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Feyman Path Integrals in Quantum Field Theory

M E N U

I.- Meaning of Planck´s constant

II.- Path Integrals: Dirac (1933), Feynman (1942)

III.- Path Integrals in Field Theory (Weinberg QFT book I, 1995)

IV.- Anomalies (since 1949...)

I.- **<u>Meaning of Planck´s constant</u> (1899)**

There are THREE universal physical constants, $G=G_N$, c and h.

The <u>first</u> Universal Constant, G_N has a clear meaning: it measures the strength of the Gravitational Force (Newton, 1687),

$$\mathbf{F = G_N\, m\ \ m'/r^2}$$

The <u>second</u>, c, has an even clearer meaning: there is a *maximum allowed velocity* (carrying objects and/or information) in any physical system, and this is the velocity of light in vacuum, c (Maxwell; Einstein).

But the <u>third</u>, h, has not such a clear meaning; it was first introduced by Planck (1899) for dimensional reasons (trying to clarify Wien´s law for black-body radiation). Then it became the irreducible "size" of the energy quantum (still Planck; the "true" radiation formula is on 14-XII-1900). Rediscovered by Einstein as energy- frequency relation ($E = h\nu$) in his "Lichtquanten" paper (April 1905). Also h is a *label* to mark quantization of orbits as applied to atoms in the Old Quantum Theory: A. Sommerfeld 1911 (1st Solvay), J. W. Nicholson, 1912; Niels Bohr, 1913.

Nicholson in particular wrote

(angular momentum) J = n <u>h</u> (<u>h</u> is my convention for h/2π)

The "microscopic" aspect of h is measured by its value in "human", antropocentric units:

$$\mathbf{\underline{h} \approx 10^{-27}\ \ cgs}$$

The "size" of h, by itself, has nothing to do with the "size" of the atoms ($\approx 10^{-8}$ cgs) or nuclei ($\approx 10^{-13}$ cgs). As c is "large", $c \approx 3 \cdot 10^{10}$ cgs, the fine structure is NOT that small: $v/c \approx e^2/\underline{h}c \approx (137)^{-1}$.

Later, Sommerfeld rules for multiperiodic systems (1915)

$$\int p_i dq = n_i h$$

emphasized the "action" nature (dimension) of h. It is *curious* that in dimensions Hamilton´s *action* and *angular momentum* are the same:

$$[\text{action}] = [\text{angular momentum}] = ML^2T^{-1} = [h]$$

It is still mysterious that $[e^2] = [\underline{h}c]$, **so** $\alpha = e^2/\underline{h}c \approx 1/137$ is a *pure* number... Einstein realized this, before anybody else...

Action as a *dynamical magnitude* was known since Hamilton (ca. 1830).

In the (modern), Heisenberg´s, Quantum Mechanics: June, 1925, h just measures the extent of non-commutatibility of canonical variables,

$$[q_i, p_j] = i\, \underline{h}\, \delta_{ij} \quad \text{(i, j: 1 to f degrees of freedom)}$$

but this is not a very "sexy" statement... with respect to the meaning of h.

A <u>representation</u> of these commutation relations (a particular one, but as good as any other, so long as $f < \infty$) is the *Schrödinger representation* (January, 1926)   (he did not find out this!; he followed other route: de Broglie)

$$p \text{ operates as } -i\underline{h}\, \partial/\partial q,, \qquad q \text{ operates as multiplication, } q\times$$

on a *wave function* $\psi = \psi(q)$.

II.- Path Integrals: Dirac (1933), Feynman (1942)

The mathematical frame of Quantum Mechanics, was established by von Neumann at that time (completed in the book, 1932): a separable (and complete) Hilbert space H, with physical states as rays, and dynamical variables (observables) as hermitian operators..., etc. The constant h is equally mysterious!.

Why H complex? Probably (I think) because the "symplectic" structure of the classical counterpart "infiltrates" the quantum framework, trading the 2-form $\mathbf{\Omega} \in \wedge \mathbf{T}^0{}_2$ by the complex structure, $\mathbf{J} \in \mathbf{T}^1{}_1$ **(with** $\mathbf{J}^2 = \mathbf{-1}$**)** via the hermitian metric (J. Clemente (?)).

