MODIFIED GRAVITY:NEW PARADIGM for UNIFIED EVOLUTION HISTORY of the UNIVERSE.

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reviews: Nojiri-SDO, hep-th 0601213; arXiv:1011.0544

Contents:

Proposed Universe evolution

Big Bang
String inflationary era
Quantum Gravity - Unknown Era

Inflationary Universe:

Scenarios: most popular
Λ_l - cosmological constant,

or Scalar field,

or ideal fluid $p = -\rho$.

possibility of quintessence and (or) phantom inflation.

Intermediate Universe:

$a(t) = t^{\alpha}$,
radiation / matter dominance.

Late Universe: Dark energy era

Almost de Sitter $a(t) = e^{Ht}$.

Scenarios:
Λ_D - cosmological constant,
scalar fields,
ideal fluid: $p = w\rho$, $w \simeq -1$ (up to 2 percent).
Possibility of phantom $w < -1$ or quintessence: $-1 < w < -\frac{1}{3}$.
Oscillating Universe?

Possible future evolution

ΛCDM most probably continues to be ΛCDM epoch.

If $p = f(\rho)$, where p is negative the following future singularity is possible:

Type I. $t \to t_s$, $a(t) \to \infty$, ρ, $|p| \to \infty$, $a(t) \sim \frac{1}{(t-t_s)}$

Type II. $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$, $|p| \to \infty$

Type III. $t \to t_s$, $a(t) \to a_s$, $\rho \to \infty$, $|p| \to \infty$,

Type IV. Only higher derivatives of H diverge.

Advantages

1. Modified gravity provides the very natural gravitational alternative for dark energy.
2. Modified gravity presents very natural unification of the early-time inflation and late-time acceleration.
3. It may serve as the basis for unified explanation of dark energy and dark matter.
4. Assuming that universe is entering the phantom phase, modified gravity may naturally describe the transition from non-phantom phase to phantom one without necessity to introduce the exotic matter.
5. Modified gravity quite naturally describes the transition from decceleration to acceleration in the universe evolution.
6. The effective dark energy dominance may be assisted by the modification of gravity.
7. Modified gravity is expected to be useful in high energy physics.
8. Despite quite stringent constraints from Solar System tests, there are versions of modified gravity which may be viable theories competing with General Relativity at current epoch.

I. Class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion

Nojiri, SDO, arXiv:0707.1941, 0710.1738; Cognola, Elizalde, Nojiri, SDO, Sebastiani, Zerbini, PRD77:046009,2008

Starting action:

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left[R + f(R) \right] + S_{(m)} \tag{1}$$

Here $f(R)$ is a suitable function, which defines the modified gravitational part of the model. The general equation of motion in $F(R) \equiv R + f(R)$ gravity with matter is given by

$$\frac{1}{2} g_{\mu\nu} F(R) - R_{\mu\nu} F'(R) - g_{\mu\nu} \Box F'(R) + \nabla_\mu \nabla_\nu F'(R) = -\frac{\kappa^2}{2} T_{(m)\mu\nu} \,. \tag{2}$$

We investigate 'viable' modified gravitational models what means, roughly speaking, they have to incorporate the vanishing (or fast decrease) of the cosmological constant in the flat ($R \to 0$) limit.

This simple model reads

$$f(R) = -2\Lambda_{\text{eff}}\,\theta(R - R_0)\,, \tag{3}$$

where $\theta(R - R_0)$ is Heaviside's step distribution.

The other class of modified gravitational models contains a sort of 'switching on' of the cosmological constant as a function of the scalar curvature R.

A simplest version of this kind reads:

$$f(R) = 2\Lambda_{\rm eff}(e^{-bR} - 1)\,. \qquad (4)$$

Here the transition is smooth. The two above models may be combined in a natural way, if one is also interested in the phenomenological description of the inflationary epoch. For example, a two-steps model may be the smooth version of

$$f(R) = -2\Lambda_0\,\theta(R - R_0)\, - 2\Lambda_I\,\theta(R - R_I)\,, \qquad (5)$$

with $R_0 << R_I$, the latter being the inflation scale curvature.

The typical, smooth behavior of $f(R)$ associated with the one- and two-step models is given, in the smooth case, in Figs. 1 and 2, respectively.

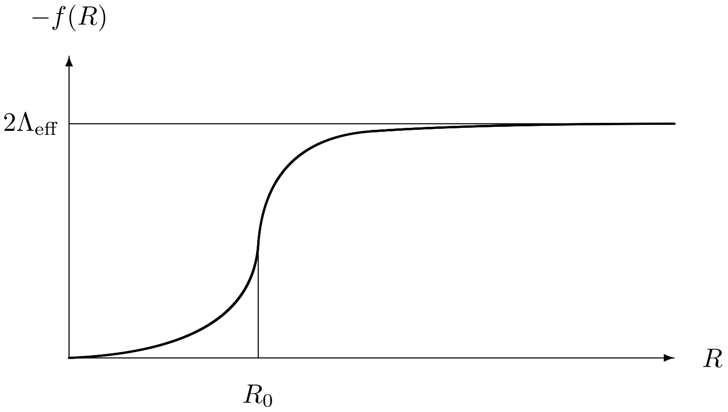

Figure: 1. Typical behavior of $f(R)$ in the one-step model.

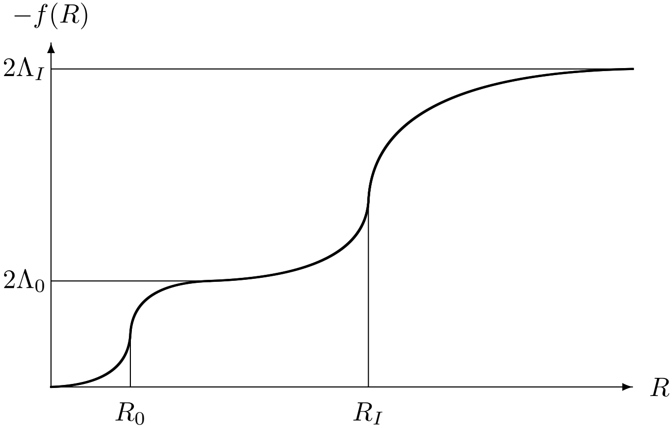

Figure: 2. Typical behavior of $f(R)$ in the two-step model.

Let us recall the two sufficient conditions which often lead to realistic models

$$f(0) = 0\,, \qquad \lim_{R\to R_1} f(R) = -\alpha\,, \tag{6}$$

where α is a suitable curvature scale which represents an effective cosmological constant, being $R_1 >> R_0$, with $R_0 > 0$, the transition point. The condition $f(0) = 0$ ensures the disappearance of the cosmological constant in the limit of flat space-time.

By using these conditions, some models in this class are seen to be able to pass the local tests (with some extra bounds on the theory parameters) and are also capable to explain the observed recent acceleration of the universe expansion, provided that $\alpha = \Lambda_0 = 2H_0^2$, H_0 being the Hubble constant at the epoch of reference.

