GLOBAL GAUGE ANOMALIES in 2D

Krzysztof Gawędzki, Benasque, September 2011

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To Manolo with best wishes

"L'homme arrive novice à chaque âge de la vie" - *Chamfort*

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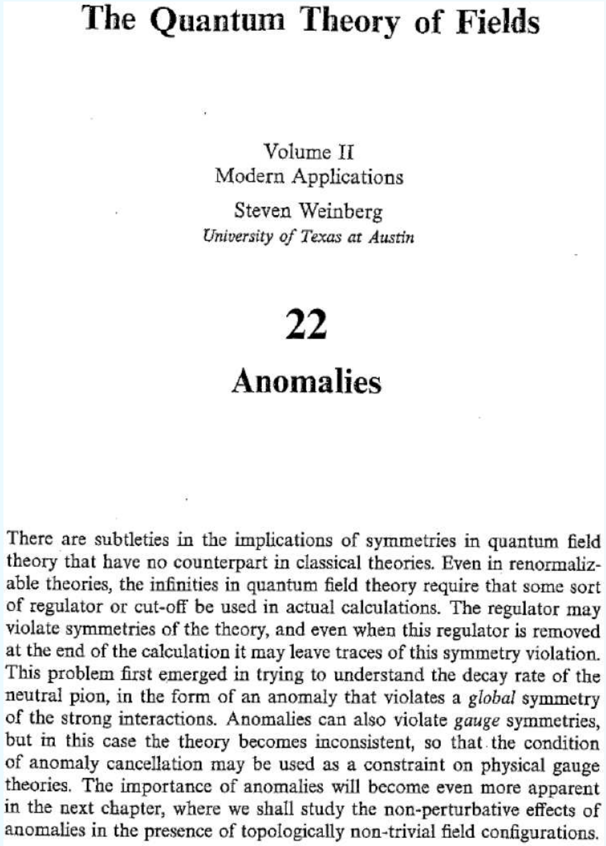
- What is **QFT ?** It is **gauge invariance !**
- May suffer from **gauge anomalies** in presence of (chiral) fermions
  - **Local** gauge anomalies obstruct **infinitesimal** gauge invariance
    - history starting with: **Schwinger** (1951), **Johnson** (1963), **Adler**, **Bell-Jackiw**, **Bardeen** (1969), **Fujikawa** (1979), ...
    - related to (**family**) **Index Theorem** of **Atyiah-Singer** (1968) and to **BRST cohomology** as clarified in the 1970-80's

- **Local** gauge anomalies are a text-book material:

