THE EQUIVALENCE THEOREMS AND MANIFESTLY LORENTZ INVARIANT NONPERTURBATIVE FORMULATION OF NON-ABELIAN GAUGE THEORIES.

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Progress in physics was usually related to the introduction of new symmetries.

Recent examples are given by gauge theories. QED may be formulated in the Coulomb gauge, however much more transparent formulation is presented by the quantization in a manifestly covariant gauge. Yang-Mills theory became really popular only after its formulation in the Lorentz covariant terms and explicit proof of its renormalizability. The gauge invariance of the Higgs model allows to give a manifestly renormalizable theory describing a massive gauge theory.

In this talk I wish to make a propaganda for a new class of symmetries, which were introduced in my paper rather long ago (A.A.S., 1991), but recently were applied successfully to the nonperturbative quantization of non-Abelian gauge theories.

Equivalence theorems: canonical transformations, point transformations $\varphi = \varphi' + f(\varphi')$

More general transformations:

$$\varphi = \frac{\partial^n \varphi'}{\partial t^n} + f(\frac{\partial^{n-1}\varphi'}{\partial t^{n-1}}, \ldots \frac{\partial \varphi'}{\partial t}) = \tilde{f}(\varphi') \qquad (1)$$

The spectrum is changed. What about the unitarity?

Path integral representation for the scattering matrix

$$S=\int \exp\{i\int L(\varphi)dx\}d\mu(\varphi); \quad lim_{t\to\pm\infty}\varphi(x)=\varphi_{out,in}(x) \tag{2}$$

If the change (1) does not change the asymptotic conditions, then the only effect of such transformation is the appearance of a nontrivial Jacobian

$$L(\varphi)\to \tilde{L}(\varphi')=L[\varphi(\varphi')]+\bar{c}^a\frac{\delta\varphi^a}{\delta\varphi'^b}c^b \tag{3}$$

For all new excitations one should take the vacuum boundary conditions.

Unitarity?

The new Lagrangian is invariant with respect to the supertransformations

$$\delta\varphi'_a = c_a\varepsilon$$
$$\delta c_a = 0; \quad \delta\bar{c}_a = \frac{\delta L}{\delta\varphi_a}(\varphi')\varepsilon \tag{4}$$

On mass shell these transformations are nilpotent and generate a conserved charge Q. In this case there exists an invariant subspace of states annihilated by Q, which has a semidefinite norm. (A.A.S.,1991). For asymptotic space this condition reduces to

$$Q_0|\phi>_{as} = 0 \tag{5}$$

The scattering matrix is unitary in the subspace which contains only excitations of the original theory. However the theories described by the L and the $\tilde{L}$ are different, and only expectation values of the gauge invariant operators coincide. In gauge theories the transition from one gauge to another may be considered as such a change.

A very nontrivial generalization is obtained if one transforms the $\tilde{L}$ further shifting the fields φ' by constants. It is not an allowed change of variables in the path integral as it changes the asymptotic of the fields. The unitarity of the "shifted"theory is not guaranteed and a special proof (if possible) is needed.

Using this method one can construct a renormalizable formulation of nonabelian gauge theories free of the Gribov ambiguity.

A.A.Slavnov,JHEP,0808(2008)047; Theor.Math.Phys,161(2009)204; A.Quadri, A.A.Slavnov, JHEP, 1007 (2010); A.Quadri, A.A.Slavnov, Theor.Math.Phys, 166(2011)201

A problem of unambiguos quantization of nonabelian gauge theories beyond perturbation theory remains unsolved. Even in classical theory the equation

$$D_\mu F_{\mu\nu} = 0 \tag{6}$$

does not determine the Cauchi problem. Gauge invariance results in existence of many solutions of this equation. To define the classical Cauchi problem and subsequently to quantize the model one imposes a gauge condition, e.g. Coulomb gauge $\partial_i A_i = 0$.

Differential gauge conditions: $L(A_\mu, \varphi) = 0 \rightarrow$ Gribov ambiguity.

Algebraic gauge conditions: $\tilde{L}(A_\mu, \varphi) = 0 \rightarrow$ absence of the manifest Lorentz invariance and other problems.

Coulomb gauge

$$\partial_i A_i = 0$$
$$A'_i = (A^{\Omega})_i$$
$$\triangle \alpha^a + ig\varepsilon^{abc}\partial_i(A_i^b \alpha^c) = 0 \tag{7}$$

This equation has nontrivial solutions fastly decreasing at spatial infinity→Gribov ambiguity.

