Renormalization Group Flows and Supersymmetry

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What is Quantum Field Theory?
Manolo-Fest, Benasque, September 2011

Research Training Group
Quantum and Gravitational Fields

seit 1558

1. Functional renormalization flows
2. Manifest supersymmetric flows for non-gauge models
3. Scale-dependence of super-potential
4. Masses, Phases, Fixed Points, ...
5. Exact solution for $O(N)$-models in large N limit

- strong-coupling phenomena:

  spectrum, phase transitions, collective condensations, etc.
- exact solutions:

  low dimensions, extended symmetry, integrable models, ...
- strong coupling expansions, large N, ...
- lattice simulations:

  problems: symmetries, doublers, chiral fermions, sign-problem, ...
  $\Longrightarrow$ wish complement to lattice studies
- functional renormalization group:

  phase transitions, condensed matter systems, infrared sector of gauge theories, quantum gravity, renormalizability-proofs, ...

QFT = solution of functional renormalization group equation

Functional renormalization flow

Wilson, Wegner-Houghton, Polchinsky, Wetterich

- generating functional for $n-$point functions: formally

$$Z[j]=\int\mathcal{D}\phi\,\mathrm{e}^{-S[\phi]+(j,\phi)}\quad,\quad W[j]=\log Z[j]$$

- add momentum- and scale-dependent regulator term to S

$$\Delta S_k[\phi]=\frac{1}{2}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\phi^*(p)R_k(p)\phi(p)=\frac{1}{2}(\phi,R_k\phi)$$

$\Longrightarrow$ scale-dependent generating functionals

$$Z_k[j]=\int\mathcal{D}\phi\,\mathrm{e}^{-S[\phi]+(j,\phi)-\Delta S_k[\phi]}\quad,\quad W_k[j]=\log Z_k[j]$$

Requirements

- R_k regulates QFT in UV and IR
- recover full generating functionals in IR:

$$R_k(p) \stackrel{k\to 0}{\longrightarrow} 0 \quad \text{fixed} \quad p$$

- recover classical theory at cutoff Λ: $R_k \stackrel{k\to\Lambda}{\longrightarrow} \infty$

  $\Longrightarrow$ interpolation between classical and quantum theory
- IR regularization

$$R_k(p) > 0 \quad \text{for} \quad p \to 0$$

- e.g. optimized regulator $R_k(p) = (k^2 - p^2)\,\theta\left(k^2 - p^2\right)$

- compatible with symmetries
- flow equation for $W_k[j]$: Polchinsky equation

$$\partial_k W_k = -\frac{1}{2}\mathrm{tr}\left(\partial_k R_k W_k^{(2)}\right) - \frac{1}{2}\left(W_k^{(1)}, \partial_k R_k W_k^{(1)}\right)$$

$W_k(n)$: n'th functional derivative of W_k

- Legendre transform $\Rightarrow$ scale dependent effective action

$$\Gamma_k[\varphi] = (\mathcal{L}W_k)[\varphi] - \Delta S_k[\varphi]$$

- Γ_k not necessarily convex, $\Gamma = \Gamma_{k\to 0}$ convex
- average field $\leftrightarrow$ source

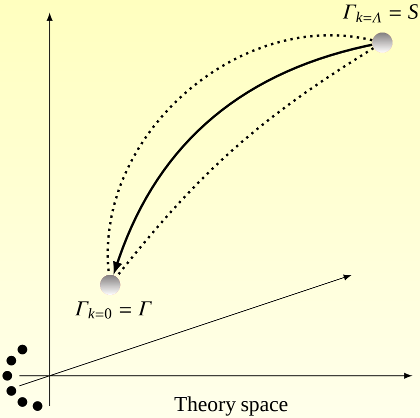
$$\varphi(x) = \frac{\delta W_k[j]}{\delta j(x)} \quad , \quad \frac{\delta \Gamma_k}{\delta \varphi(x)} = j(x) - (R_k\varphi)(x)$$

- exact flow equation for $\Gamma_k[\varphi]$: Wetterich equation

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{tr}\left(\frac{\partial_k R_k}{\Gamma_k^{(2)}[\varphi] + R_k}\right)$$