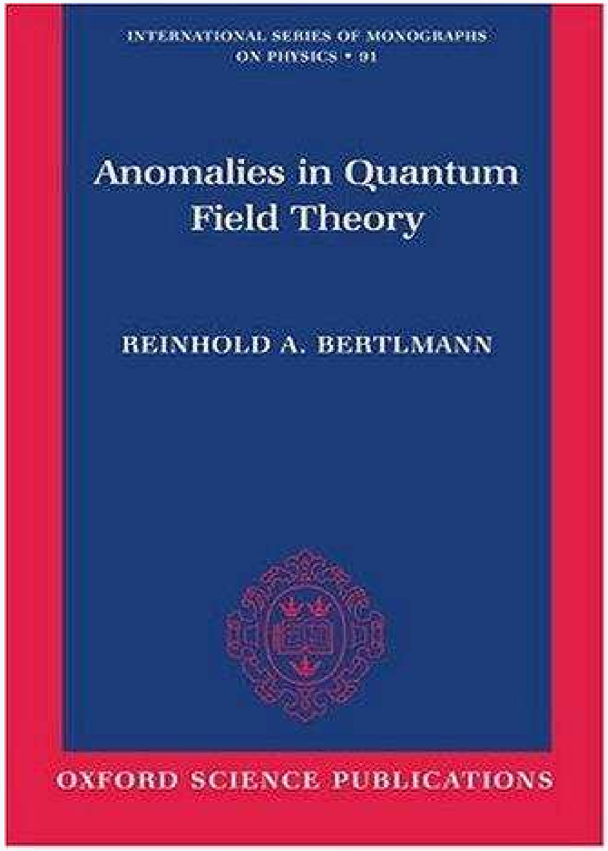


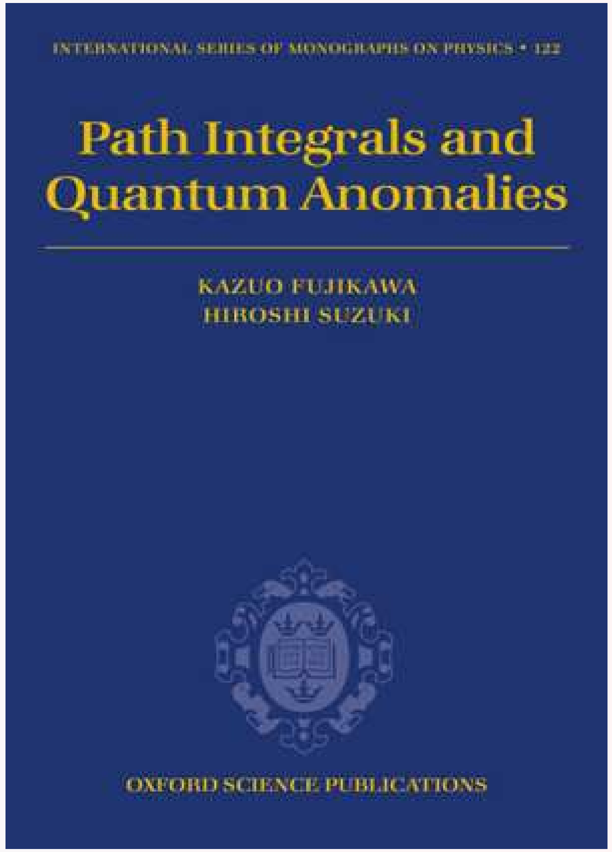
The Quantum Theory of Fields

Volume II
Modern Applications
Steven Weinberg
University of Texas at Austin

22
Anomalies

There are subtleties in the implications of symmetries in quantum field theory that have no counterpart in classical theories. Even in renormalizable theories, the infinities in quantum field theory require that some sort of regulator or cut-off be used in actual calculations. The regulator may violate symmetries of the theory, and even when this regulator is removed at the end of the calculation it may leave traces of this symmetry violation. This problem first emerged in trying to understand the decay rate of the neutral pion, in the form of an anomaly that violates a *global* symmetry of the strong interactions. Anomalies can also violate *gauge* symmetries, but in this case the theory becomes inconsistent, so that the condition of anomaly cancellation may be used as a constraint on physical gauge theories. The importance of anomalies will become even more apparent in the next chapter, where we shall study the non-perturbative effects of anomalies in the presence of topologically non-trivial field configurations.





and also:

"**Introduction to gauge anomalies**", by **Manuel Asorey**,
in: *Nuevas Perspectivas en Teoras Cunticas de Campos*, Eds. J.L. Alonso et al, U. de Zaragoza (1985)

- In absence of local, **global** gauge anomalies may obstruct invariance under **large** gauge transformations non-homotopic to identity

  - **Witten**'s Phys. Lett. **B 117** (1982) example of $SU(2)$ gauge theory in 4 Euclidian dimensions compactified to S^4

  - reviewed in: **Fabbrichesi**, Pramana **62** (2004)

- Gauge anomalies descend to bosonic **low energy effective** theories of **Goldstone modes**, producing **Wess-Zumino** (1971) terms in the action

  - reviewed in: **Petersen**, Acta Phys. Polon. **B 16** (1985)

- Cancellation of **gauge anomalies** leads to **selection rule** in High Energy Physics model building

  - reviewed e.g. in: **Weinberg**, *The Quantum Theory of Fields*, vol. 2

AN SU(2) ANOMALY

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Received 14 July 1982

A new restriction on fermion quantum numbers in gauge theories is derived. For instance, it is shown that an SU(2) gauge theory with an odd number of left-handed fermion doublets (and no other representations) is mathematically inconsistent.

It has been a long-standing puzzle to elucidate the properties of an SU(2) gauge theory with a single doublet of left-handed (Weyl) fermions. This theory defies simple phenomenological descriptions. There is no obvious attractive channel in which a fermion condensate could form, consistent with Fermi statistics and Lorentz invariance. But it is hard to believe that the fermions could remain massless in the presence of strong SU(2) gauge forces at long distances.

This puzzle persists (in the absence of other representations) whenever the number of elementary fermion doublets is odd. An even number of doublets, even if they have zero bare mass, could pair up and become massive Dirac fermions through spontaneous chiral symmetry breaking. With an odd number of elementary doublets, however, there would always be one massless doublet left over after any assumed chiral symmetry breaking, as long as the SU(2) gauge symmetry remains unbroken.

Of course, there is no real paradox here. Perhaps our heuristic pictures of strongly interacting gauge theories are inadequate. However, the facts noted above do suggest that something is strange about an SU(2) gauge theory with an odd number of elementary doublets. The purpose of this paper is to determine precisely what is strange about these theories; we will see that they are mathematically inconsistent! The inconsistency arises from a problem somewhat analogous to the Adler–Bell–Jackiw anomaly.