In perturbation theory the only solution is $\alpha = 0$.

A remedy: new (equivalent) formulation of the Yang-Mills theory using more ghost fields.

Let us consider the classical ($SU(2)$)Lagrangian

$$\tilde{L} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + (D_\mu\varphi)^*(D_\mu\varphi) - (D_\mu\chi)^*(D_\mu\chi) \\ +i[(D_\mu b)^*(D_\mu e) - (D_\mu e)^*(D_\mu b)] \tag{8}$$

The scalar fields (φ, χ are commuting, e, b are anticommuting) are parametrized by the Hermitean components

$$\Phi = \left(\frac{i\Phi_1 + \Phi_2}{\sqrt{2}}, \frac{\Phi_0 - i\Phi_3}{\sqrt{2}}\right) \tag{9}$$

Integrating over the fields φ, χ, b, e with vacuum boundary conditions one gets

$$\int \exp\{i \int \tilde{L} dx\} d\tilde{\mu} = \int \exp\{i \int L dx\} (\det D^2)^2 (\det D^2)^{-2} d\mu \qquad (10)$$

Here the measure $d\mu$ includes the gauge fixing factor and Faddeev-Popov ghosts and the measure $d\tilde{\mu}$ includes also differentials of the fields φ, χ, b, e. The integral reduces to the usual path integral for the Yang-Mills scattering matrix. The lagrangian $\tilde{L}$ gives for the gauge invariant correlators the same result as the standard Yang-Mills Lagrangian:

$$L = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \qquad (11)$$

Now we consider a different lagrangian, which may be obtained from $\tilde{L}$ by the shift

$$\varphi \to \varphi - g^{-1}\widehat{m}; \quad \chi \to \chi + g^{-1}\widehat{m} \tag{12}$$

The constant field $\widehat{m}$ has a form:

$$\widehat{m} = (0, m) \tag{13}$$

The new Lagrangian looks as follows

$$\begin{aligned} L = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + (D_\mu\varphi)^*(D_\mu\varphi) - (D_\mu\chi)^*(D_\mu\chi) \\ -g^{-1}[(D_\mu\varphi)^* + (D_\mu\chi)^*](D_\mu\widehat{m}) - g^{-1}(D_\mu\widehat{m})^*[D_\mu\varphi + D_\mu\chi] \\ +i[(D_\mu b)^*(D_\mu e) - (D_\mu e)^*(D_\mu b)] \end{aligned} \tag{14}$$

Note that because of the negative sign of the χ kinetic term this field posesses negative energy. This is crucial to insure the cancellation of the terms quadratic in m in the shifted Lagrangian and provide the zero mass for the Yang-Mills field.

Higgs model.

The model we consider in many respects reminds the Higgs model. Instead of one scalar fields we have two scalar fields with different signs of energy and two more anticommuting scalar fields. The presence of two commuting scalar fields with different signs of energy allows to avoid the mass generation for the vector field. As in the Higgs model these scalar fields become gauge fields, that is by the gauge transformation they are shifted by arbitrary function.

In the Higgs model one starts with the Lagrangian

$$L = L_{YM} + (D_\mu \varphi)^*(D_\mu \varphi) - \lambda^2(\varphi^* \varphi - \mu^2)^2 \tag{15}$$

After the shift $\varphi = \varphi' + \widehat{\mu}, \quad \widehat{\mu} = \{0, \quad \mu\}$ $\varphi'_a, \quad a = 1,2,3$ becomes a gauge field:$\varphi'_a \to \varphi'_a + \mu\eta^a(x) + \ldots$. Unitary gauge $\varphi'_a = 0$ is algebraic, but Lorentz invariant. However this gauge is nonrenormalizable.

Is it possible to invent Lorentz invariant algebraic gauge for the Yang-Mills theory in which the theory is renormalizable?

The Lagrangian (14) may be obtained from the gauge invariant Lagrangian, describing the interaction of the complex scalar doublets with the Yang-Mills field by the shift

$$\varphi \to \varphi - g^{-1}\hat{m}; \quad \chi \to \chi + g^{-1}\hat{m} \tag{16}$$

Hence the Lagrangian (14) is invariant with respect to the"shifted"gauge transformations.

