Lower Dimensional Field Theories and Ocurrences of Topological Defects in Condensed matter

J. Sanchez-Guillen

Department of Particle Physics
University of Santiago de Compostela

What is Quantum Field Theory?
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Contents

- Introduction: Working on $2+1$ Field Theory ... : "has a serious risk of wasting time or even getting the wrong impressions, but the ease of a much better visualization makes it worth" R. P. Feynman 81.
- Oldies and latest hits. Planar ("Baby") Skyrme .
- Giant Baby Skyrmions in Magnetic Materials.
- Conclusions (and Surprise).

[based on work with C. Adam, P. Klimas, C. Naya, J.M. Queiruga, A. Wereszczynski...and discusions with M. Asorey]

1. Why Baby Skyrmions

- Baby Skyrme Lagrangian ($d = 2+1$)

$$L = \frac{\lambda_2}{2}\partial_\mu \vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{\lambda_4}{4}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2 - \lambda_0 V(\phi_3)$$

  $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$; $\vec{\phi}^2 = 1$, stereogr. projection of complex u field. λ_i couplings. Potential (λ_0) now mandatory.
- Same topology as full, but more simple: meromorphic BPS static solutions (or $1+1$ instantons) (Belavin Polyakov 75).
- $\mathcal{L}_4 + \mathcal{L}_0$ is Derrick stable alternative to conformal $\mathcal{L}_2$ of BP.

2.1 New Solutions, numerical.

Baby Skyrme with new potential $L_0 = (1 - {\phi_3}^2)^{1/2}$ has:

- compact non top- Q-balls and Q-shells, spinning and non-spinning, from Ansatz $u = e^{i(\omega t + n\phi)} f(r)$, with bounded ω_c, contrary to Signum Gordon $\Leftarrow$ (BS for small $\mid u \mid$).
- Peakon solutions: jump in 1st derivative, extremely large 2nd. Generic featrue of BS, rather indep. of potential.
- Topological Compacton BS solutions for different Q_T, generalizing Hen and Karliner. Stable multisoliton solutions, if separated enough. $\Rightarrow$ Consider restricted action $\mathcal{L}_4 + \mathcal{L}_0$, PRD 80, (2009)and 81 (2010)

2.2 Analytical Results. Restricted action.

$L = \frac{1}{2}(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi})^2 + \mu^2 \, V(\phi^3)$, generalizes Gisiger Paranjape .

- Hughe Symmetries: Area preserving diffeos on base and Abelian subgroup on target $\Rightarrow$ generalized integrable.
- Exact (compact) solitions of Restricted (R) saturate new BPS bound, a generalization of a tighter one for the full BS (Ward): $E_{bS} = E_{O(3)} + E_R \geq 4\pi|Q|(1+\alpha\mu)$
- multisolitons in full BS exist only for potentials with compactons in restricted, which has no Qballs.
- Energies of RBS approximate reasonably full BS.
- Seed of $3+1$ BPS Skyrme for Hadrons and Nuclei: $E = 2\lambda\mu\pi^2 < \sqrt{V} >_{S^3} |B|$. Phys.Lett.B691 105, 2010.

2.3 Generalized Integrability (GI).

G I is a proposal (Orlando Alvarez, L.A. Ferreira and J.S.-G. Nucl. Phys. B 529, 689 (1998) and Int. J. Mod. Phys. A 24, 1825 (2009)) to extend to higher dimensions the wealth of $2d$ ideas based on holonomies in loop spaces LP with gauge connections and nonabelian Stokes calculus.

- GI : means Infinite conservation laws in nonlinear Field Theories when their equations o.m. expresable as $D_A B = 0$ with a 1-form $A \in g$ and a $d-1$ form B in a an abelian ideal, which are suff. conds for flatness in the LP.
- Flat connections $\mathcal{A}$ in LP are Diff. (reparam) invariance.$\Leftrightarrow$ Locality of curvature of $\mathcal{A}$.
- Holonomy group of rep-flat connections is abelian.

- The 0(3) model has an integrable sector ($\equiv$infinite conserved currents) given by $(\partial_\mu u)^2 = 0$ and $\partial^2 u = 0$, which are the (static) Cauchy-Riemann eqs.
- the full baby model possesses an integrable submodel defined just by the eikonal equation $(\partial_\mu u)^2 = 0$.
- the restricted model IS integrable (in this generalized sense)

$$J_\mu = \frac{\delta G}{\delta \bar{u}} \mathcal{K}_\mu - \frac{\delta G}{\delta u} \bar{\mathcal{K}}_\mu, \quad G = G(u\bar{u}),$$

$$\mathcal{K}^\mu = \frac{K^\mu}{(1+|u|^2)^2}, \quad K^\mu = (u_\nu \bar{u}^\nu)\bar{u}^\mu - \bar{u}_\nu^2 u^\mu$$