Although a hamiltonian approach exists, let us first look at this problem from the point of view of euclidean functional integrals. The starting point is the fact [1] that the fourth homotopy group of SU(2) is nontrivial,

$$\pi^4(\mathrm{SU}(2)) = \mathrm{Z}_2. \qquad (1)$$

[Note that we are dealing with the *fourth* homotopy group, while the *third* homotopy group, $\pi^3(\mathrm{SU}(2)) = \mathrm{Z}$, has entered in instanton studies [2]. The analogue of π^4 has entered in some recent studies [3] of 2 + 1 dimensional models.] Eq. (1) means that in four-dimensional euclidean space, there is a gauge transformation $U(x)$ such that $U(x) \to 1$ as $|x| \to \infty$, and $U(x)$ "wraps" around the gauge group in such a way that it cannot be continuously deformed to the identity. The fact that the homotopy group is Z_2 means that a gauge transformation that wraps twice around SU(2) in this way can be deformed to the identity. We will not need the detailed form of $U(x)$.

The existence of the topologically non-trivial mapping $U = U(x)$ means that when we carry out the euclidean path integral

$$\int (\mathrm{d}A_\mu) \exp\left(-\frac{1}{2g^2}\int \mathrm{d}^4x \ \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}\right), \qquad (2)$$

we are actually double counting. For every gauge field A_μ, there is a conjugate gauge field

$$A_\mu^{\mathrm{U}} = U^{-1}A_\mu U - \mathrm{i}U^{-1}\partial_\mu U,$$

which makes exactly the same contribution to the functional integral. There is no way to eliminate this

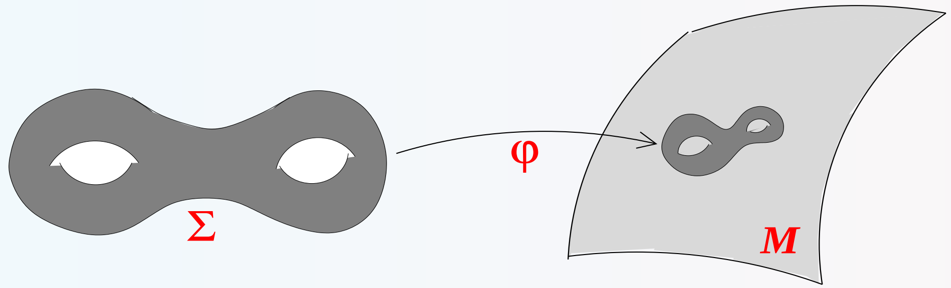
[1] Supported in part by the National Science Foundation under Grant No. PHY80-19754.

Topic of this talk:

- **global gauge anomalies** in two-dimensional bosonic sigma models with general **Wess-Zumino** (**WZ**) terms
- These provide a good laboratory and were not much discussed
- We tackle the problem with mathematical theory of **gerbes**
- Applications include **WZW** and **coset models** of **CFT** and, potentially, global aspects of **T-duality**
- **Joint work** with **Rafał Suszek** & **Konrad Waldorf** CMP **302** (2011), 513-580, and with **Paul Defromont** (unpublished)

General 2D WZ action

Consider a 2D sigma model with classical fields φ mapping **closed** **worldsheet** Σ to **target space** M



The **WZ**-action is formally given by the symbolic formula involving a **closed** 3-form H on M :

$$S_{WZ}(\varphi) = \int_{\varphi(\Sigma)} d^{-1}H$$

- If there exists a global 2-form B on M such that $H = dB$, then one may set $d^{-1}H = B$

- If $H \neq dB$ but has **periods** in $2\pi\mathbf{Z}$ then the **WZ Feynman amplitude** is defined as a (2-dimensional) **holonomy** of a **bundle gerbe** $\mathcal{G}$ with **connection of curvature** H :

$$\boxed{\mathrm{e}^{i\,S_{WZ}(\varphi)} \;:=\; Hol_{\mathcal{G}}(\varphi)}$$

- The **WZ** action usually accompanies the standard sigma model action

$$S_\sigma(\varphi) \;=\; \|d\varphi\|^2_{L^2} \;=\; \int\limits_\Sigma g_{\mu\nu}(\varphi)\,\partial_i\varphi^\mu\,\partial_j\varphi^\nu\,h^{ij}\sqrt{h}$$

involving **Riemannian** metrics g on M and h on Σ and the total (**Euclidian**) **Feynman** amplitudes have the form

$$\mathscr{A}(\varphi) \;=\; \mathrm{e}^{-S_\sigma(\varphi)+iS_{WZ}(\varphi)} \;=\; \mathrm{e}^{-S_\sigma(\varphi)}\,Hol_{\mathcal{G}}(\varphi)$$

