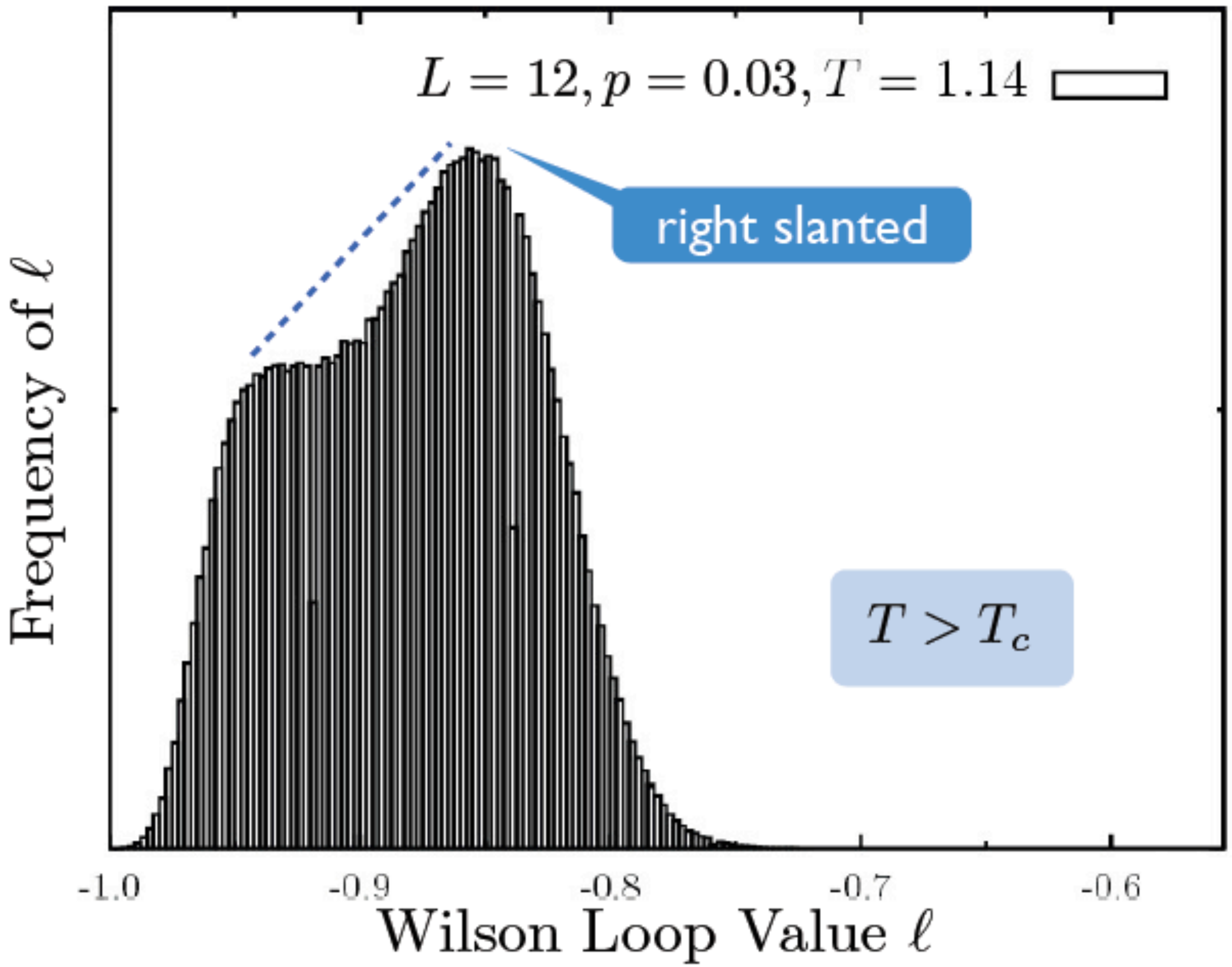