2.4. Susy extensions of Baby Skyrmions

- Baby Skyrme Lagrangian ($d = 2 + 1$)

$$L = \frac{\lambda_2}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{\lambda_4}{4}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2 - \lambda_0 V(\phi_3)$$

- SUSY extension: superfield constraint $\vec{\Phi}^2 = 1 \;\Rightarrow$

$$\vec{\phi}\cdot\vec{\phi} = 1\,,\;\vec{\phi}\cdot\vec{\psi}_\alpha = 0\,,\;\vec{\phi}\cdot\vec{F} = \frac{1}{2}\vec{\psi}^\alpha\cdot\vec{\psi}_\alpha$$

E. Witten, Phys. Rev. D**16**, 2991 (1977), C. A., J.Q., J.S-G and A. W., Phys. Rev. D 84, 025008 (2011)

- Bosonic sector $\psi = 0$:

$$\vec{\phi}\cdot\vec{\phi} = 1\,,\;\vec{\phi}\cdot\vec{F} = 0$$

- SUSY extension: same "building blocks"
- O(3) sigma model term

$$(\mathcal{L}_2)_{\psi=0} = \frac{1}{2}[(D^\alpha \Phi^i D_\alpha \Phi^i)||]_{\theta^2;\psi=0} = F^i F^i + \partial^\mu \phi^i \partial_\mu \phi^i$$

- Potential

$$(\mathcal{L}_0)_{\psi=0} = -[P(\Phi_3)||]_{\theta^2;\psi=0} = F_3 P'(\phi_3)$$

- Quartic (Skyrme) term

$$
\begin{aligned}
(\mathcal{L}_4)_{\psi=0} &= \frac{1}{2}\epsilon_{ijk}\epsilon_{i'j'k}[(D^\alpha\Phi^i D_\alpha\Phi^{i'} D^2\Phi^j D^2\Phi^{j'} + \\
&\qquad D^\alpha\Phi^j D_\alpha\Phi^{j'} D^2\Phi^i D^2\Phi^{i'})|]_{\theta^2;\psi=0} \\
&\quad - \frac{1}{8}\epsilon_{ijk}\epsilon_{i'j'k}[(D^\alpha\Phi^i D_\alpha\Phi^{i'} D^\gamma D^\beta\Phi^j D_\gamma D_\beta\Phi^{j'} + \\
&\qquad + D^\alpha\Phi^j D_\alpha\Phi^{j'} D^\gamma D^\beta\Phi^i D_\gamma D_\beta\Phi^{i'})|]_{\theta^2;\psi=0} \\
&= \epsilon_{ijk}\epsilon_{i'j'k}(F^i F^{i'} F^j F^{j'} - \partial_\mu\phi^i\partial^\mu\phi^{i'}\partial_\nu\phi^j\partial^\nu\phi^{j'}) \\
&= -(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2
\end{aligned}
$$

- $\Rightarrow$ Bosonic Lagrangian

$$\mathcal{L}_b = \frac{\lambda_2}{2}[(\vec{F})^2 + \partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}] - \frac{\lambda_4}{4}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2$$
$$+\lambda_0 F_3 P' + \mu_F(\vec{F}\cdot\vec{\phi}) + \mu_\phi(\vec{\phi}^2 - 1)$$

μ_F and μ_ϕ ... Lagrange multipliers enforcing constraints

- Field equ. for F^i: Solution

$$F^i = \frac{\lambda_0}{\lambda_2}(\phi_3\phi^i - \delta^{i3})P'$$

$$\mu_F = -\lambda_0\phi_3 P'$$

- Resulting Lagrangian, Baby Skyrme with potential ≥ 0

$$\begin{aligned} \mathcal{L}_b &= \frac{\lambda_2}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{\lambda_4}{4}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2 \\ &\quad -\frac{\lambda_0^2}{2\lambda_2}(1-\phi_3^2)P'^2 + \mu_\phi(\vec{\phi}^2 - 1) \end{aligned}$$

- $\Rightarrow$ standard (BS) $\mathcal{L}_b$ with $V \geq 0$ allow for SUSY extensions, (aswell as just $\mathcal{L}_2$ and $\mathcal{L}_4$) . . . .
- . . . BUT NOT the BPS restricted model $\mathcal{L}_4 + \mathcal{L}_0$, since the only solution with $\lambda_2 = 0$ is $\lambda_0 = 0$.
- Building block procedure generalizable arX11074370, PRD

3. BSkyrmions in magnetic materials

Action $\mathcal{L}_2 + \mathcal{L}_0$ taken as continuum limit of Heisenberg model

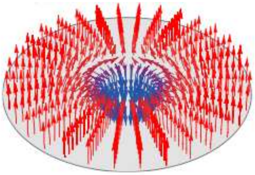
$$H_J = \frac{1}{2}\Gamma \int d^2x [(\partial_k \vec{n})(\partial_k \vec{n}) + \xi^{-2}(1 - n_z^2)], \qquad \Gamma = (1/2)zS^2J$$

z denotes the number of nearest neighbours, S the spin per atom, J the exchange constant, ξ is the single-ion anisotropy.