- nonlinear functional differential equation, 1-loop structure
- denominator: IR regularization
- numerator: $\partial_k R_k$ peaked at $p = k$

  $\Rightarrow$ quantum fluctuations near momentum shell $p \approx k$
- integrate 'down': collect quantum fluctuations below cutoff Λ
- nonperturbative, applicable to QFT's and statistical systems
- in particular: supersymmetric QFT's


$\Gamma_{k=\Lambda} = S$
$\Gamma_{k=0} = \Gamma$
Theory space

Challenges

- supersymmetry: Poincare algebra, fermionic supercharges

$$\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}^j_{\dot\alpha}\} = 2\delta^{ij}\sigma^\mu_{\alpha\dot\alpha}P_\mu \quad , \quad \{\mathcal{Q}^i_\alpha, \mathcal{Q}^j_\beta\} = 2\varepsilon_{\alpha\beta}Z^{ij}$$

- symmetry of renormalization flow $\Longrightarrow$
  Ward identities $\Rightarrow$ mass degeneracy, non-renormalization theorems, ...
- dynamical susy breaking $\Longrightarrow$ phase transitions
- fixed-point structure of susy theories
- finite temperature effects
- here: simple susy Yukawa models
  - $d = 2$: infinitely many fixed point solutions: superconformal QFT's
  - $d = 3$: two fixed point solutions; susy breaking, finite T, ...

Wess-Zumino models in 2 and 3 dimensions

- $\mathcal{N} = 1$: real superfield
- field content: scalar ϕ, Majorana ψ, auxiliary F
- off-shell Yukawa model

$$\mathcal{L}_\mathrm{E} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}F^2 + \frac{1}{2}W''(\phi)\bar{\psi}\gamma_*\psi - FW'(\phi)$$

- eliminate dummy $F \Longrightarrow$ on-shell Yukawa model

$$\mathcal{L}_\mathrm{E} = \frac{1}{2}(\partial\phi)^2 + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}W'^{\,2}(\phi) + \frac{1}{2}W''(\phi)\,\bar{\psi}\gamma_*\psi$$

- classical model $\Longleftrightarrow$ superpotential W
- Witten index $\Rightarrow W(\phi) \sim \phi^m$: m even: susy always unbroken
  m odd: susy breaking possible

- supersymmetric regulator
- on-shell: no quadratic regulator
- off-shell: quadratic regulators exist
- most general solution known, here

$$\Delta S_k = \frac{1}{2}\int d^d x\, \left(\phi\, p^2 r(p^2)\, \phi - \bar{\psi}\, \not{p} r(p^2)\, \psi - r(p^2)\, F^2\right)$$

- susy ⇒ cut-off functions for component fields related ☺
- truncation: leading order local potential approximation

$$\Gamma_k = \int d^d x \left(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{i}{2}\bar{\psi}\not{\partial}\psi - \tfrac{1}{2}F^2 + \tfrac{1}{2}W_k''(\phi)\bar{\psi}\gamma_*\psi - W_k'(\phi)F\right)$$

Georg Bergner, Jens Braun, Holger Gies, Daniel Litim, Marianne Mastaler, Franziska Synatschke-Czerwonka