In particular the transformation of the field $\varphi_-^a = \frac{\varphi - \chi}{\sqrt{2}}$ is

$\delta\varphi_-^a = m\eta^a + \frac{g}{2}\varepsilon^{abc}\varphi_-^b\eta^c + \frac{g}{2}\varphi_-^0\eta^a$

This Lagrangian is also invariant with respect to the supersymmetry transformations

$$\begin{aligned} \delta\varphi_\alpha^-(x) &= 2i\epsilon b_\alpha(x) \\ \delta e_\alpha(x) &= \epsilon\varphi_\alpha^+(x) \\ \delta b(x) &= 0 \end{aligned} \tag{17}$$

where ϵ is a constant anticommuting parameter.

This invariance plays a crucial role in the proof of the equivalence of the model described by the Lagrangian (8) to the standard Yang-Mills theory. It provides the unitarity of the scattering matrix in the subspace which includes only three dimensionally transversal components of the Yang-Mils field.

The field φ_-^a is shifted under the gauge transformation by an arbitrary function $m\eta^a$. It allows to impose Lorentz invariant algebraic gauge condition $\varphi_-^a = 0$.

However imposing the Lorentz invariant gauge condition $\varphi_-^a = 0$ does not solve the problem of ambiguity completely. The field φ_-^a satisfying the condition $\varphi_-^a = 0$ is transformed by the gauge transformation to $\varphi_-'^a = (m + \frac{g}{2}\varphi_-^0)\eta^a$. For some x the factor $(m + \frac{g}{2}\varphi_-^0(x))$ may vanish, leading to nonuniqueness of the gauge fixing. Moreover calculation of the divergency index of Feynman diagrams shows that there are divergent diagrams with arbitrary numbers of external φ_-^0 lines, that is the theory is not renormalizable in the usual sense.

To avoid the problem of ambiguity completely we redefine the fields as follows

$$\varphi_-^0 = \frac{2m}{g}(\exp\{\frac{gh}{2m}\} - 1); \quad \varphi_-^a = \tilde{M}\tilde{\varphi}_-^a$$
$$\varphi_+^a = \tilde{M}^{-1}\tilde{\varphi}_+^a; \quad \varphi_+^0 = \tilde{M}^{-1}\tilde{\varphi}_+^0$$
$$e = \tilde{M}^{-1}\tilde{e}; \quad b = \tilde{M}\tilde{b} \tag{18}$$

where

$$\tilde{M} = 1 + \frac{g}{2m}\varphi_-^0 = \exp\{\frac{gh}{2m}\} \tag{19}$$

The new Lagrangian has the form

$$\tilde{L} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \partial_\mu h \partial_\mu \tilde{\varphi}^0_+ - \frac{g}{2m}\partial_\mu h \partial_\mu h \tilde{\varphi}^0_+$$
$$+m\tilde{\varphi}^a_+\partial_\mu A^a_\mu - [((D_\mu\tilde{b})^* + \frac{g}{2m}\tilde{b}^*\partial_\mu h)(D_\mu\tilde{e} - \frac{g}{2m}\tilde{e}\partial_\mu h) + h.c.]$$
$$+\frac{mg}{2}A^2_\mu\tilde{\varphi}^0_+ + g\partial_\mu h A^a_\mu \tilde{\varphi}^a_+ \ldots \quad (20)$$

Here ... denote the terms $\sim \tilde{\varphi}^a_-$. By construction this Lagrangian is invariant with respect to the gauge transformations written in terms of the new variables. In particular $\delta\varphi^a_- = \eta^a$, and the ambiguity is absent.

Obviously the lagrangian is also invariant with respect to the supersymmetry transformations written in terms of the transformed variables. However imposing the gauge condition $\varphi_-^a = 0$ we break the invariance of the effective action with respect to the supersymmetry transformation (17). The transition from one gauge to the other one may be achieved by a gauge transformation, and in the gauge $\partial_i A_i = 0$ the effective action is invariant with respect to the supertransformation (17). Therefore in the gauge $\varphi_-^a = 0$ it also must be invariant with respect to some supertransformation. The corresponding gauge function is a solution of the equation

$$\int d^4x \lambda^a(x) \partial_i (A^\Omega)_i^a(x) = \int d^4x \lambda^a(x) \varphi_-^a(x) \tag{21}$$

The solution of this equation may be found explicitly.

For asymptotic states it is sufficient to solve this equation to zero order in g. In this approximation the supersymmetry transformation acquires the form:

$$
\begin{aligned}
\delta\tilde{A}^a_\mu(x) &= \frac{1}{m}\partial_\mu\tilde{b}^a(x) \\
\delta\tilde{e}^\alpha(x) &= \tilde{\varphi}^\alpha_+(x) \\
\delta h &= -\tilde{b}^0(x)
\end{aligned}
\tag{22}
$$

Other fields do not change in this approximation

The spectrum:

Ghost exitations: $\varphi_{\pm}, b, e,$ longitudinal and temporal components of A^a_μ

Physical exitations: three dimensionally transversal components of the Yang-Mills field.