Thus, one might also reasonably require that

$$f(0) = 0\,, \quad \lim_{R \to R_2} f(R) = -(\alpha + \alpha_I)\,, \tag{7}$$

where $\alpha_I >> \alpha$ is associated with the inflation cosmological constant, Λ_I, and where $R_2 >> R_I >> R_0$, R_I being the corresponding transition large scalar curvature.

Further restrictions, like small corrections to Newton's law and the stability of planet-like gravitational solutions need to be fulfilled too.

The starting point is the trace of the equations of motion, which is trivial in the Einstein theory but gives precious dynamical information in the modified gravitational models. It reads

$$3\nabla^2 f'(R) = R + 2f(R) - Rf'(R) - \kappa^2 T\,. \tag{8}$$

The above trace equation can be interpreted as an equation of motion for the non trivial 'scalaron' $f'(R)$ (since it is indeed associated with the corresponding scalar field in the other frame). For solutions with constant scalar curvature R_*, the scalaron field is constant and one obtains the following vacuum solution:

$$R_* + 2f(R_*) - R_* f'(R_*) = 0\,. \tag{9}$$

Furthermore, we can describe the degree of freedom associated with the scalaron by means of a scalar field χ, defined by $F'(R) = 1 + f'(R) = e^{-\chi}$. If we consider a perturbation around the vacuum solution of constant curvature R_*, given by $R = R_* + \delta R$, where

$$\delta R = -\frac{1 + f'(R_*)}{f''(R_*)}\delta\chi\,, \tag{10}$$

then the equation of motion for the scalaron field is

$$\Box\delta\chi - \frac{1}{3}\left(\frac{1 + f'(R_*)}{f''(R_*)} - R_*\right)\delta\chi = -\frac{\kappa^2}{6(1 + f'(R_*)}T\,. \tag{11}$$

As a result, in connection with the local and with the planetary tests, the following effective mass plays a very crucial role:

$$M^2 \equiv \frac{1}{3}\left(\frac{1 + f'(R_*)}{f''(R_*)} - R_*\right)\,. \tag{12}$$

If $M^2 < 0$, a tachyon appears and this leads to an instability. Even if $M^2 > 0$, when M^2 is small, it is $\delta R \neq 0$ at long ranges, which generates a large correction to Newton's law. As a result, M^2 has to be positive and very large in order to pass both the local and the astronomical tests.

In order to arrive to a stability condition, we can start by noting that the scalaron equation can be rewritten in the form

$$\Box R + \frac{f'''(R)}{f''(R)} \nabla_\rho R \nabla^\rho R + \frac{(1 + f'(R)R}{3f''(R)} - \frac{2(R + f(R))}{3f''(R)} = \frac{\kappa^2}{6f''(R)} T \,. \tag{13}$$

If we now consider a perturbation, δR, of the Einstein gravity solution $R = R_e = -\frac{k^2 T}{2} > 0$, we obtain

$$0 \simeq (-\partial_t^2 + U(R_e))\delta R + C\,, \tag{14}$$

with the effective potential

$$\begin{aligned} U(R_e) \quad \equiv \quad & \left(\frac{F''''(R_e)}{F''(R_e)} - \frac{F'''(R_e)^2}{F''(R_e)^2}\right)\nabla_\rho R_e \nabla^\rho R_e + \frac{R_e}{3} - \\ & -\frac{F'(R_e)F'''(R_e)R_e}{3F''(R_e)^2} - \frac{F'(R_e)}{3F''(R_e)} + \\ & +\frac{2F(R_e)F'''(R_e)}{3F''(R_e)^2} - \frac{F'''(R_e)R_e}{3F''(R_e)^2}\,. \end{aligned} \tag{15}$$

If $U(R_e)$ is positive, then the perturbation δR becomes exponentially large and the whole system becomes unstable. Thus, the matter stability condition is, in this case,

$$U(R_e) < 0\,. \tag{16}$$

We will here present some viable $f(R)$ models. We start with a most simple one

$$f(R) = \alpha(e^{-bR} - 1)\,. \tag{17}$$

Since $f(0) = 0$ and $f(R) \to -\alpha$ for large R, conditions (6) are satisfied. Moreover,

$$f'(R) = -b\alpha e^{-bR}\,, \quad f''(R) = b^2\alpha e^{-bR}\,. \tag{18}$$

With regard to the trivial fixed point $R_* = 0$, this model has the properties

$$1 + f'(0) = 1 - \alpha b\,, \qquad f''(0) = \alpha b^2\,. \tag{19}$$

Thus, the effective mass for $R_* = 0$ is

$$M^2(0) = \frac{1 - \alpha b}{3\alpha b^2}\,, \tag{20}$$

and then Minkowski space time is stable as soon as $\alpha b < 1$. Such condition is equivalent to $1 + f'(0) > 0$.

A simple modification of the above model which incorporates the inflationary era, namely the requirement (7), is

$$f(R) = \alpha(e^{-bR} - 1) - \alpha_I \frac{e^{bR} - 1}{e^{bR} + e^{bR_I}}\,, \tag{21}$$

or, as a two-step model,

$$f(R) = -\alpha \frac{e^{bR} - 1}{e^{bR} + e^{bR_0}} - \alpha_I \frac{e^{bR} - 1}{e^{bR} + e^{bR_I}}\,. \tag{22}$$

Again, $f(0) = 0$ and, at the value $R = R_I$, there is a transition to a higher constant value $-(\alpha + \alpha_I)$ which can be related to inflation.
A possible modification of the previous model is the following:

$$f(R) = -\alpha(e^{-bR} - 1) + cR^N \frac{e^{bR} - 1}{e^{bR} + e^{bR_I}}\,, \tag{23}$$

with $N > 2$ and $c > 0$. In this variant, during the inflationary era at $R > R_I$, $f(R)$, the model acquires also a power dependence on the scalar curvature, which may help to exit from the inflationary stage.

We now investigate the correction to the Newton's law and the matter instability issue. In the solar system domain, on or inside the earth, where $R \gg R_0$, $f(R)$ can be approximated by

$$f(R) \sim -2\Lambda_{\mathrm{eff}} + 2\alpha \mathrm{e}^{-b(R-R_0)} \,. \tag{24}$$

On the other hand, since $R_0 \ll R \ll R_I$, it could be also approximated by

$$f(R) \sim -2\Lambda_0 + 2\alpha \mathrm{e}^{-b_0(R-R_0)} \,, \tag{25}$$

which has the same expression, after having identified $\Lambda_0 = \Lambda_{\mathrm{eff}}$ and $b_0 = b$. Then, we may check the case of (24) only.

We find that the effective mass has the following form

$$M^2 \sim \frac{e^{b(R-R_0)}}{4\alpha b^2}\,, \tag{26}$$

which could be very large again, and the correction to Newton's law can be made negligible. We also find that $U(R_b)$ in (15) has the form

$$U(R_e) = -\frac{1}{2\alpha b}\left(2\Lambda + \frac{1}{b}\right) e^{-b(R_e-R_0)}\,, \tag{27}$$

which could be negative, what would suppress any instability.
The first modified gravity to pass these constraints is introduced in *Nojiri-SDO, hep-th/0307288* The perturbations story?

Elizalde, Nojiri, SDO, Sebastiani and Zerbini, arXiv:1012.2280.