What are (bundle) **gerbes** (with connection) **?**

- **Line bundles with connection** may be given by **local data** $(A_\alpha, g_{\alpha\beta})$ relative to an open covering $\mathcal{O}_\alpha$ with

$$dA_\alpha = F\,, \qquad A_\beta - A_\alpha = i\,d\ln g_{\alpha\beta}\,, \qquad g_{\alpha\beta}\,g_{\alpha\gamma}^{-1}\,g_{\beta\gamma} = 1$$

  but also possess a geometric description

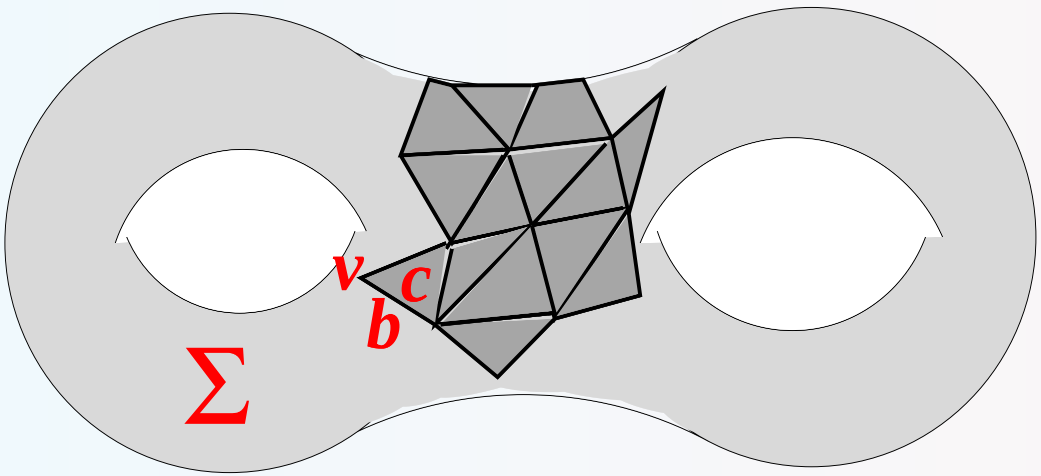
- Similarly, **gerbes** may be given by **local data** $(B_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma})$ satisfying descent equations

$$dB_\alpha = H\,, \quad B_\beta - B_\alpha = dA_{\alpha\beta}\,, \quad A_{\alpha\beta} - A_{\alpha\gamma} + A_{\beta\gamma} = i\,d\ln g_{\alpha\beta\gamma}\,,$$
$$g_{\alpha\beta\gamma}\,g_{\alpha\beta\delta}^{-1}\,g_{\alpha\gamma\delta}\,g_{\beta\gamma\delta}^{-1} = 1$$

  but have a geometric definition given by **Murray** 1994 & **Murray-Stevenson** 1999

- **Gerbe holonomy** may be expressed by integrals of local data $(B_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma})$ over triangles c, edges b, and vertices v of a **triangulation** of Σ (**O. Alvarez** (1985), **G.** (1988))

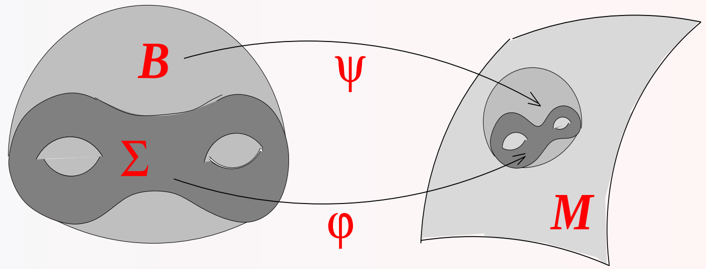
$$Hol_{\mathcal{G}}(\varphi) \;=\; \exp\Big[\,i\sum_{c}\int_{\varphi(c)} B_{i_c} + i\sum_{b\subset c}\int_{\varphi(b)} A_{i_c i_b}\Big]\prod_{v\in b\subset c} g_{i_c i_b i_v}(\varphi(v))^{\pm 1}$$



- **Main property** of the gerbe holonomy:

If $\psi : B \to M$ for an oriented 3-manifold B with boundary $\partial B = \Sigma$ and $\varphi = \psi|_{\partial B}$ then

$$Hol_{\mathcal{G}}(\varphi) \;=\; \exp\Big[\, i \int_{\psi(B)} H \,\Big]$$



(i.e. $Hol_{\mathcal{G}}$ is a **Cheeger-Simons differential character**)