A new development occurred in Richard P. FEYNMAN Ph. D. Thesis (1942):

Feynman found (if right!) the *true* meaning of "h" as fundamental constant:

"The laws of motion of a (quantum) system are fundamentally *probabilistic*, not strictly causal; in particular, in configuration space C {q, q˙} any classical path γ occurs with a probability *amplitude* A(γ), where, if S(γ) is the classical action *along path* γ, the amplitude, with a given Lagrangian L, is

$$\mathbf{S(\gamma) \equiv \int_\gamma L\,(q,\ \dot{q})\,dt,, \qquad \mathsf{A}(\gamma) := \exp iS(\gamma)/\underline{h}}$$

The observation (the observable) is *only* to find the particle detected at the boundary-of-path point Q′ at time t when originated at the point Q at time t=0, where the boundary is $\partial\gamma = \{Q', Q\}$, and it is given, as a probability, by the path integral mod-squared

$$\mathbf{P[\partial\gamma] = |\int_\gamma D\gamma\ \mathsf{A}\ (\gamma)\ |^2} \text{ ''}$$

This, Feynman formulation of the laws of Quantum Mechanics (Richard FEYNMAN Ph. D. Thesis, Princeton 1942; J.A.Wheeler advisor), has SEVEN notorious advantages:

1) The *probabilistic* nature of Quantum Theory is set-up at the outset, not after, as for example, in the Max Born interpretation of the wave function: in Heisenberg´s original matrix formulation there was no “uncertainty”, and Schrödinger *opposed* the probabilistic interpretation of ψ. Now if the paths are *not* observed, the theory should be constructed in such a way that *all* paths should enter (or not!) *on equal footing*, somehow, in the formulation: this is the minimum “positivistic” attitude (This principle, also from Feynman, leads also to “right” counting for identical particles). Of course, “realistic” people like Einstein *rejected* this point of view. ECG Sudarshan is also not very keen on it... E. Santos Corchero (U. de Cantabria) also would not “buy” it... But I do...

2) The *meaning* of the Planck constant h (or, rather h) is very clear: the weight of any path γ potentially (or "virtually") “explored” by the particle is exp[iS(γ)/h], with modulus=1, where S is the classical Action (or Hamilton´s principal function) and it is measured against the constant h, hence S/h is a kind of *angle*: So the phase (= angle) ∝ action/action, is dimensionless, as it should. Neither in matrix mechanics nor in wave mechanics is the meaning of h so clear...

We hope THIS will be the way Quantum Mechanics will be taught in the future... (to our grandchildren...)

3) *Democracy* of paths: the absolute value of A(γ) is always =1, regardless which (individual) path we choose. 3) and 1) are related. Interference occurs because the addition of two mod 1 complex numbers is not mod one in general …

4) The *classical theory* is recovered in the most sensible way: when the actions "S" in the game are much, much bigger than h, S >> h, only the extremal path contributes, and by the stationary phase theorem this is the classical trajectory, defined by $\delta \int L \, dt = 0$. This makes h analogous conceptually to c: a system is non-relativistic, when the pertinent velocities $\overline{v}$ are much less than c, that is: $\overline{v} \ll c$. The argument $h \to 0$, as the analogous $c \to \infty$ is *vitiated*: c and h are constants, and *constants* do not *change*... by definition (just a *pedagogical* point).

5) In *nonabelian gauge theories* (Yang-Mills 1954; ´t Hooft 1971) sum over histories is (by far) the best mode of making calculations, in particular avoiding redundancies due to gauge (invariance) choices, ghosts, etc. (Bryce de Witt, Fadeev, etc.). This is a God´s gift: when you express the theory correctly, you can also perform calculations! J. Schwinger emphasized also this point of view... (in a different context).

6) *Anomalies* are better understood also in the path-integral formalism (Fujikawa). To recall: anomalies are symmetries (of the classical theory), which are *violated* in the quantization process. In terms of path integrals, the path measure is *not* invariant, and it acquires Jacobian terms $\neq 1$. There are relations between divergences and anomalies... More on this later.

7) In the new *interpretation* of Quantum Mechanics, due to Griffiths, Gell-Mann & Hartle, Omnès etc., path integrals are part of the game also.... : It is sometimes called "*sum over histories*" interpretation...

So we see the several advantages of path integrals, *both* as matter of principle and as a practical tool.

The Feynman approach faces the paradoxical aspect of Quantum Mechanics *even better* that the conventional formalism:

1) *Ad* <u>randomness</u>, e.g. radioactive decay, say $^{238}U \to {}^{234}Th + \alpha$,: impredictible moment and direction of the outgoing α: as said, probability is set up at the beginning, axiomatically. In other words, causality is "relaxed": there is NO strict causality (Einstein abhorred this... and wrote : "God does not play dice...", letter to M. Born, 1926).

2) *The two slit* experiment, and wave-particle duality: the amplitude for particles is computed as it were a (de Broglie) wave... [We disagree with R. Feynman that it is the *only* mystery of Q.M. ...]. Low-intensity experiments (Tonomura...) are well explained.