Viable conditions in $F(R)$-gravity
In order to avoid anti-gravity effects, it is required that $F'(R) > 0$, namely, the positivity of the effective gravitational coupling.

Existence of a matter era and stability of cosmological perturbations.
On the critical points, $\dot{F}'(R) = 0$ and

$$\rho_{\rm eff} = \frac{1}{F'(R)} \left\{ \rho + \frac{1}{2\kappa^2} \left[(F'(R)R - F(R)) \right] \right\} , \tag{28}$$

$$p_{\rm eff} = \frac{1}{F'(R)} \left\{ p + \frac{1}{2\kappa^2} \left[-(F'(R)R - F(R)) \right] \right\} . \tag{29}$$

During matter era, $p_{\rm eff} \simeq 0$ and $\rho_{\rm eff} \simeq \rho/F'(R)$. As a consequence,

$$\frac{RF'(R)}{F(R)} = 1 , \qquad \frac{d}{dR} \left(\frac{RF'(R)}{F(R)} \right) = 0 . \tag{30}$$

This leads to

$$\frac{F''(R)}{F'(R)} = 0 \Rightarrow F''(R) \simeq 0 . \tag{31}$$

Since if $F''(R) < 0$ the theory is strongly unstable, $F''(R) \simeq 0^+$.

Local tests

The typical value of the curvature in the Solar System far from sources is $R = R^*$, where $R^* \simeq 10^{-61}\text{eV}^2$. If a Schwarzshild-de Sitter solution exists, it will be stable provided

$$\frac{F'(R^*)}{R^* F''(R^*)} > 1\,. \tag{32}$$

The stability of the solution is necessary in order to find the post-Newtonian parameters as in GR.

Exponential gravity

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-R/R_0}\right)\,. \tag{33}$$

Here, $\Lambda \simeq 10^{-66}\text{eV}^2$ is the cosmological constant and $R_0 \simeq \Lambda$ a curvature parameter. In flat space one has $F(0) = 0$ and recovers the Minkowski solution. The model satisfied all viable conditions and it is consistent with the results of ΛCDM Model.

Inflation

A quite natural possibility is

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right) - \Lambda_i\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right) + \gamma R^\alpha\,. \tag{34}$$

This is the function discussed above with another one-step function reproducing the cosmological function during inflation AND a power term necessary to obtain the exit from inflation ($\gamma \simeq 1/(4\Lambda_i)^{\alpha-1}$).
By taking into account all the viability conditions, the simplest choice of parameters to introduce in the function of Eq. (34) is

$$n = 4\,, \qquad \alpha = \frac{5}{2}\,, \tag{35}$$

while the curvature R_i is set as

$$R_i = 2\Lambda_i\,. \tag{36}$$

In this way, since $n > \alpha > 1$, we avoid the contribute of inflation and undesirable instability effects in the small-curvature regime. No anti-gravity effects. The unstable de Sitter solution describing inflation is

$$R_{\mathrm{dS}} = \frac{2\Lambda_i}{3-\alpha} \equiv 4\Lambda_i\,. \tag{37}$$

Dark energy evolution

We will now be interested in the cosmological evolution of the dark energy density $\rho_{\rm DE} = \rho_{\rm eff} - \rho/F'(R)$ in the case of the two-step model of Eq. (34), near the late-time acceleration era.

We use the variable

$$y_{\rm H} \equiv \frac{\rho_{\rm DE}}{\rho_m^{(0)}} = \frac{H^2}{\tilde{m}^2} - a^{-3} - \chi a^{-4} \,. \tag{38}$$

Here, $\rho_m^{(0)}$ is the energy density of matter at present time, $\tilde{m}^2$ is the mass scale

$$\tilde{m}^2 \equiv \frac{\kappa^2 \rho_m^{(0)}}{3} \simeq 1.5 \times 10^{-67} \text{eV}^2 \,, \tag{39}$$

and χ is defined as

$$\chi \equiv \frac{\rho_r^{(0)}}{\rho_m^{(0)}} \simeq 3.1 \times 10^{-4} \,, \tag{40}$$

where $\rho_r^{(0)}$ is the energy density of current radiation.

The EoS-parameter $\omega_{\rm DE}$ for dark energy is

$$\omega_{\rm DE} = -1 - \frac{1}{3} \frac{1}{y_{\rm H}} \frac{d y_{\rm H}}{d(\ln a)} \,. \tag{41}$$

By combining the Equations of motion of modified gravity theories, one gets

$$\frac{d^2 y_{\rm H}}{d(\ln a)^2} + J_1 \frac{d y_{\rm H}}{d(\ln a)} + J_2 y_{\rm H} + J_3 = 0\,, \tag{42}$$

where

$$J_1 = 4 + \frac{1}{y_{\rm H} + a^{-3} + \chi a^{-4}} \frac{1 - F'(R)}{6\tilde{m}^2 F''(R)}\,,$$

$$J_2 = \frac{1}{y_{\rm H} + a^{-3} + \chi a^{-4}} \frac{2 - F'(R)}{3\tilde{m}^2 F''(R)}\,,$$

$$J_3 = -3a^{-3} - \frac{(1 - F'(R))(a^{-3} + 2\chi a^{-4}) + (R - F(R))/(3\tilde{m}^2)}{y_{\rm H} + a^{-3} + \chi a^{-4}} \frac{1}{6\tilde{m}^2 F''(R)}\,,$$

and thus, we have

$$R = 3\tilde{m}^2 \left(\frac{d y_{\rm H}}{d \ln a} + 4 y_{\rm H} + a^{-3} \right)\,. \tag{43}$$

We will study our model,

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right) - \Lambda_i\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right) + \gamma R^\alpha\,. \tag{44}$$

The parameters of Eq. (44) are chosen as follows:

$$\begin{gathered}
\Lambda = (7.93)\tilde{m}^2\,, \\
\Lambda_i = 10^{100}\Lambda\,, \\
R_i = 2\Lambda_i\,, \qquad n = 4\,, \\
\alpha = \frac{5}{2}\,, \qquad \gamma = \frac{1}{(4\Lambda_i)^{\alpha-1}}\,, \\
R_0 = 0.6\Lambda\,, \quad 0.8\Lambda\,, \quad \Lambda\,.
\end{gathered} \tag{45}$$

Eq. (42) can be solved in a numerical way, in the range of $R_0 \ll R \ll R_i$ (matter era/current acceleration). y_{H} is then found as a function of the red shift z,

$$z = \frac{1}{a} - 1 \,. \tag{46}$$

In solving Eq. (42) numerically we have taken the following initial conditions at $z = z_i$

$$\left.\frac{dy_{\mathrm{H}}}{d(z)}\right|_{z_i} = 0 \,,$$

$$\left. y_{\mathrm{H}}\right|_{z_i} = \frac{\Lambda}{3\tilde{m}^2} \,, \tag{47}$$

which correspond to the ones of the ΛCDM model. This choice obeys to the fact that in the high red shift regime the exponential model is very close to the ΛCDM Model. The values of z_i have been chosen so that $RF''(z = z_i) \sim 10^{-7}$, assuming $R = 3\tilde{m}^2(z+1)^3 + 4\Lambda$. We have $z_i = 1.5$, 2.2, 2.5 for $R_0 = 0.6\Lambda$, 0.8Λ, Λ, respectively. In setting the parameters, we have used the last results of the WMAP, BAO and SN surveys (*Komatsu et al. [WMAP Collaboration], arXiv:0803.0547*).