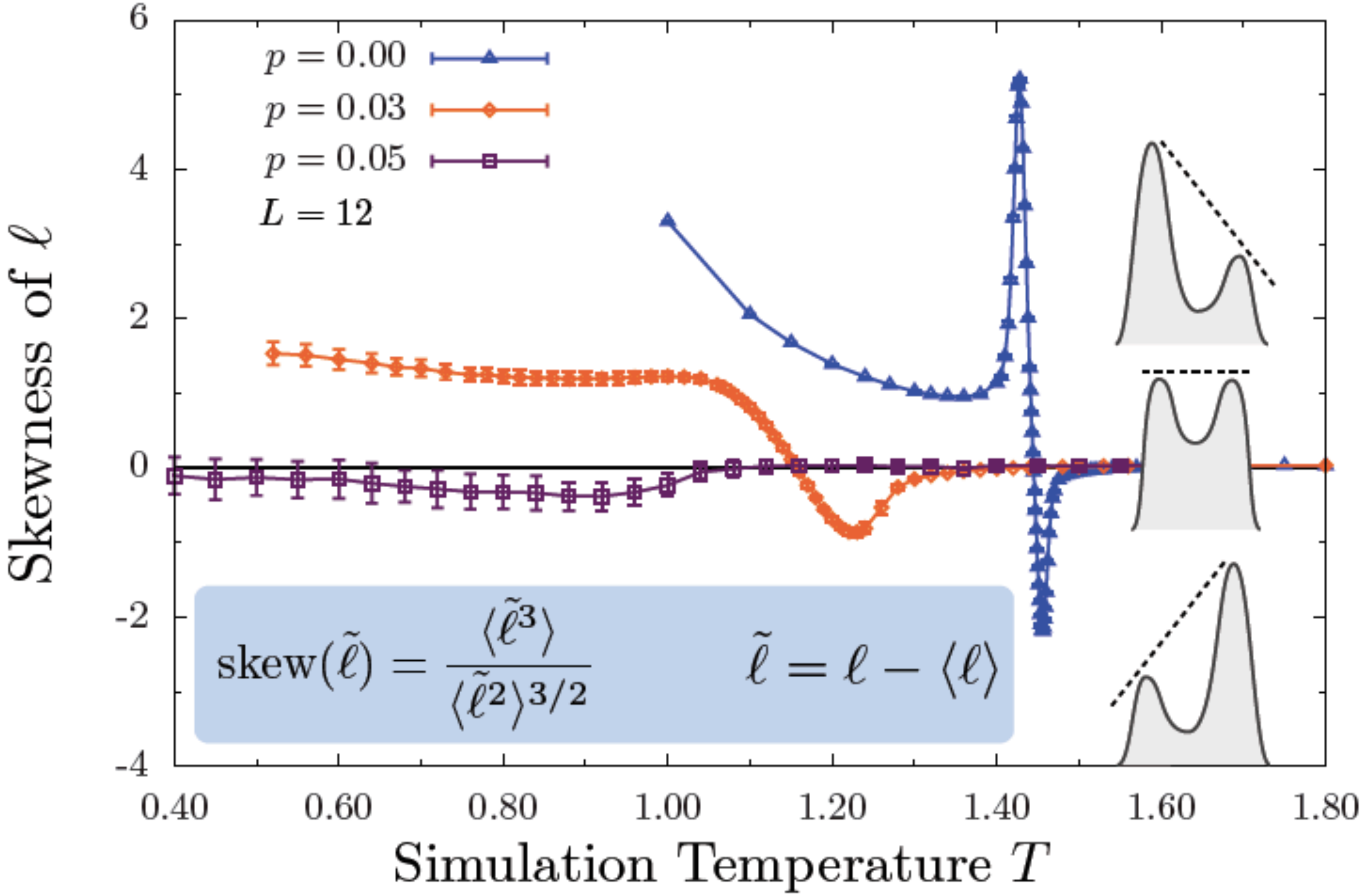