Phys. Rev. **D81** (2010) 125001; **D80** (2009) 085007; **D80** (2009) 101701

JHEP **0903** (2009) 028; ArXiv 1107.3011; including NLO

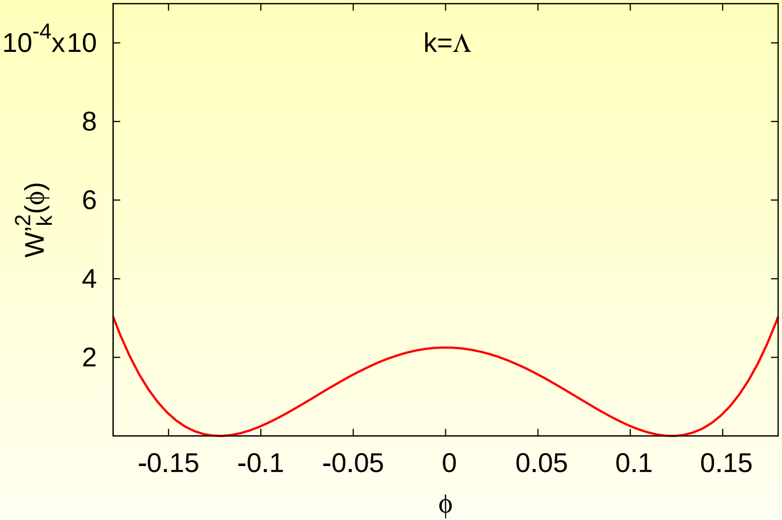
Flow of super potential

- project flow onto $F \Longrightarrow$ flow equation for W_k:

$$\partial_k W_k(\phi) = -\frac{k^{d-1}}{A_d} \frac{W_k''(\phi)}{k^2 + W_k''(\phi)^2}$$

- nonlinear PDE
- $W_k \Longrightarrow$ flow for scalar potential $V_k = \frac{1}{2} W_k'^2$
- $k = \Lambda$: classical potential

  $k \to 0$: convex effective potential
- flow into susy broken phase:



10^-4x10
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W'^2_k(ϕ)
k=Λ
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-0.1
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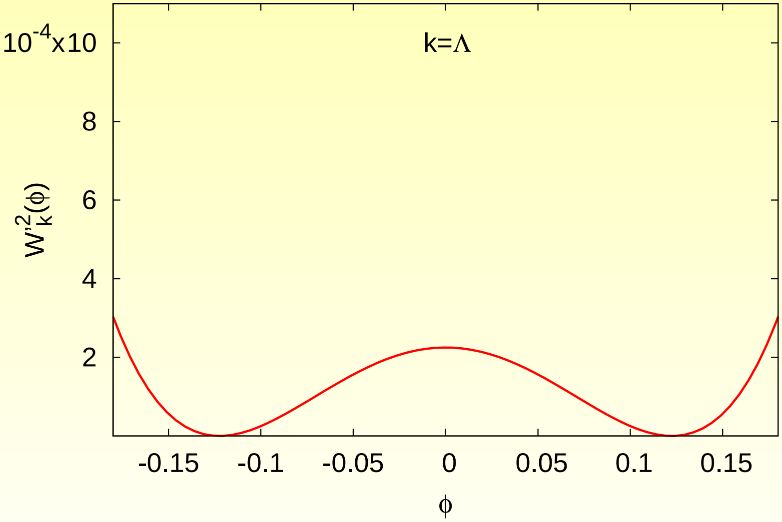
ϕ