Few math facts about gerbes (similar to facts about line bdles):

- **gerbes** over manifold M form a **category** with morphisms that may also be described by local data

- **gerbes** are classified up to isomorphism by their **holonomy**

- **gerbes** have duals (with opposite curvature), tensor products (with curvatures adding) and pullbacks (with curvatures pulling back)

- a global 2-**form** B defines canonically a **gerbe** $\mathcal{I}_B$ with curvature dB and holonomy $Hol_{\mathcal{I}_B}(\varphi) = \exp\left[\, i \int_{\varphi(\Sigma)} B\right]$

- **flat gerbes** (i.e. with vanishing curvature) are classified up to isomorphism by cohomology classes in $H^2(M, U(1))$

Rigid symmetries of sigma models

- Suppose that a **Lie group** Γ acts on the **target space** M

- $$S_\sigma(\gamma\varphi) \;=\; S_\sigma(\varphi)$$

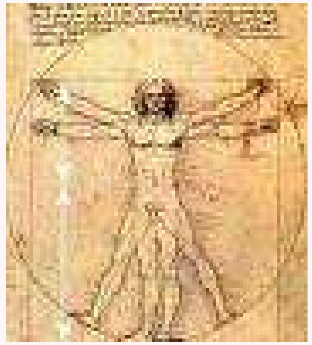
  for all $\gamma \in \Gamma$, all **Riemann** surfaces Σ and all $\varphi : \Sigma \to M$ **iff** the metric g on M is Γ-invariant

- $$\mathrm{e}^{\,i\,S_{WZ}(\gamma\varphi)} \equiv Hol_{\mathcal{G}}(\gamma\varphi) = \boxed{Hol_{\gamma^*\mathcal{G}}(\varphi) \;=\; Hol_{\mathcal{G}}(\varphi)} \equiv \mathrm{e}^{\,i\,S_{WZ}(\varphi)}$$

  for all $\gamma \in \Gamma$, all Σ and all $\varphi : \Sigma \to M$ **iff** all **gerbes** $\gamma^*\mathcal{G}$ (pullbacks of $\mathcal{G}$ by the action of $\gamma \in \Gamma$) are **isomorphic**

- Γ-invariance of total **Feynman** amplitudes $\mathscr{A}(\varphi) = \mathrm{e}^{-S_\sigma(\varphi)}\,Hol_{\mathcal{G}}(\varphi)$ requires both conditions

Infinitesimal rigid symmetries



- t^a - generators of $Lie(\Gamma)$ (with $[t^a, t^b] = i\, f^{abc} t^c$)
  $\bar{t}^a$ - vector fields on M induced by action of t^a

- Γ-invariance of the metric g on M implies that $\bar{t}^a$ are **Killing** vector fields

- Isomorphism of gerbes $\gamma^* \mathcal{G}$ with curvature H for $\gamma \in \Gamma$ implies that there exist 1-forms v^a on M s.t.

$$\underbrace{\iota^a}_{\text{contraction with } \bar{t}^a} H \;=\; d v^a$$

- 1-forms v^a will play important role in what follows**!**

Gauging of rigid symmetries

- In many situations, one would like to promote rigid symmetries of the theory to **local** ones by **gauging**

  - Standard sigma-model action is gauged by **minimal coupling** to **gauge fields** given by $Lie(\Gamma)$-valued 1-forms $A = t^a A^a$ on Σ

$$d\varphi \;\mapsto\; d\varphi - \bar{t}^a A^a \qquad \text{i.e.} \qquad \partial_i \varphi^\mu \;\mapsto\; \partial_i \varphi^\mu - (\bar{t}^a)^\mu A^a_i$$

  - Such coupling assures invariance under all gauge transformations $h : \Sigma \to \Gamma$:

$$S_\sigma({}^h\varphi, {}^hA) \;=\; S_\sigma(\varphi, A)$$

  for $({}^h\varphi)(x) = h(x)\varphi(x)$ and ${}^hA = hAh^{-1} + i\,hdh^{-1}$

Gauging of rigid symmetries in **WZ**-term

- Minimal coupling doesn't work for the (non-local) **WZ**-term **!**
- Instead, take a 2-form on $\Sigma \times M$

$$\boxed{\rho_A \; = \; -v^a A^a + \frac{1}{2}(\iota^a v^b) A^a A^b}$$

(dropping the $\wedge$ sign)

Proposition (**Jack-Jones-Mohammedi-Osborn**, **Hull-Spence** 1989)