Skewness as a “Binder parameter”



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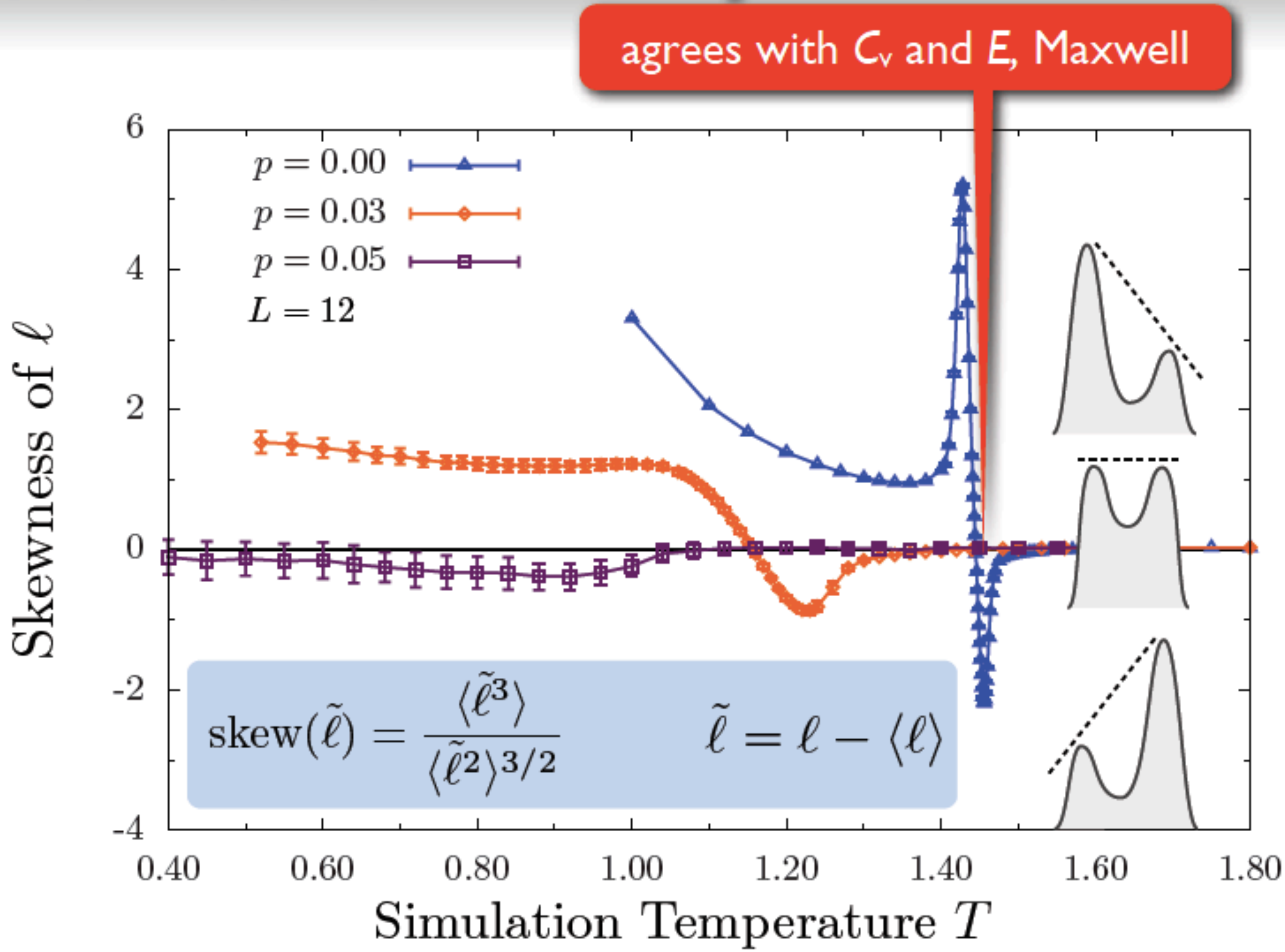
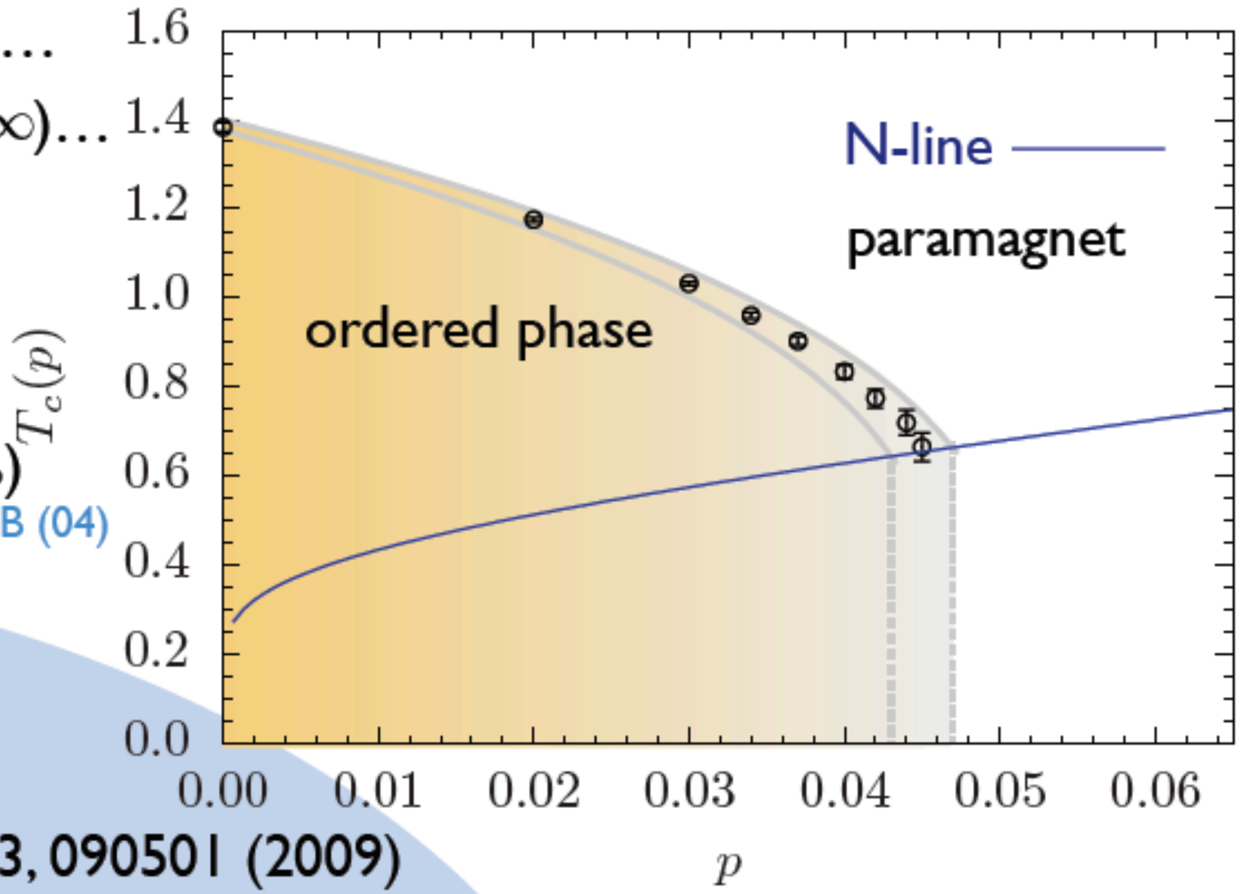