We can also extrapolate the behavior of the density parameter of dark energy, $\Omega_{\rm DE}$,

$$\Omega_{\rm DE} \equiv \frac{\rho_{\rm DE}}{\rho_{\rm eff}} = \frac{y_{\rm H}}{y_{\rm H} + (z+1)^3 + \chi\,(z+1)^4}\,. \tag{48}$$

The data we have found are in accordance with the last and very accurate observations of our present universe, where:

$$\omega_{\rm DE} = -0.972^{+0.061}_{-0.060}\,,$$

$$\Omega_{\rm DE} = 0.721 \pm 0.015\,. \tag{49}$$

At the redshift $z = 0$ we obtain $\omega_{DE} = -0.994, -0.975, -0.950$ and $\Omega_{DE} = 0.726, 0.728, 0.732$ in the cases of $R_0 = 0.6\Lambda, 0.8\Lambda, \Lambda$, respectively.

The de Sitter solution is a final attractor of our system and describes an eternal accelerating expansion.

II. Cosmological reconstruction of modified F(R) gravity

Nojiri-SDO-Saez-Gomez, arXiv:0908.1269, Nojiri-SDO, hep-th/0611071, hep-th/0608008

Let us demonstrate that any FRW cosmology may be realized in specific $F(R)$ gravity.

The starting action of the $F(R)$ gravity is given by

$$S = \int d^4 x \sqrt{-g} \left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\rm matter} \right) \ . \qquad (50)$$

The field equation corresponding to the first FRW equation is:

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + \dot{H}\right) F'(R) - 18(\left(4H^2\dot{H} + H\ddot{H}\right) F''(R) + \kappa^2 \rho \ . \qquad (51)$$

We now rewrite Eq.(51) by using a new variable (which is often called e-folding) instead of the cosmological time t, $N = \ln \frac{a}{a_0}$. The variable N is related with the redshift z by $\mathrm{e}^{-N} = \frac{a_0}{a} = 1 + z$.
Since $\frac{d}{dt} = H \frac{d}{dN}$ and therefore $\frac{d^2}{dt^2} = H^2 \frac{d^2}{dN^2} + H \frac{dH}{dN} \frac{d}{dN}$,
one can rewrite (51) by

$$\begin{aligned} 0 &= -\frac{F(R)}{2} + 3\left(H^2 + HH'\right) F'(R) \\ &\quad -18(\left(4H^3H' + H^2\left(H'\right)^2 + H^3H''\right) F''(R) + \kappa^2 \rho \, . \end{aligned} \quad (52)$$

Here $H' \equiv dH/dN$ and $H'' \equiv d^2H/dN^2$.

If the matter energy density ρ is given by a sum of the fluid densities with constant EoS parameter w_i,

$$\rho = \sum_i \rho_{i0} a^{-3(1+w_i)} = \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N} \, . \tag{53}$$

Let the Hubble rate is given in terms of N via the function $g(N)$ as

$$H = g(N) = g\left(-\ln\left(1+z\right)\right) \, . \tag{54}$$

Then scalar curvature takes the form: $R = 6g'(N)g(N) + 12g(N)^2$, which could be solved with respect to N as $N = N(R)$.

Then by using (53) and (54), one can rewrite (52) as

$$\begin{aligned} 0 &= -18\Big(4g\left(N\left(R\right)\right)^3 g'\left(N\left(R\right)\right) \\ &\quad + g\left(N\left(R\right)\right)^2 g'\left(N\left(R\right)\right)^2 + g\left(N\left(R\right)\right)^3 g''\left(N\left(R\right)\right)\Big) \frac{d^2F(R)}{dR^2} \\ &\quad + 3\Big(g\left(N\left(R\right)\right)^2 + g'\left(N\left(R\right)\right) g\left(N\left(R\right)\right)\Big) \frac{dF(R)}{dR} - \frac{F(R)}{2} \\ &\quad + \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N(R)} \,, \end{aligned} \tag{55}$$

which constitutes a differential equation for $F(R)$, where the variable is scalar curvature R.

Instead of g, if we use $G(N) \equiv g\left(N\right)^2 = H^2$, the expression (55) could be a little bit simplified:

$$\begin{aligned} 0 \; = \; & -9G\left(N\left(R\right)\right)\left(4G'\left(N\left(R\right)\right) + G''\left(N\left(R\right)\right)\right)\frac{d^2F(R)}{dR^2} \\ & + \left(3G\left(N\left(R\right)\right) + \frac{3}{2}G'\left(N\left(R\right)\right)\right)\frac{dF(R)}{dR} \\ & - \frac{F(R)}{2} + \sum_i \rho_{i0} a_0^{-3(1+w_i)} \mathrm{e}^{-3(1+w_i)N(R)} \; . \end{aligned} \tag{56}$$

Note that the scalar curvature is given by $R = 3G'(N) + 12G(N)$.
Hence, when we find $F(R)$ satisfying the differential equation (55) or (56), such $F(R)$ theory admits the solution (54). Hence, such $F(R)$ gravity realizes above cosmological solution.

As an example, we reconstruct the $F(R)$ gravity which reproduces the ΛCDM-era but without real matter.
In the Einstein gravity, the FRW equation for the ΛCDM cosmology is given by

$$\frac{3}{\kappa^2}H^2 = \frac{3}{\kappa^2}H_0^2 + \rho_0 a^{-3} = \frac{3}{\kappa^2}H_0^2 + \rho_0 a_0^{-3}\mathrm{e}^{-3N} \ . \quad (57)$$

Here H_0 and ρ_0 are constants.
The (effective) cosmological constant Λ in the present universe is given by $\Lambda = 12H_0^2$. Then one gets

$$G(N) = H_0^2 + \frac{\kappa^2}{3}\rho_0 a_0^{-3}\mathrm{e}^{-3N} \ , \quad (58)$$

and $R = 3G'(N) + 12G(N) = 12H_0^2 + \kappa^2\rho_0 a_0^{-3}\mathrm{e}^{-3N}$, which can be solved with respect to N as follows,

$$N = -\frac{1}{3}\ln\left(\frac{\left(R - 12H_0^2\right)}{\kappa^2\rho_0 a_0^{-3}}\right) \ . \quad (59)$$

Eq.(56) takes the following form:

$$0=3\left(R-9H_0^2\right)\left(R-12H_0^2\right)\frac{d^2F(R)}{d^2R}-\left(\frac{1}{2}R-9H_0^2\right)\frac{dF(R)}{dR}-\frac{1}{2}F(R)\,. \tag{60}$$

By changing the variable from R to x by $x=\frac{R}{3H_0^2}-3$, Eq.(60) reduces to the hypergeometric differential equation:

$$0=x(1-x)\frac{d^2F}{dx^2}+\left(\gamma-\left(\alpha+\beta+1\right)x\right)\frac{dF}{dx}-\alpha\beta F\ . \tag{61}$$

Here

$$\gamma=-\frac{1}{2}\ ,\alpha+\beta=-\frac{1}{6}\ ,\quad \alpha\beta=-\frac{1}{6}\ , \tag{62}$$

Solution of (61) is given by Gauss' hypergeometric function $F(\alpha,\beta,\gamma;x)$:

$$F(x)=AF(\alpha,\beta,\gamma;x)+Bx^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x)\ . \tag{63}$$

Here A and B are constant.