The **coupled amplitudes**

$$\exp\left[i\, S_{WZ}(\varphi, A)\right] \; := \; Hol_{\mathcal{G}}(\varphi) \; \exp\left[\, i \int_{(Id,\varphi)(\Sigma)} \rho_A \,\right]$$

are invariant under **infinitesimal** gauge transformations $h(x) = e^{i\delta\Lambda(x)}$ **iff**

$$\iota^a H = dv^a\,, \qquad \mathcal{L}^a v^b = f^{abc} v^c\,, \qquad \iota^a v^b = -\iota^b v^a$$

Lie derivative $= d\iota^a + \iota^a d$

Remarks

- The conditions on the 1-forms v^a on M assure the absence of the **local** gauge anomalies of the gauged theory

- They mean that the form $\widehat{H}(X) = H + X^a v^a$ depending on $X = X^a t^a \in Lie(\Gamma)$ is a **(Cartan-)equivariantly closed** extension of H (**Witten** 1992, **Figueroa-O'Farrill-Stanciu** 1994)

- Infinitesimal gauge invariance is equivalent to the one under **small** gauge transformations $h : \Sigma \to \Gamma$ homotopic to identity:

$$\exp\left[i\, S_{WZ}({}^h\varphi, {}^hA)\right] \;=\; \exp\left[i\, S_{WZ}(\varphi, A)\right]$$

What about invariance under large gauge transformations ?

For $\ell : \Gamma \times M \to M$ the action of Γ on M and $pr_2 : \Gamma \times M \to M$ the projection on the 2^{nd} factor, consider two **gerbes** over $\Gamma \times M$:

$$\ell^* \mathcal{G} =: \mathcal{G}_{12} \qquad\qquad pr_2^* \mathcal{G} =: \mathcal{G}_2$$

Their curvatures differ by the exact 3-form $d\rho_\theta$ where $\theta = i\,\gamma^{-1} d\gamma = t^a \theta^a$ is the **Maurer-Cartan** connection 1-form on Γ and $\rho_\theta = -v^a \theta^a + \frac{1}{2} (\iota^a v^b) \theta^a \theta^b$

Hence :

- Gerbes $\mathcal{G}_{12}$ and $\mathcal{G}_2 \otimes \mathcal{I}_{\rho_\theta}$ have the same curvature
- Their inequality is measured by the “quotient” gerbe $\mathcal{G}_{12} \otimes (\mathcal{G}_2 \otimes \mathcal{I}_{\rho_\theta})^* \equiv \mathcal{F}$ which is a **flat gerbe** over $\Gamma \times M$

Proposition

For arbitrary gauge transformations $h : \Sigma \to \Gamma$

$$\boxed{\mathrm{e}^{\,i\,S_{WZ}({}^h\varphi,{}^hA)} \;=\; \mathrm{e}^{\,i\,S_{WZ}(\varphi,A)}\; Hol_{\mathcal{F}}(h,\varphi)}$$

where $\mathcal{F}$ is the flat gerbe over $\Gamma \times M$ just defined and $(h,\varphi) : \Sigma \to \Gamma \times M$

Remarks

- $Hol_{\mathcal{F}}(h,\varphi) \equiv 1$ **iff** the isomorphism class $[\mathcal{F}] \in H^2(\Gamma \times M, U(1))$ is trivial

- Class $[\mathcal{F}]$ is the **global gauge anomaly:** **large** gauge transformations act trivially **iff** $[\mathcal{F}] = 0$

Example of the **WZW** **models**

- $M = G$ for $G = \widetilde{G}/Z$ with $\widetilde{G}$ a semi-simple compact simply-connected **Lie** group and Z a subgroup of its center $\widetilde{Z}$ (e.g. $G = SU(N)/Z\,,\quad Z \subset \boldsymbol{Z}_N$)

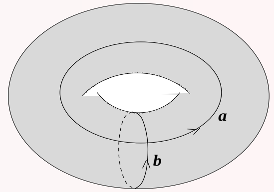
- The **WZ** amplitude of fields $\varphi : \Sigma \to G$ uses a **gerbe** $\mathcal{G}_k$ over G with curvature

$$H_k \;=\; \frac{k}{12\pi}\,\mathrm{tr}\,(g^{-1}dg)^3$$

- Symmetry group: $\Gamma = \widetilde{G}/\widetilde{Z}$ with the **adjoint** action on G (adjoint action is free of local gauge anomalies)

- The **WZ** action is gauged by the **JJMO-HS** prescription with $\boxed{v_k^a \;=\; \frac{ik}{4\pi}\,\mathrm{tr}\,t^a\,(g^{-1}dg - gdg^{-1})}$ and no local gauge anomalies