3) *Stern-Gerlach* experiment: <u>realism</u> appears "mitigated", not lost, and the Stern-Gerlach set-up is a typical case of loss of classical realism (the spin does not "exist" until is measured...). Einstein exaggerated ("It is the moon there, when nobody looks?").

4) (Enhaced) α + α sacattering at 90º degrees, twice as "naïve": identity of particles in Q.M. implies indistinguishibility, another case of loss of realism... In *this* case, we´re dealing with bosons...

But ... Feynman procedure also has its own problems; we remark at once just two and ½ (!):

I) The mathematics of path integral is ill-defined (M. Asorey &)

II) The Lagrangian must be substituted by “the Hamiltonian” (S. Weinberg)

III) (Not a problem, but an obscurity). Why the theory defines amplitudes, not probabilities...? It will take us too far afield to discuss justly with this (important) question...

Ad (I): M. Asorey once told me he “hated” path integrals, because they are mathematically ill-defined. Indeed, Dan Fried (Austin, TX) has assured me that $\int D\gamma$ does not make *any* mathematical sense! The Feynman-Kac integrals make sense... but then the exponential does not have “i”.

ECG S. also *dislikes* path Integrals... Niels BOHR is dead (since 1962), but he disliked that also (together with the whole QED development of Tomonaga, R.F. and Schwinger), to the point of blocking the Nobel Prize to the three of them, which was awarded (1965) when he had died... (Y. Ne´eman *dixit*).

My answer (about $\int D\gamma$) is double: i) No mathematical technique used in physics has been well defined from the beginning: Newton “calculus of fluxions” was incorrect (the definition of limit has to wait until Cauchy, well within the XIX century); as for classical fields (Maxwell etc.) the true mathematical nature, as sections in vector bundles with connections, was found not early than 1940! C. N. Yang and T. T. Wu realized the importance of this *for physics* in 1976!!

ii) Feynman himself said: “The absence of a correct mathematical theory would have delayed the development of physics *just a day”* !!

Ad Point II): Hamiltonian not Lagrangian as the "object" was also shown to me by S. Weinberg: he said the original Feynman formulation was *wrong*: the Hamiltonian should be used (in the form pdq – Hdt, of course), not the Lagrangian; and for nonabelian gauge theories, they are not equivalent! Of course, the naïve equivalence is, as just above

$$\mathbf{H = p\dot{q} - H}$$

Please note (J. Cariñena) the Lagrangian is to be integrated, i.e., it is *a form*, not a function...

As said, we shall not discuss point III: why we define amplitudes, not just probabilities?... because I do not know the answer...

Feynman was a very original thinker: faced with some divergences in classical electromagnetism, like $\lim_{r\to 0} e^2/r$ for the (electron) self-energy, he supposed (`*cutting the Gordian knot*´) a particle does not *act* in itself. Wheeler told him this is barely sustainable, as at any rate it would imply all radiation in the universe have to be eventually absorbed; (advanced and retarded waves in proportion 1:1).

Let us briefly recall Lagrangian and Hamiltonian mechanics, first classically. In configuration space, the Lagrangian $L(q, \dot{q})$ rules by extremizing the action:

$$\boldsymbol{\delta\int L(q, \dot{q})\, dt = 0 \quad \Longleftrightarrow \quad \partial L/\partial q - d/dt\, (\partial L/\partial \dot{q}) = 0}$$

The Hamiltonian formulation starts by defining (*carelessly*)

$$H := p\dot{q} - L,$$

and the equations become Hamilton´s:

$$\dot{q} = \partial H/\partial p,, \qquad \dot{p} = -\,\partial H/\partial q$$

Now, around 1950 people (who?) realized the Hamiltonian presentation is equivalent to finding a flow in a symplectic manifold:

Let M(q, p) be a *symplectic* manifold, with a (regular, closed) 2-form Ω. Then generate the flow via the known steps (J. Sancho: Barcelona 1963)

(The energy is a function H on the manifold with real values); then

$$H \to dH \to (\Omega^{-1})(dH) = X_H \to \text{flow} \quad (\text{i.e., } q_i(t), p_i(t))$$

That is, the 2-form Ω is *regular* (non-degenerate), and returns a vector field $X \in T^1{}_0$ from any 1-form $T^0{}_1$, whose integration is the searched for flow, τ_t= "exp"(t X_H): from any *ordinary* point (for the vector field X_H) starts the trajectory (of the possible motion).