Thus, we demonstrated that modified $F(R)$ gravity may describe the ΛCDM epoch without the need to introduce the effective cosmological constant.

III. The formulation of modified gravity as General Relativity with generalized fluid and finite time future singularities

Bamba-Nojiri-SDO,JCAP 0810:045,2008; Nojiri-SDO, PRD 78,046006, 2008

Let us start from the general modified gravity with the action:

$$S = \int d^4x\sqrt{-g} \left\{ \frac{1}{2\kappa^2}\left(R + f\left(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \Box R, \Box^{-1}R, \cdots\right)\right) + L_{\rm m} \right\}, \tag{64}$$

where all combinations of local and non-local terms are possible, $L_{\rm m}$ is matter Lagrangian and the function $f(R,\cdots)$ may also contain gravitational partner (say, dilatons, axion, etc. in string-inspired gravity). In all cases for theory (64), it is possible to write the gravitational field equations in the form of standard FRW equations with effective energy-density $\rho_{\rm eff}$ and pressure $p_{\rm eff}$ produced by the extra gravitational terms $F(R,\cdots)$ and $L_{\rm m}$.

For instance, when $f = f(R)$, one gets

$$\begin{aligned}\rho_{\rm eff} &= \frac{1}{\kappa^2}\left(-\frac{1}{2}f(R) + 3\left(H^2 + \dot{H}\right)f'(R)\right. \\ &\left. -18\left(4H^2\dot{H} + H\ddot{H}\right)f''(R)\right) \\ &+\rho_{\rm matter}\,, \end{aligned} \tag{65}$$

$$\begin{aligned}p_{\rm eff} &= \frac{1}{\kappa^2}\left(\frac{1}{2}f(R) - \left(3H^2 + \dot{H}\right)f'(R)\right. \\ &+6\left(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \dddot{H}\right)f''(R) \\ &\left. +36\left(4H\dot{H} + \ddot{H}\right)^2 f'''(R)\right) \\ &+p_{\rm matter}\,. \end{aligned} \tag{66}$$

In case of Gauss-Bonnet modified gravity:

$$\begin{aligned}
\rho_{\rm eff} &= \frac{1}{2\kappa^2}\left[\mathcal{G} f_{\mathcal{G}}'(\mathcal{G}) - f_{\mathcal{G}}(\mathcal{G}) - 24^2 H^4 \left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right) f_{\mathcal{G}}''\right] \\
&\quad + \rho_{\rm matter} \ , \\
p_{\rm eff} &= \frac{1}{2\kappa^2}\left[f_{\mathcal{G}}(\mathcal{G}) + 24^2 H^2 \left(3H^4 + 20H^2\dot{H}^2 + 6\dot{H}^3 + 4H^3\ddot{H} + H^2\dddot{H}\right)\right. \\
&\quad \left. f_{\mathcal{G}}''(\mathcal{G}) - 24^3 H^5 \left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right)^2 f_{\mathcal{G}}'''(\mathcal{G})\right] + p_{\rm matter} \ . \quad (67)
\end{aligned}$$

In the same way one can get the effective gravitational pressure and energy density so that the equations of motion for arbitrary modified gravity can be rewritten in the universal FRW form typical for General Relativity:

$$\frac{3}{\kappa^2}H^2 = \rho_{\rm eff}\ , \quad p_{\rm eff} = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right)\ . \quad (68)$$

There are just standard FRW gravitational equations.

- Type I ("Big Rip") : For $t \to t_s$, $a \to \infty$, $\rho_{\rm eff} \to \infty$ and $|p_{\rm eff}| \to \infty$.
  This also includes the case of $\rho_{\rm eff}$, $p_{\rm eff}$ being finite at t_s.

- Type II ("sudden") : For $t \to t_s$, $a \to a_s$, $\rho_{\rm eff} \to \rho_s$ and $|p_{\rm eff}| \to \infty$

- Type III : For $t \to t_s$, $a \to a_s$, $\rho_{\rm eff} \to \infty$ and $|p_{\rm eff}| \to \infty$

- Type IV : For $t \to t_s$, $a \to a_s$, $\rho_{\rm eff} \to 0$, $|p_{\rm eff}| \to 0$ and higher derivatives of H diverge.
  This also includes the case in which $p_{\rm eff}$ ($\rho_{\rm eff}$) or both of $p_{\rm eff}$ and $\rho_{\rm eff}$ tend to some finite values, while higher derivatives of H diverge.

Curing singularity with R^2-term: *M.Abdalla, Nojiri, SDO, CQG22,L35(2005)*

IV. Modified non-local-F(R) gravity as the key for the inflation and dark energy

Nojiri-SDO, PLB 569, 821, 2008

The starting action of the non-local gravity is given by

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R \left(1 + f(\Box^{-1} R) \right) + \mathcal{L}_{\rm matter} \right\} . \tag{69}$$

The above action can be rewritten by introducing two scalar fields ϕ and ξ:

$$\begin{aligned} S &= \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left\{ R \left(1 + f(\phi) \right) + \xi \left(\Box\phi - R \right) \right\} + \mathcal{L}_{\rm matter} \right] \\ &= \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left\{ R \left(1 + f(\phi) \right) - \partial_\mu \xi \partial^\mu \phi - \xi R \right\} + \mathcal{L}_{\rm matter} \right] \end{aligned} \tag{70}$$

Varying (70) with respect to the metric tensor $g_{\mu\nu}$ gives

$$\begin{aligned} 0 \quad = \quad & \frac{1}{2} g_{\mu\nu} \left\{ R \left(1 + f(\phi) - \xi \right) - \partial_\rho \xi \partial^\rho \phi \right\} - R_{\mu\nu} \left(1 + f(\phi) - \xi \right) \\ & + \frac{1}{2} \left(\partial_\mu \xi \partial_\nu \phi + \partial_\mu \phi \partial_\nu \xi \right) \\ & - \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) \left(f(\phi) - \xi \right) + \kappa^2 T_{\mu\nu} \ . \end{aligned} \tag{71}$$

On the other hand, the variation with respect to ϕ gives

$$0 = \Box \xi + f'(\phi) R \ . \tag{72}$$

Now we assume the FRW metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 \ , \tag{73}$$

and the scalar fields ϕ and ξ only depend on time. Then Eq.(71) has the following form:

$$\begin{aligned} 0 &= -3H^2 \left(1 + f(\phi) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\phi} - 3H\left(f'(\phi)\dot{\phi} - \dot{\xi}\right) + \kappa^2 \rho \ , \quad (74) \\ 0 &= \left(2\dot{H} + 3H^2\right)\left(1 + f(\phi) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\phi} + \\ &\quad \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)\left(f(\phi) - \xi\right) + \kappa^2 p \ . \quad (75) \end{aligned}$$