- Global gauge anomalies are signaled by non-trivial holonomy of flat gerbe $\mathcal{F}_k$
- It is enough to calculate holonomy of $\mathcal{F}_k$ for $\Sigma = S^1 \times S^1$ and



$$h(\mathrm{e}^{i\,\sigma_1}, \mathrm{e}^{i\,\sigma_2}) \;=\; \mathrm{e}^{i\,\sigma_1 \tilde{p}}\,, \qquad \varphi(\mathrm{e}^{i\,\sigma_1}, \mathrm{e}^{i\,\sigma_2}) \;=\; \mathrm{e}^{i\,\sigma_2\, p}$$

  with $\tilde{p}$ and p in the **Cartan** subalgebra $\mathfrak{t}$ of $Lie(G)$ s.t. $\mathrm{e}^{2\pi i\,\tilde{p}} \in \widetilde{Z}$ and $\mathrm{e}^{2\pi i\,p} \in Z \subset \widetilde{Z}$

  The result is
$$\boxed{Hol_{\mathcal{F}_k}(h, \varphi) \;=\; \mathrm{e}^{-2\pi i\,k\,\mathrm{tr}\,\tilde{p}\,p}}$$

- **Global anomaly** is absent **iff** $k\,\mathrm{tr}\,\tilde{p}\,p$ is an integer for all $\tilde{p}$, p ("k-**congruence**" between $\widetilde{Z}$ and Z)
- If $\Gamma = \widetilde{H}/(\widetilde{Z} \cap \widetilde{H})$ for a subgroup $\widetilde{H} \subset \widetilde{G}$ then one should only consider $\tilde{p}$ s.t. $\mathrm{e}^{2\pi i\,\tilde{p}} \in \widetilde{Z} \cap \widetilde{H}$

- **Examples of anomalous cases :**
  - $G = \Gamma = SU(N)/\boldsymbol{Z}_N$ with odd N at level 1 or even $N \geq 4$ at level 2
  - $G = SO(N)$, $\Gamma = SO(N)/\boldsymbol{Z}_2$ with N even at level 1
    (**equivalent to** N **massless** **Majorana** (**non−chiral !**) **fermions**)
  - $G = (SU(3) \times SU(3))/(\boldsymbol{Z}_3 \times \boldsymbol{Z}_3)$, $\Gamma = \mathrm{diag}(SU(3)/\boldsymbol{Z}_3)$ at level 1

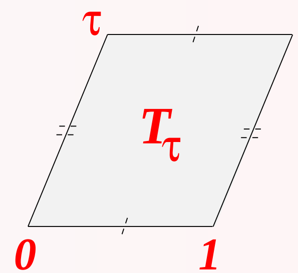
- **Cases without global anomaly :**
  - Simply connected G with whatever Γ
  - Non-simply connected G with $\widetilde{G} = B_r$ or $\widetilde{G} = C_n$
  - $G = (SU(2) \times SU(2))/Z$ with $\Gamma = \mathrm{diag}\, SO(3)$
    (used in **minimal** coset models)

- **Full classification** for **regular** subalgebras $\mathfrak{h} \equiv \mathrm{Lie}(\Gamma) \subset \mathrm{Lie}(G) \equiv \mathfrak{g}$ in **Dynkin**'s classification has been obtained (**de Fromont-G.**)

- **Does the global anomaly of classical-field amplitudes** $\mathscr{A}(\varphi, A)$ **carry over to quantum theory ?**

  - **Yes** if the values of functional integrals are fixed by symmetry properties like in **WZW** and **coset models** of **CFT**

  - E.g. in **WZW model** the partition functions on torus $T_\tau = \mathbf{C}/(\boldsymbol{Z} + \tau\boldsymbol{Z})$



$$\mathcal{Z}^G(\tau, A_u) \;=\; \int\limits_{\varphi: T_\tau \to M} \mathscr{A}(\varphi, A_u)\; D\varphi$$

  in flat external gauge field $A_u = \frac{\bar{u}dz - ud\bar{z}}{\tau - \bar{\tau}}$ with $u \in \mathfrak{t} \subset \mathfrak{g}$ is an explicit sesquilinear combination of **characters** of **current algebra** $\widehat{\mathfrak{g}}$

  - One may check that in the presence of global anomalies

$$\mathcal{Z}^G(\tau, A_u) \;\neq\; \mathcal{Z}^G(\tau, \underbrace{A_{u+\tilde{p}_1+\tau\tilde{p}_2}}_{\text{gauge transform of } A_u})$$

  in a way predicted by the classical argument

Can one live with **global gauge anomalies**?