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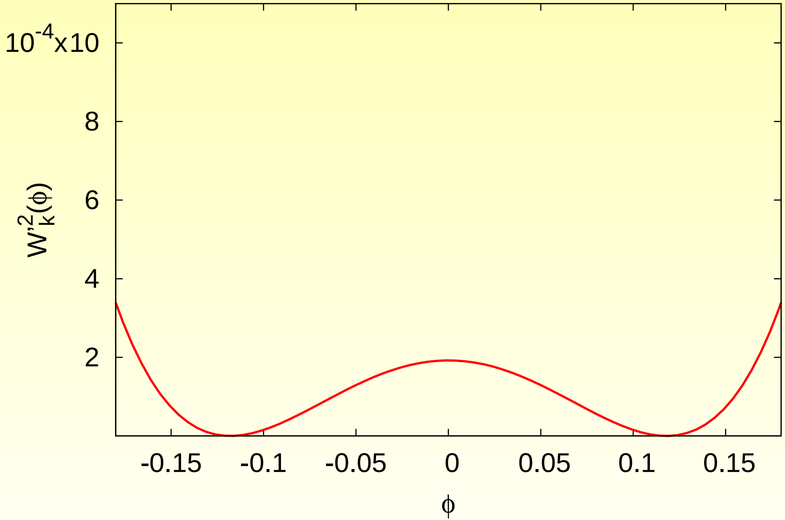
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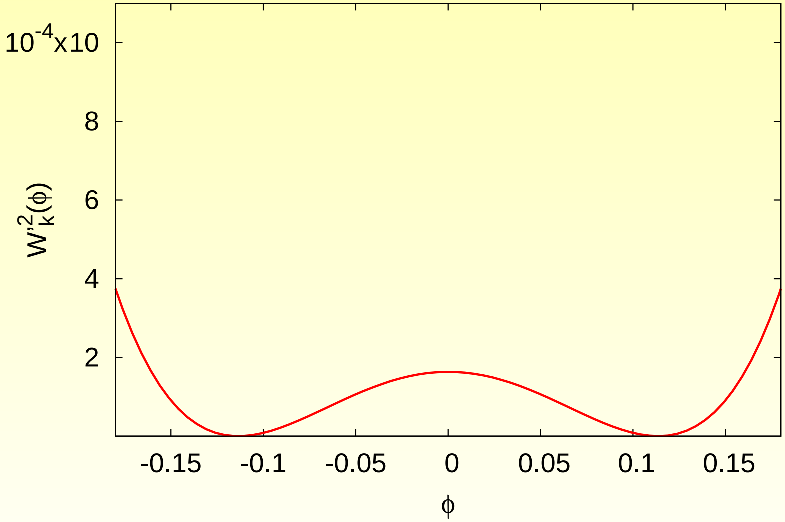
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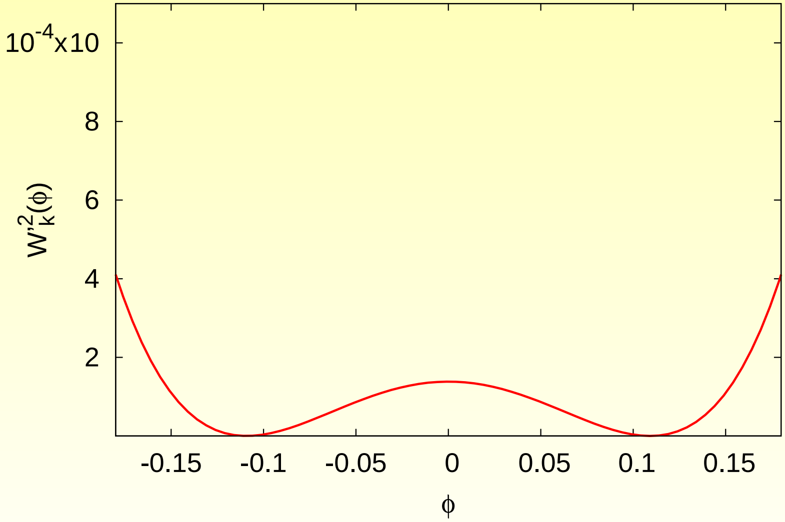
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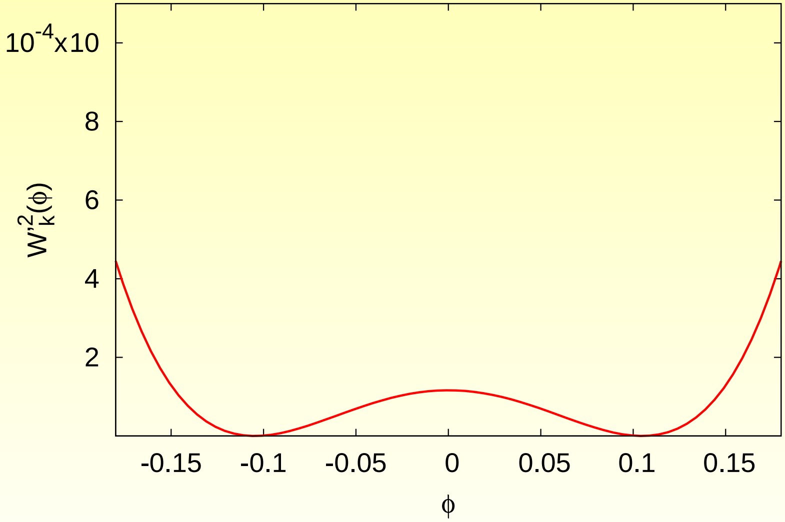