Error threshold with measurement errors

- 23 CPU years later…
- Extrapolate ($L \to \infty$)…
- Threshold:

$$p_c = 0.045(2)$$

- Revisit TC (p_c = 3%)
  Ohno *et al.*, Nuc Phys B (04)



arXiv:quant-phys/1005.0777

- See also:
  - Phys. Rev. Lett. 103, 090501 (2009)
  - Phys. Rev. A 81, 012319 (2010)

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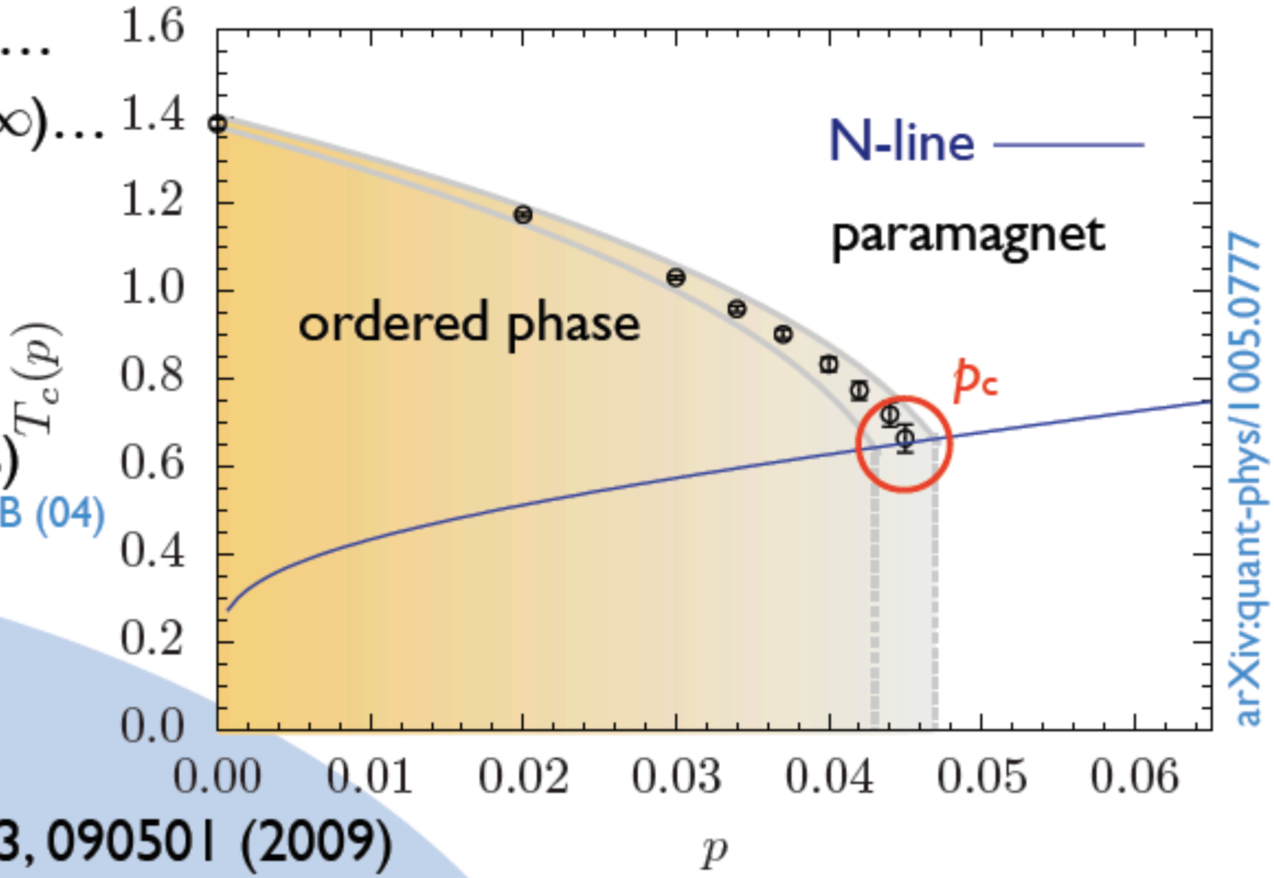