Dirac (November, 1925) was the first to notice: the "Poisson brackets" of two functions A, B in classical mechanics:

$$\{A, B\} = \Omega^{-1}\,(dA, dB)$$

become commutators in quantum mechanics,

$$\{q_i, p_j\} \to [q_i, p_j] = i\underline{h}\,\delta_{ij}$$

Recall the *contrast* between Riemannian and symplectic manifolds: the first have no obstructions to be constructed: namely, as long as you include the condition of paracompactness ($\exists$ a partition of the unity; J. Dieudonné) in the definition of manifold M, any such M should admit a Riemann metric g: the (frame) principal bundle of the tangent bundle descends to the orthogonal bundle. On the contrary, there *are* known obstructions to set up a symplectic structure in a manifold (two easy ones are: the manifold should be even-dimensional and orientable; but there are more...). However, there is a complement: Riemannian manifolds can have a *local* obstruction to be flat (namely, the Riemann-Christoffel curvature tensor Riem(g) $\in T^1{}_3$ being zero), whereas any symplectic manifold is automatically "locally flat" (Darboux Theorem, obtained using Ω is closed, $d\Omega=0$).

On the other hand, it is an interesting question to pursue the relation between classical mechanics as symplectic theory, and Quantum Mechanics with its Hilbert space; we mention just one relation, the first one, discovered by Dirac (and alluded to above also): The symplectic character of Classical Mechanics. The relation it is shown, *inter alia*, in the Poisson bracket, which becomes the commutator under the quantization; there are many more analogies... (G. Marmo, J. Clemente, etc. M. de Gosson: the Wigner-Moyal phase space formulation of Q. M. ...). Out of ignorance, I´ll stop the topic here...

It is time to deduce the path-integral formalism. Write $<q \mid \psi(t)>$ as

$$<q \mid \psi(t)> = < q \mid \exp(-iHt/\underline{h} \mid \psi(0)> = \int dq' <q| \exp(-iHt/\underline{h} \mid q'> <q'| \psi\ (0)> = \int dq' U(q, q'; Ht)$$

We use here $\int dq <q \mid q> =1$, hence for the p´s the normalization is different, as $[q, p] = i\underline{h}$:

$$\int dp\ |p><p| = h = 2\pi\underline{h},$$

Now Feynman divides the time interval in N segments, where the integral reduces, and introduces the kets for p, | p´>, with the known result

$$<q_{fin}|\exp\{-iH(q, p)t\}|\ q_{ini}> = \iint \mathbf{D}q\ \mathbf{D}p \exp\{\ i/h \int dt\ [p\dot{q} - H]$$

where the path integral extends for *all* paths with fixed extremes. Of course, the above integrand is just the lagrangian for the systems,

$$L = \Sigma\ p\dot{q} - H,$$ so one can write, finally (for the moment!)

$$< q_f, t_f \mid q_i, t_i > = \Sigma_{\text{paths}} \exp [\ iS(qt; q't')/h$$

In the precise calculation (see e.g. [FS]) use is made of the duality

$$<q|p> = \exp\{ipq/\underline{h}\}$$

between conjugate variables.

III. PATH INTEGRALS IN FIELD THEORY

The utility of the path integrals is best seen in the case of (quantum) fields. In particular, *fermions* can be included easily, changing commutators by anticommutators: if b, $b^\dagger$ are fermion operators (quantization of "ordinary" Grassmann numbers, it is

$$\{b, b\} = \{b^\dagger, b^\dagger\} = 0,, \qquad \{b^\dagger, b\} = \{b, b^\dagger\} = 1$$

We skip elaborating. We only mention another important development due to Schwinger (1951): The Quantum Action Principle.

We establish it following again [FS]. Let ϕ be a quantum scalar field, and take the labels {f, i} stand for final and initial positions of the field amplitudes; the principle asserts that the variation of the transition amplitude $< \phi_f, t_f | \phi_i, t_i >$ is given in terms of the variation of the Lagrangian density L for the scalar field, namely

$$\delta < \phi_f, t_f | \phi_i, t_i > = \int d^4x < \phi_f, t_f | i \delta L / \underline{h} | \phi_i, t_i >$$

Notice the variation of the Lagrangian density has to be compatible with the extremes $t_{f,i}$ fixed, as corresponding to the observation: particle originates and ends up at fixed points, while describes in between *any* path.

The Schwinger action principle therefore unites the old operation presentation of Quantum field Theory with the new path integral development of Feynman...

As I said, Weinberg holds the original path integral expressions with the Lagrangian density was wrong. He [cfr. W-QFT-I p. 376ff.] deduce the new (paths) formalism from the "old" operator formalism to ensure unitarity of the S-matrix. He also shows that the "naïve" Feynman approach is wrong in the case of the sigma model (that we refrain to explain), in the sense that the Feynman vertices are not complete... This reminds one that the nonabelian gauge theories also have "ghosts" or additional vertices, not seen for example in QED.