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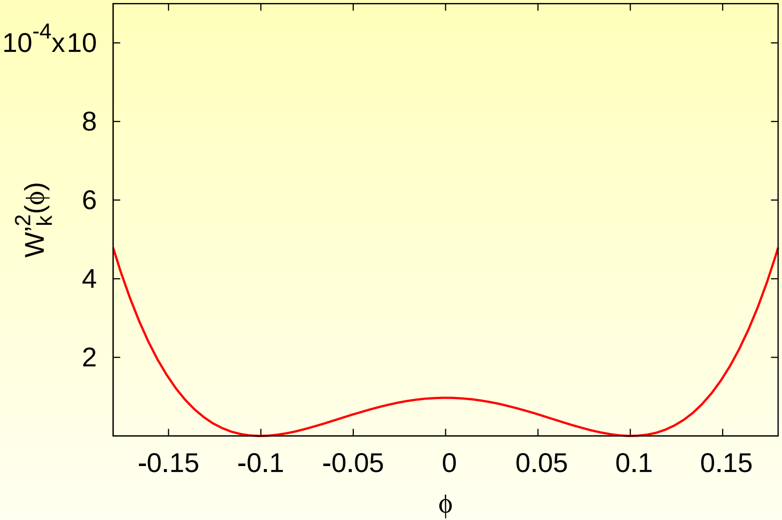

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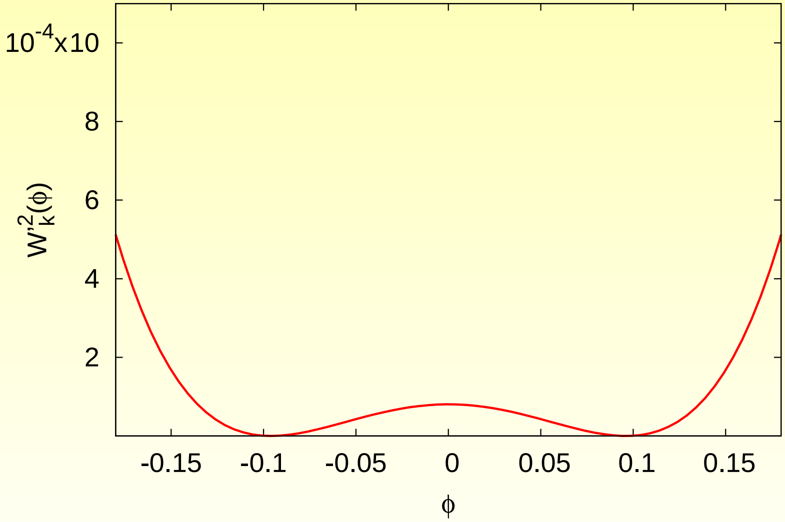
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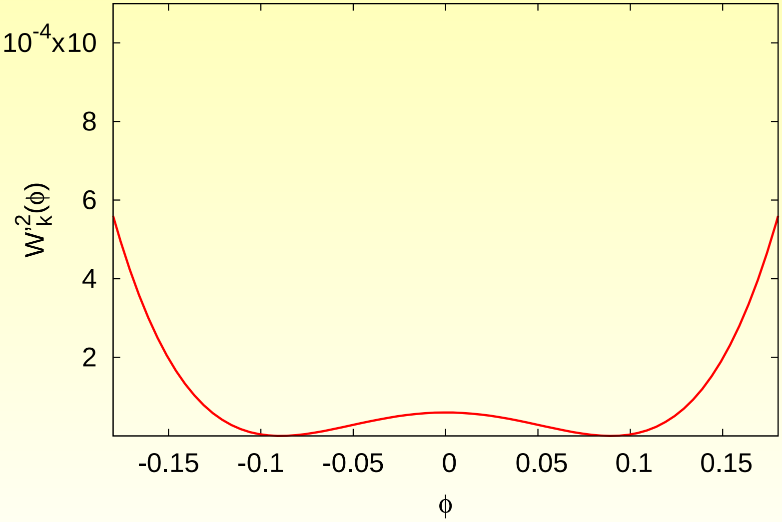
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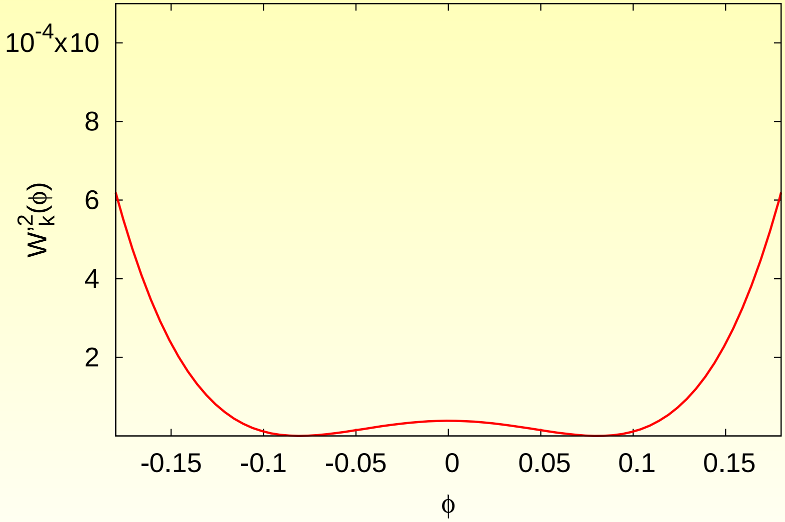