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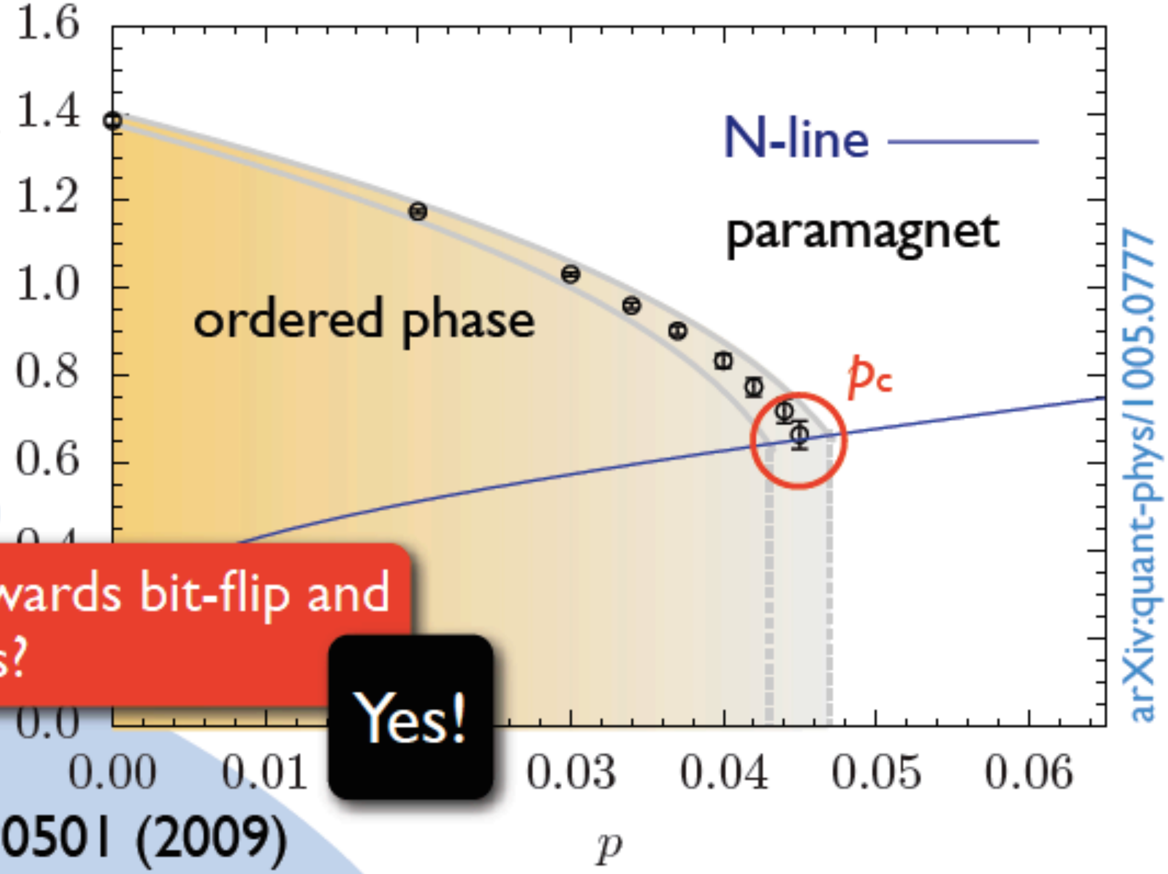

Are TCCs stable towards bit-flip and measurement errors?

Yes!

- See also:
  - Phys. Rev. Lett. 103, 090501 (2009)
  - Phys. Rev. A 81, 012319 (2010)
  - arXiv:quant-phys/1005.0777, PRL subm.

SPECIAL THANKS TO MY COLLABORATORS

- HECTOR BOMBIN
  PERIMETER INSTITUTE (CANADA)
- HELMUT KATZGRABER
  UNIVERSITY OF TEXAS A&M, ETH ZURICH
- RUBEN ANDRIST
  ETH ZURICH (SWITZERLAND)

REFERENCES

- "Tricolored Lattice Gauge Theory with Randomness: Fault-Tolerance in Topological Color Codes"
  R. ANDRIST, H. KATZGRABER, H. BOMBIN, M. A. MARTIN-DELGADO
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SUPPLEMENTARY MATERIAL

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[5] M. Kargarian, H. Bombin, M.A. Martin-Delgado, “Topology induced anomalous defect production by crossing a quantum critical point”, *New. J. Phys*, Vol. **12**, pp. 025018, (2010).


The best
Is yet
To come

For now, this is it!

MANY THANKS FOR YOUR ATTENTION

THE END