The supersymmetry of the effective action generates a conserved nilpotent charge Q. Physical states are separated by the condition

$$Q|\psi>_{ph} = 0 \tag{23}$$

the states separated by this condition describe only three dimensionally transversal components of the Yang-Mills field.

The ghost exitations decouple.

Renormalization

The field $h(\varphi_-^0)$ enters interaction only with derivative $\partial_\mu h$. Hence the divergency index of a diagram with n external $h(\varphi_-^0)$ lines decreases by n.

The index of divergency of an arbitrary diagram is

$$n = 4 - 2L_{\varphi_+^0} - 2L_{\varphi_+^0} - L_A - L_e - L_b - L_h \tag{24}$$

The theory is manifestly renormalizable.

In terms of the old (nontransformed) variables the theory is not manifestly renormalizable. Transition to the new variables simultaneously eliminates the residual ambiguity and makes the theory manifestly renormalizable. Renormalization preserves all the symmetries of the theory.

Weinberg-Salam model.

In perturbation theory all predictions fit the experiment very well. However there are certain questions to be answered

1. Where is the Higgs meson? (LHC).

2. Is the model valid beyond perturbation theory?

3. Is it possible to derive the Weinberg-Salam model from some grand-unified model?

4. Quantization of the Weinberg-Salam model beyond the perturbation theory?

An alternative formulation of the Higgs-Kibble model.

$$
\begin{aligned}
L = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + (D_\mu\varphi^+)^*(D_\mu\varphi^-) + (D_\mu\varphi^-)^*(D_\mu\varphi^+) \\
+(D_\mu\varphi)^*(D_\mu\varphi) - \lambda^2(\varphi^*\varphi - \mu^2)^2 \\
-[(D_\mu b)^*(D_\mu e) + (D_\mu e)^*(D_\mu b)]
\end{aligned} \tag{25}
$$

Here the field φ is the complex doublet describing the Higgs meson, and the fields $\varphi^\pm$ are new auxiliary fields. The fields b, e have a similar structure, but correspond to the anticommuting fields. The shift

$$
\varphi^-(x) \to \varphi^-(x) - \hat{m}; \quad \varphi(x) \to \varphi(x) - \hat{\mu} \tag{26}
$$

where $\hat{m}$ and $\hat{\mu}$ are the coordinate-independent condensates

$$
\hat{m} = (0, m/g); \quad \hat{\mu} = (0, \mu/g) \tag{27}
$$

generates the mass term for the vector field.

In the same way as before one can show that the theory described by this Lagrangian is renormalizable and unitary in the space, including only three polarizations of the vector field and the scalar Higgs meson.

Therefore the extension of the spectrum of non-Abelian gauge theories, supplemented by the corresponding extension of their symmetry (supersymmetry), makes possible to perform quantization of these theories both in the framework of perturbation theory and beyond it. It allows to study QCD and electroweak models beyond perturbation theory.

The possible counterterms may be classified on the basis of ST-Identities, associated with the symmetry, which combines the gauge invariance of the effective action and supersymmetry and which are conveniently written in the form proposed by J.Zinn-Justin

$$S(\Gamma) = \int d^4x \sum_{\Phi} \{ \frac{\delta\Gamma}{\delta\Phi^*(x)} \frac{\delta\Gamma}{\delta\Phi(x)} \} = 0 \tag{28}$$

Φ are the fields: $A_\mu, \varphi^\alpha_+, e^\alpha, b^\alpha, h$; Φ^* are the antifields introducing the variations of the fields Φ, e.g. $\int dx \{-\frac{2i}{m} A^{*a}_\mu (D_\mu b)^a\}$

All ultraviolet divergencies may be removed by a multiplicative renormalizations and redefinitions of the fields.

Conclusion.

A renormalizable manifestly Lorentz invariant formulation of the non-Abelian gauge theories which allows a canonical quantization without Gribov ambiguity is possible.

In particular the Weinberg-Salam model may be formulated in a manifestly Lorentz invariant, renormalizable and ambiguity free way.

In perturbation theory the scattering matrix and the gauge invariant correlators coincide with the standard ones.

It would be interesting to carry out semi-analytic and numerical calculations in this formalism beyond the perturbation theory and compare the results with the existing calculations.