On the other hand, scalar equations are:

$$0 = \ddot{\phi} + 3H\dot{\phi} + 6\dot{H} + 12H^2 \ , \quad (76)$$

$$0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right) f'(\phi) \ . \quad (77)$$

We now assume deSitter solution $H = H_0$, then Eq.(76) can be solved as

$$\phi = -4H_0 - \phi_0 \mathrm{e}^{-3H_0 t} + \phi_1 \ , \quad (78)$$

with constants of integration, ϕ_0 and ϕ_1. For simplicity, we only consider the case that $\phi_0 = \phi_1 = 0$. We also assume $f(\phi)$ is given by

$$f(\phi) = f_0 \mathrm{e}^{b\phi} = f_0 \mathrm{e}^{-4bH_0\phi} \ . \quad (79)$$

Then Eq.(77) can be solved as follows,

$$\xi = -\frac{3f_0}{3-4b}\mathrm{e}^{-4bH_0 t} + \frac{\xi_0}{3H_0}\mathrm{e}^{-3H_0 t} - \xi_1 \ . \quad (80)$$

Here ξ_0 and ξ_1 are constants. For the deSitter space a behaves as $a = a_0 \mathrm{e}^{H_0 t}$. Then for the matter with constant equation of state w, we find

$$\rho = \rho_0 \mathrm{e}^{-3(w+1)H_0 t} \ . \quad (81)$$

Then by substiruting (78), (80), and (81) into (74), we obtain

$$0 = -3H_0^2\left(1+\xi_1\right) + 6H_0^2 f_0\left(2b-1\right)\mathrm{e}^{-4H_0 bt} + \kappa^2\rho_0\mathrm{e}^{-3(w+1)H_0 t}\ . \quad (82)$$

When $\rho_0 = 0$, if we choose

$$b = \frac{1}{2}\ , \quad \xi_1 = -1\ , \quad (83)$$

deSitter space can be a solution. Even if $\rho \neq 0$, if we choose

$$b = \frac{3}{4}(1+w)\ , \quad f_0 = \frac{\kappa^2\rho_0}{3H_0^2\left(1+3w\right)}\ , \quad \xi_1 = -1\ , \quad (84)$$

there is a deSitter solution.

In the presence of matter with $w \neq 0$, we may have a deSitter solution $H = H_0$ even if $f(\phi)$ given by

$$f(\phi) = f_0\mathrm{e}^{\phi/2} + f_1\mathrm{e}^{3(w+1)\phi/4}\ . \quad (85)$$

Then the following solution exists:

$$\begin{aligned}
\phi &= -4H_0 t \ , \\
\xi &= 1 + 3f_0 \mathrm{e}^{-2H_0 t} + \frac{f_1}{w}\mathrm{e}^{-3(w+1)H_0 t} \ , \\
\rho &= -\frac{3(3w+1)H_0^2 f_1}{\kappa^2}\mathrm{e}^{-3(1+w)H_0 t} \ .
\end{aligned} \tag{86}$$

Note that H_0 in (78) can be arbitrary and can be determined by an initial condition. Since H_0 can be small or large, the theory with $b = 1/2$ could describe the early-time inflation or current cosmic acceleration.

Motivated by this, we may propose the following model:

$$f(\phi) = \left\{ \begin{array}{ll} f_0 \mathrm{e}^{\phi/2} & 0 > \phi > \phi_1 \\ f_0 \mathrm{e}^{\phi_1/2} & \phi_1 > \phi > \phi_2 \\ f_0 \mathrm{e}^{(\phi - \phi_2 + \phi_1)/2} & \phi < \phi_2 \end{array} \right. . \tag{87}$$

Here ϕ_1 and ϕ_2 are constants. We also assume that matter could be neglected when $0 > \phi > \phi_1$ or $\phi < \phi_2$. Since the above function $f(\phi)$ is not smooth around $\phi = \phi_1$ and ϕ_2, one may replace the above $f(\phi)$ with a more smooth function. When $0 > \phi > \phi_1$ or $\phi < \phi_2$, the universe is described by the deSitter solution although corresponding H_0 might be different.

When $\phi_1 > \phi > \phi_2$, since $f(\phi)$ is a constant, the universe is described by the Einstein gravity, where effective gravitational constant $\kappa_{\rm eff}$ is given by

$$\frac{1}{\kappa_{\rm eff}^2} = \frac{1}{\kappa^2}\left(1 + f_0 e^{\phi_1/2}\right) \ . \tag{88}$$

Then due to the matter contribution there could occur matter dominated phase. In this phase, the Hubble rate H behaves as $H = \frac{2}{3(t_0+t)}$ with a constant t_0 and the scalar curvature is given by $R = \frac{4}{3(t_0+t)^2}$. Now we assume that the universe started at $t = 0$ with a rather big but constant curvature $R = R_I = 12H_I^2$ with a constant H_I, that is, the universe is in deSitter phase. Then in the model (87), by following (78), ϕ behaves as $\phi = -4H_I t$. Subsequently, at $t = t_1 \equiv -\phi_1/4H_I$, we have $\phi = \phi_1$ and the universe enters into the matter dominated phase. If the curvature is continuous at $t = t_1$, t_0 can be found by solving

$$R = \frac{4}{3\left(t_0 + t_1\right)^2} = 12H_I^2 \ . \tag{89}$$

If ϕ and $\dot{\phi}$ are also continuous, when $\phi_1 > \phi > \phi_2$, ϕ is given by solving (76) as

$$\phi = -\frac{4}{3}\ln\left(\frac{t}{t_1}\right) - \tilde{\phi}\,(t - t_1) + \phi_1 \ , \quad \tilde{\phi} \equiv -4H_I\,(t_0 + t_1)^2 + \frac{4}{3}\,(t_0 + t_1) \ . \tag{90}$$

When $\phi = \phi_2$, the deSitter phase, which corresponds to the accelerating expansion of the present universe, could have started. The solution corresponds to deSitter space (with some shifts of parameters) and $H_0 = H_L$ could be given by solving

$$12H_L^2 = \frac{4}{3\,(t_0 + t_2)^2} \ . \tag{91}$$

if the curvature is continuous at $\phi = \phi_2$. In (91), t_2 is defined by $\phi(t_2) = \phi_2$. Thus, we got the cosmological FRW model with inflation, radiation/matter dominated phase, and current accelerating expansion.

V. Late-time cosmology in modified Gauss-Bonnet gravity $f(G)$ gravity

Nojiri-SDO, Phys.Lett.B631,1,2006

Our example is modified Gauss-Bonnet gravity.

Let us start from the action :

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + f(G) + \mathcal{L}_m\right) . \qquad (92)$$

Here $\mathcal{L}_m$ is the matter Lagrangian density and G is the GB invariant:
$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}$.