Although we shall not follow Weinberg´s arguments in detail, let´s note what he does: he deduces the path-integral with the form pq –H, and then shows that the lagrangian obtains in many, but by no means in all, cases: the Hamiltonian has to be quadratic in the momenta, for example... A special case is indeed the nonabelian gauge fields...

IV.- ANOMALIES

As introduction, consider the integral in the real line

$$\int_{-\infty}^{+\infty} \mathbf{f(x)\, dx}$$

If the traslation $x \to x+a$ is a contempable symmetry of the integrand, the result is also symmetric, because the measure in translation-invariant:

$$\mathbf{dx = d(x+a)}$$

However, for dilations ($x \to \lambda x$) the measure is *not* invariant:

$$\mathbf{d\,(\lambda x) = \lambda\ dx = (\lambda d)\ x}$$

This simple example shows the anomaly problem in Quantum Field Theory: for example, massless field theories in 4-dim tend to be conformal invariant, and con formal transformations include dilatations, under which the measure is not invariant...

In general, anomalies are not divergent quantities, just unexpected results: classically two groups (Fukuda and Miyamoto in Japan, J. Steinberger in USA) found (ca. 1949) "naively" that the neutral pion decay into photons was forbidden; however, the process

$$\pi^0 \to \gamma + \gamma$$

is the main mode of π^0 decay, with a lifetime around the picosec. J. Schwinger, without realizing it was a (first!)

case of anomaly, by pursing carefully a gauge-invariant renormalization procedure, found the decay to be possible. Incidentally, the fustration in J. Steinberg determined for him to “mutate” to an experimental physicist, and eventually he got the Nobel prize... for his experimental work (two neutrinos, 1962).

Jackiw and J. Bell studied the problem twenty years later (1969) and concluded, correctly, that was a case of anomalous (chiral) symmetry: the whole thing is more complicated, as the chiral symmetry of the strong forces is the global $SU(2)_L \times SU(2)_R$ symmetry, which is both spontaneous and explicitely broken! (with the resulting pion as the Goldstone-Nambu scalar, and e.m. violation of isospin..., S. Weinberg speaks of pseudo-Goldstone bosons...).

Besides gravitational anomalies, important in supergravity and in string theories (in higher symmetries), the two more mundane anomalies occur for the two just mentioned space-time & internal symmetries:

Chiral symmetry: e.g. QCD is globally invariant under SU(2)_L x SU(2)_R, and only the diagonal part (SU(2)_isospin is preserved.

Conformal symmetry. Bateman and Cunningham (1910) were the first to write the conformal invariance of the vaccuum Maxwell equations. Also massless quantum fields are naively conformal invariant also...

In string theory in the Polyakov form (1981), the Weyl symmetry is kind of scale symmetry, it is violated upon quantization, and the result is: the theory is anomaly free only in 26 dimensions (bosonic string) or in ten (superstring)...

We just add a commentary on the “index” of the oscillator operera operators, following [FS]. Suppose D is an “elliptic” operator (swe skip the very technical definition), in particular a linear operator with finite dimension for both kernel (ker D = {x| Dx=0} and cokernel (If D leads (linear) space E to linear space F, Coker D = F\ Im D). Then the index of d is defined (Atuyah and Singer, 1963) as

$$\mathbf{Ind\ D = dim\ Ker\ D - dim\ Coker\ D \in Z}$$

This is one of the most important theorems in the mathematics of last 50 years. The point: That index (but not separately the summands!) in a topological invariant, which can be computed from the spaces E and F (which are supposed to be vector bundles over *same* manifold, say M). In fact, one computes the index from the characteristic classes of the bundles.

The simplest example I know is the $d + \delta$ differential operators in a compact manifold, with d the ordinary exterior derivative and δ some Hodge dual. For E one takes the even-dimensional p-forms, for F the odd ones; hence Ker d are the even closed forms, and (one shows) Ker δ the coclosed ones; hence, we get the Betti numbers:

$$\mathbf{Ind}\,(d + \delta) = b_0 - b_1 + b_2 - \ldots \pm b_n = \chi(M),$$

So the index is just the Euler-Poincaré characteristic of the manifold.

Admitting the "elliptic"character of the $[a, a^\dagger]$ set, one has Ind =1, as the Coker is empty...

We stop here...

Muchas gracias por su atención

Thank you very much for your attention

SOME PERTINENT LITERATURE

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