- A Dipole-Dipole interaction with with strength $\Omega = NS^2g^2\mu_B^2\mu_0/a^4$ (N number of layers, g Landé factor, a lattice constant), dominant at large distances, plays the role of the topological terms.
- An external magnetic field can induce a phase transition between the defect lattice of up-down domains (weak) and the ferromagnetic (strong), alowing Skyrmion formation.

Skyrmion simplest spin configuration with non-zero topological charge for the spin configuration

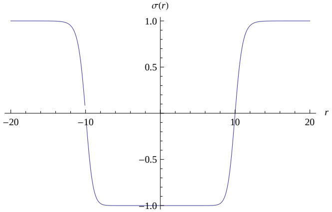
$$n_x = -\sqrt{1-\sigma^2(r)}\cos(\theta+\theta_0)$$
$$n_y = -\sqrt{1-\sigma^2(r)}\sin(\theta+\theta_0)$$
$$n_z = \sigma(r)$$



with θ_0 a constant and $\sigma(r)$ a trial function with boundary conditions

$$\sigma(r) \xrightarrow[r\to\infty]{} 1 \qquad \sigma(r) \xrightarrow[r\to 0]{} -1$$

$$\Rightarrow Q_{sky} = \frac{1}{8\pi}\int d^2x\, \epsilon_{ij}\vec{n}(\vec{x})\cdot(\partial_i\vec{n}(\vec{x})\times\partial_j\vec{n}(\vec{x})) = 1$$



The trial function

$$\sigma(r) = \tanh[(R/\xi)\log(r/R)]$$

gives a circular domain of radius R separated by a domain wall so inside, $\sigma(r) \approx -1$, whereas outside, $\sigma(r) \approx 1$.

Near the domain wall, $r \approx R$, tends to the kink solution

$$\sigma(r) \approx \tanh[(r-R)/\xi]$$

The qualitative proposal by Ezawa P R L **105,** (2010) can be put on firm ground: So the total energy of the Skyrmion is

$$E_{sky} = \frac{4\pi\Gamma R}{\xi} - 4\Omega[R\log(R/d_F) - R] + 2\pi\frac{R^2}{a^2}\Delta_Z$$

Minimizing with respect to R, $\frac{dE_{sky}}{dR} = 0$

$$\Rightarrow R = -\frac{a^2\Omega}{\pi\Delta_Z}W_{-1}\left(-\frac{\Delta_Z}{e\Delta_Z^s}\right)$$

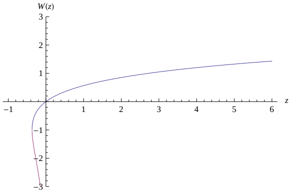
where $W_{-1}(z)$ is the Lambert function and

$$\Delta_Z^s = e\Delta_Z^s = e\frac{a^2\Omega}{\pi d_F e^2}e^{-\frac{\pi\Gamma}{\xi\Omega}}$$

The Lambert function, W(z)

$$ze^z = a \Longrightarrow z = W(a)$$

- Two real branches and branch point, $z = -1/e$, where $W(-1/e) = -1$.
- Real values for $z \geq -1/e$.



Minimun radius

There is an upper bound for Δ_Z and h.

$$\Delta_Z < \Delta_Z^s \qquad \text{or} \qquad h < h_s$$

$$\Rightarrow R_{min} = \frac{a^2\Omega}{\pi\Delta_Z} = \frac{a^2\Omega}{\pi\Delta_Z^s} = \frac{d_F e^2}{e} e^{\frac{\pi\Gamma}{\xi\Omega}} = \frac{\ell_S}{e}$$

(where ℓ_S is periodicity),

- Under weak external magnetic field the ground state is a lattice of alternating up-down domains, while for strong external magnetic field there is a ferromagnetic ground state allowing the formation of Skyrmions.
- There exists in fact a magnetic field window $h_c < h < h_s$ in which the Skyrmion can be created, as transition between the alternating lattice with $\vec{n} = (0, 0, -1)$ shrinking and the regions $(0, 0, 1)$ growing with the strong field.
- The Skyrmion radius presents a minimum value:

$$R_{min} = e d_F e^{\frac{\pi \Gamma}{\xi \Omega}} = \frac{\ell_S}{e}$$

in qualitative agreement ($\sim 100nm$) with experiments (Yu et al. Nature 465, 880 (2010)).

4. Conclusions

- $2+1$ Nonlinear Field Theories very useful to visualize and understand basic problems with geometric and topological approaches, from Susy extension to BPS and other analytic solutions
- Direct practical applications like defects in magnetic materials, Quantum Hall and structure formation in the very early universe (if SUSY is still unbroken)
- Further work: fermions, dynamics (time dependence and scattering) and deeper Generalized Integrability.