By variation over $g_{\mu\nu}$ one gets:

$$
\begin{aligned}
&0 = \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu} + \frac{1}{2}g^{\mu\nu}f(G) \\
&-2f'(G)RR^{\mu\nu} + 4f'(G)R^{\mu}{}_{\rho}R^{\nu\rho} - 2f'(G)R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} \\
&-4f'(G)R^{\mu\rho\sigma\nu}R_{\rho\sigma} + 2\left(\nabla^{\mu}\nabla^{\nu}f'(G)\right)R \\
&-2g^{\mu\nu}\left(\nabla^2 f'(G)\right)R - 4\left(\nabla_{\rho}\nabla^{\mu}f'(G)\right)R^{\nu\rho} \\
&-4\left(\nabla_{\rho}\nabla^{\nu}f'(G)\right)R^{\mu\rho} + 4\left(\nabla^2 f'(G)\right)R^{\mu\nu} \\
&+4g^{\mu\nu}\left(\nabla_{\rho}\nabla_{\sigma}f'(G)\right)R^{\rho\sigma} \\
&-4\left(\nabla_{\rho}\nabla_{\sigma}f'(G)\right)R^{\mu\rho\nu\sigma}.
\end{aligned} \tag{93}
$$

The equation corresponding to the first FRW equation has the following form:

$$0 = -\frac{3}{\kappa^2} H^2 + G f'(G) - f(G) - 24 \dot{G} f''(G) H^3 + \rho_m \ , \qquad (94)$$

where ρ_m is the matter energy density. When $\rho_m = 0$, Eq. (94) has a deSitter universe solution where H, and therefore G, are constant. For $H = H_0$, with constant H_0, Eq. (94) turns into

$$0 = -\frac{3}{\kappa^2} H_0^2 + 24 H_0^4 f'\left(24 H_0^4\right) - f\left(24 H_0^4\right) \ . \qquad (95)$$

For a large number of choices of the function $f(G)$, Eq. (95) has a non-trivial ($H_0 \neq 0$) real solution for H_0 (deSitter universe).

We now consider the case $\rho_m \neq 0$. Assuming that the EoS parameter $w \equiv p_m/\rho_m$ for matter (p_m is the pressure of matter) is a constant then, by using the conservation of energy: $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, we find $\rho = \rho_0 a^{-3(1+w)}$. The function $f(G)$ is chosen as

$$f(G) = f_0 |G|^\beta \ , \tag{96}$$

with constant f_0 and β. If $\beta < 1/2$, $f(G)$ term becomes dominant compared with the Einstein term when the curvature is small. If we neglect the contribution from the Einstein term in (94), the following solution may be found

$$\begin{aligned} h_0 &= \frac{4\beta}{3(1+w)} \ , \\ a_0 &= \left[-\frac{f_0(\beta-1)}{(h_0-1)\rho_0}\left\{24\left|h_0^3(-1+h_0)\right|\right\}^\beta (h_0-1+4\beta)\right]^{-\frac{1}{3(1+w)}} \end{aligned} \tag{97}$$

Then the effective EoS parameter $w_{\rm eff}$ is less than -1 if $\beta < 0$, and for $w > -1$ is

$$w_{\rm eff} = -1 + \frac{2}{3h_0} = -1 + \frac{1+w}{2\beta} \ , \tag{98}$$

which is again less than -1 for $\beta < 0$. Thus, if $\beta < 0$, we obtain an effective phantom with negative h_0 even in the case when $w > -1$. Near this Big Rip , however, the curvature becomes dominant and then the Einstein term dominates, so that the $f(G)$ term can be neglected. Therefore, the universe behaves as $a = a_0 t^{2/3(w+1)}$ and as a consequence the Big Rip does not eventually occur. The phantom era is transient. Unification is againe possible.

VI. Modified $F(R)$ Hořava-Lifshitz gravity

Chaichian, Nojiri,SDO, Oksanen, Tureanu, arXiv:1001.4102

$$S_{F(R)} = \int d^4x\sqrt{-g}F(R). \tag{99}$$

Here F is a function of the scalar curvature R. By using the ADM decomposition.

$$ds^2 = -N^2dt^2 + g^{(3)}_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \quad i = 1,2,3\,. \tag{100}$$

Here N is called the lapse variable and N^i's are the shift variables. Then the scalar curvature R has the following form:

$$R = K^{ij}K_{ij} - K^2 + R^{(3)} + 2\nabla_\mu\left(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu\right) \tag{101}$$

and $\sqrt{-g} = \sqrt{g^{(3)}}N$. Here $R^{(3)}$ is the three-dimensional scalar curvature defined by the metric $g^{(3)}_{ij}$ and K_{ij} is the extrinsic curvature defined by

$$K_{ij} = \frac{1}{2N}\left(\dot{g}^{(3)}_{ij} - \nabla^{(3)}_i N_j - \nabla^{(3)}_j N_i\right), \quad K = K^i{}_i\,. \tag{102}$$

n^μ is a unit vector perpendicular to the three-dimensional hypersurface Σ_t defined by $t =$ constant and $\nabla^{(3)}_i$ expresses the covariant derivative on the hypersurface Σ_t.

Degenerated theory:

$$S_{F_{\mathrm{HL}}(R)} = \int d^4x\sqrt{g^{(3)}}NF(R_{\mathrm{HL}}), \quad R_{\mathrm{HL}} \equiv K^{ij}K_{ij} - \lambda K^2 - E^{ij}G_{ijkl}E^{kl}. \tag{103}$$

Here λ is a real constant in the "generalized De Witt metric" or "super-metric" ("metric of the space of metric"),

$$G^{ijkl} = \frac{1}{2}\left(g^{(3)ik}g^{(3)jl} + g^{(3)il}g^{(3)jk}\right) - \lambda g^{(3)ij}g^{(3)kl}, \tag{104}$$

defined on the three-dimensional hypersurface Σ_t, E^{ij} can be defined by the so called *detailed balance condition* by using an action $W[g^{(3)}_{kl}]$ on the hypersurface Σ_t

$$\sqrt{g^{(3)}}E^{ij} = \frac{\delta W[g^{(3)}_{kl}]}{\delta g_{ij}}, \tag{105}$$

and the inverse of G^{ijkl} is written as

$$G_{ijkl} = \frac{1}{2}\left(g^{(3)}_{ik}g^{(3)}_{jl} + g^{(3)}_{il}g^{(3)}_{jk}\right) - \tilde{\lambda}g^{(3)}_{ij}g^{(3)}_{kl}, \quad \tilde{\lambda} = \frac{\lambda}{3\lambda - 1}. \tag{106}$$

In the ultraviolet (high energy) region, the time coordinate and the spatial coordinates are assumed to behave as

$$\mathbf{x} \to b\mathbf{x} \ , \quad t \to b^z t \ , \quad z = 2, 3, \cdots \ , \tag{107}$$

under the scale transformation.