- **Yes**, if the gauge field is **external**
  - in **WZW model** in external gauge field the global anomaly is mearly a statement about the dependence of quantum amplitudes on the latter

- **No**, if the gauge field is **dynamic** (or integrated out) in which case global anomalies lead to **destructive interferences**
  - e.g. in the **coset models** of **CFT**

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VIRASORO ALGEBRAS AND COSET SPACE MODELS

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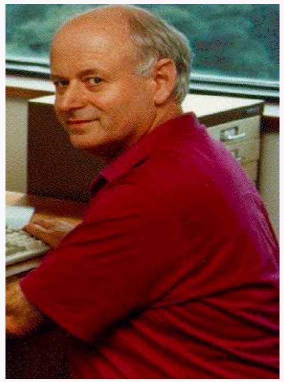
A previous construction of unitary representations of the Virasoro algebra is extended and interpreted physically in terms of a coset space quark model. The quaternionic projective spaces HP^{n-1} yield the complete range of possible values for the central charge when it is less than unity, namely $1 - 6/(n+1)(n+2)$. The supersymmetric extension is also found.



- **Coset models** are **WZW models** with a part of adjoint symmetries gauged and the gauge fields integrated out in functional integrals (**Bardakci** *et al.*, **G.-Kupiainen**, **Karabali** *et al.*, 1988)

- They represent a large class of 2D conformal field theories, e.g. the unitary **BPZ minimal models** for $\widetilde{G} = SU(2) \times SU(2)$ at level $(k, 1)$ and $\Gamma = \mathrm{diag}\, SO(3)$

- Toroidal partition functions of coset models are explicit combinations of the so called **branching functions** for two current algebras $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}}$

- **Destructive interferences** caused by **global gauge anomaly** lead to fractional multiplicities of branching functions destroying **Hilbert**-space interpretation of partition functions!

- **Simplest example of such inconsistency:**

In the G/G **topological** coset theory with $G = SU(3)/\boldsymbol{Z}_3$ at $k=1$ the toroidal **partition functions** are:

$$\mathcal{Z}^G = |\hat{\chi}_{\mathbf{1}}|^2 + \hat{\chi}_{\mathbf{3}}\overline{\hat{\chi}_{\bar{\mathbf{3}}}} + \hat{\chi}_{\bar{\mathbf{3}}}\overline{\hat{\chi}_{\mathbf{3}}}\,, \qquad \mathcal{Z}^{G/G} = \tfrac{1}{3}(1+0+0) = \tfrac{1}{3}$$

(labeling $k=1$ $\widehat{su(3)}$ characters by the $su(3)$ representations $\mathbf{1},\ \mathbf{3},\ \bar{\mathbf{3}}$)

But $Z^{G/G}$ should be an integer **!** to assure **Hilbert** space interpretation as in topological field theory

toroidal partition function $=$ **dimension of space of states**

This is not the end of the story !

- One may couple **WZ** amplitudes to gauge fields that are connections in topologically non-trivial **principal** Γ**-bundles** P over Σ

- **K. Hori** (1996) indicated that inclusion of such gauge fields should solve the so called **field identification problem** in the **coset models**

- More **gerbish** structure, a $\boxed{\Gamma\textbf{-equivariant structure} \text{ on gerbe } \mathcal{G}}$, is needed to couple **WZ** amplitudes to such gauge fields in a fully gauge invariant way

- $\Rightarrow$ a new source of "**discrete torsion**" ambiguity in quantum amplitudes

Conclusions and open problems

- Gauged 2D sigma models with **WZ** term may suffer from **global gauge anomalies** classified by theory of **gerbes**
- Such anomalies render inconsistent many **coset models** of **CFT** based on **WZW models** with non-simply connected groups
- Some open problems:
  - inclusion of **boundaries** and of **conformal defects** via the theory of **equivariant gerbe (bi-)modules** (essentially completed)
  - extension to **super-symmetric** and **non-compact** versions of **CFT** models, etc.
  - applications to global aspects of **T-duality**
  - precise relation to **Chern-Simons** theory and **multiplicative gerbes**
  - higher dimensional generalizations (e.g. in the toroidal geometry) via theory of n-**gerbes**

Paul Maxwell
"Anomaly" (2004)

Some revolutions are large, like those associated with the names of Copernicus, Newton, or Darwin, but most are much smaller, like the discovery of oxygen or the planet Uranus. The usual prelude to changes of this sort is, I believed, the awareness of anomaly, of an occurrence or set of occurrences that does not fit existing ways of ordering phenomena.

Thomas S. Kuhn in "The Essential Tension"

Most so-called anomalies don't seem anomalous to me at all. They seem like nuggets from a gold mine, found by one of the thousands of miners all over the world.

Fischer Black (of **Black-Scholes model**)