$W[g^{(3)}_{kl}]$ is explicitly given for the case $z = 2$,

$$W = \frac{1}{\kappa_W^2} \int d^3\mathbf{x} \sqrt{g^{(3)}} (R - 2\lambda_w). \tag{108}$$

In the Hořava-Lifshitz-like $F(R)$-gravity, we assume that N can only depend on the time coordinate t, which is called the *projectability condition*. The reason is that the Hořava-Lifshitz gravity does not have the full diffeomorphism invariance, but is invariant only under "foliation-preserving" diffeomorphisms, i.e. under the transformations

$$\delta x^i = \zeta^i(t, \mathbf{x}) \, , \quad \delta t = f(t) \, . \tag{109}$$

The FRW universe with a flat spatial part,

$$ds^2 = -N^2 dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 \, . \tag{110}$$

Then

$$R = \frac{12H^2}{N^2} + \frac{6}{N}\frac{d}{dt}\left(\frac{H}{N}\right) = -\frac{6H^2}{N} + \frac{6}{a^3 N}\frac{d}{dt}\left(\frac{Ha^3}{N}\right) \, ,$$
$$R_{\mathrm{HL}} = \frac{(3 - 9\lambda)\, H^2}{N^2} . \tag{111}$$

Note

$$\begin{aligned}\int d^4x\sqrt{-g}R &= \int d^4x\ a^3N\left\{-\frac{6H^2}{N}+\frac{6}{a^3N}\frac{d}{d_t}\left(\frac{Ha^3}{N}\right)\right\}= \\ &= \int d^4x\left\{-6H^2a^3+6\frac{d}{d_t}\left(\frac{Ha^3}{N}\right)\right\}. \qquad (112)\end{aligned}$$

Now we propose a new and very general Hořava-Lifshitz-like $F(R)$-gravity by

$$S_{F(\tilde{R})} = \int d^4x \sqrt{g^{(3)}} N F(\tilde{R}) \,,$$
$$\tilde{R} \equiv K^{ij}K_{ij} - \lambda K^2 + 2\mu \nabla_\mu \left(n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu \right) - E^{ij} G_{ijkl} E^{kl} . \tag{113}$$

In the FRW universe with the flat spatial part, $\tilde{R}$ has the following form:

$$\tilde{R} = \frac{(3-9\lambda) H^2}{N^2} + \frac{6\mu}{a^3 N} \frac{d}{dt}\left(\frac{Ha^3}{N}\right) = \frac{(3-9\lambda+18\mu) H^2}{N^2} + \frac{6\mu}{N}\frac{d}{dt}\left(\frac{H}{N}\right) . \tag{114}$$

The case one obtains with the choice of parameters $\lambda = \mu = 1$ corresponds to the usual $F(R)$-gravity as long as we consider spatially-flat FRW cosmology. The $\mu = 0$ version corresponds to some degenerate limit of the above general Hořava-Lifshitz $F(R)$-gravity.

For the action (113), the FRW equation given by the variation over $g_{ij}^{(3)}$ has the following form:

$$\begin{aligned} 0 &= F\left(\tilde{R}\right) - 2\left(1 - 3\lambda + 3\mu\right)\left(\dot{H} + 3H^2\right) F'\left(\tilde{R}\right) \\ &\quad -2\left(1 - 3\lambda\right) H \frac{dF'\left(\tilde{R}\right)}{dt} + 2\mu \frac{d^2 F'\left(\tilde{R}\right)}{dt^2} + p. \end{aligned} \tag{115}$$

On the other hand, the variation over N gives the global constraint:

$$\begin{aligned} 0 &= \int d^3\mathbf{x} \\ &\quad \left[F\left(\tilde{R}\right) - 6\left\{\left(1 - 3\lambda + 3\mu\right) H^2 + \mu \dot{H}\right\} F'\left(\tilde{R}\right) + 6\mu H \frac{dF'\left(\tilde{R}\right)}{d_t} - p \right] \end{aligned} \tag{116}$$

after setting $N = 1$.

Eq.(115) can be integrated to give

$$0 = F\left(\tilde{R}\right) - 6\left\{\left(1 - 3\lambda + 3\mu\right)H^2 + \mu\dot{H}\right\}F'\left(\tilde{R}\right) + 6\mu H \frac{dF'\left(\tilde{R}\right)}{dt} - \rho - \frac{C}{a^3}\,. \tag{117}$$

Here C is the integration constant.

Note that Eq. (117) corresponds to the first FRW equation and (115) to the second one. Specifically, if we choose $\lambda = \mu = 1$ and $C = 0$, Eq. (117) reduces to

$$\begin{aligned} 0 &= F\left(\tilde{R}\right) - 6\left(H^2 + \dot{H}\right)F'\left(\tilde{R}\right) + 6H\frac{dF'\left(\tilde{R}\right)}{dt} - \rho \\ &= F\left(\tilde{R}\right) - 6\left(H^2 + \dot{H}\right)F'\left(\tilde{R}\right) + 36\left(4H^2\dot{H} + \ddot{H}\right)F''\left(\tilde{R}\right) - \rho\,, \end{aligned} \tag{118}$$

which is identical to the corresponding equation in the standard $F(R)$-gravity.

The Example:

$$F\left(\tilde{R}\right) \propto \tilde{R} + \beta \tilde{R}^2. \tag{119}$$

Then

$$0 = H_0^2 \left\{1 - 3\lambda + 9\beta\left(1 - 3\lambda + 6\mu\right)\left(1 - 3\lambda + 2\mu\right) H_0^2\right\} . \tag{120}$$

In the case of usual $F(R)$-gravity, where $\lambda = \mu = 1$ and therefore $1 - 3\lambda + 2\mu = 0$, there is only the trivial solution $H_0^2 = 0$, although the R^2-term could generate the inflation when more gravitational terms, like $R_{\mu\nu}R^{\mu\nu}$ etc., are added. For our general case, however, there exists the non-trivial solution

$$H_0^2 = -\frac{1 - 3\lambda}{\beta\left(1 - 3\lambda + 6\mu\right)\left(1 - 3\lambda + 2\mu\right)} , \tag{121}$$

as long as the r.h.s. of (121) is positive. If the magnitude of this non-trivial solution is small enough, this solution might correspond to the accelerating expansion in the present universe.

Instead of (119) one may consider the following model:

$$F\left(\tilde{R}\right) \propto \tilde{R} + \beta\tilde{R}^2 + \gamma\tilde{R}^3 \,. \qquad (122)$$

Then

$$0 = H_0^2 \left\{1 - 3\lambda + 9\beta\left(1 - 3\lambda + 6\mu\right)\left(1 - 3\lambda + 2\mu\right)H_0^2 + 9\gamma\left(1 - 3\lambda + 6\mu\right)^2\left(5 - \right.\right. \qquad (123)$$

which has the following two non-trivial solutions,

$$H_0^2 = -\frac{\left(1 - 3\lambda + 2\mu\right)\beta}{2\left(1 - 3\lambda + 6\mu\right)\left(5 - 15\lambda + 12\mu\right)\gamma} \left(1 \pm \sqrt{1 - \frac{4\left(1 - 3\lambda\right)\left(5 - 15\lambda + 12\mu\right)\gamma}{9\left(1 - 3\lambda + 2\mu\right)^2\beta^2}}\right) \,, \qquad (124)$$

as long as the r.h.s. is real and positive.

If

$$\left|\frac{4\,(1-3\lambda)\,(5-15\lambda+12\mu)\,\gamma}{9\,(1-3\lambda+2\mu)^2\,\beta^2}\right| \ll 1\,, \tag{125}$$

one of the two solutions is much smaller than the other solution.

Possibility:
of unification
of inflation with DE?

Summary

Unification of inflation with DE is natural in modified gravity!