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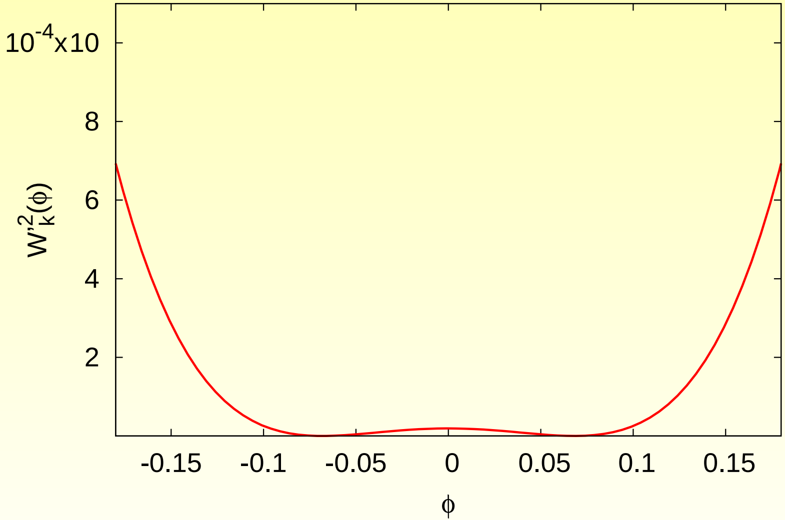
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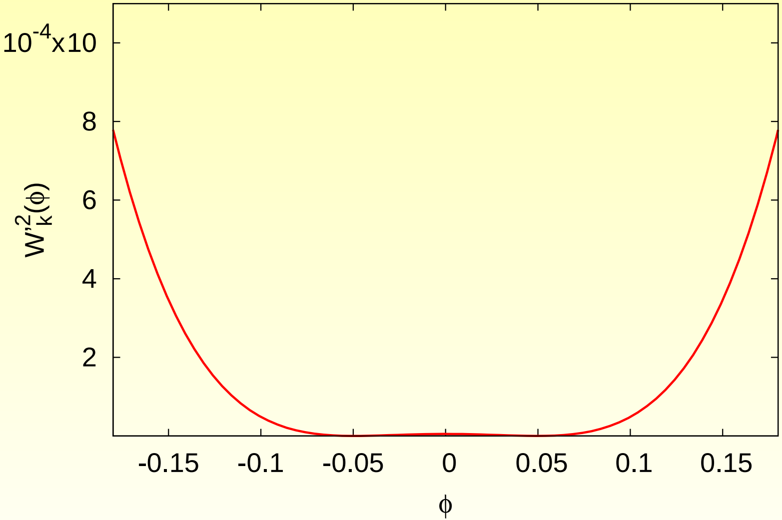
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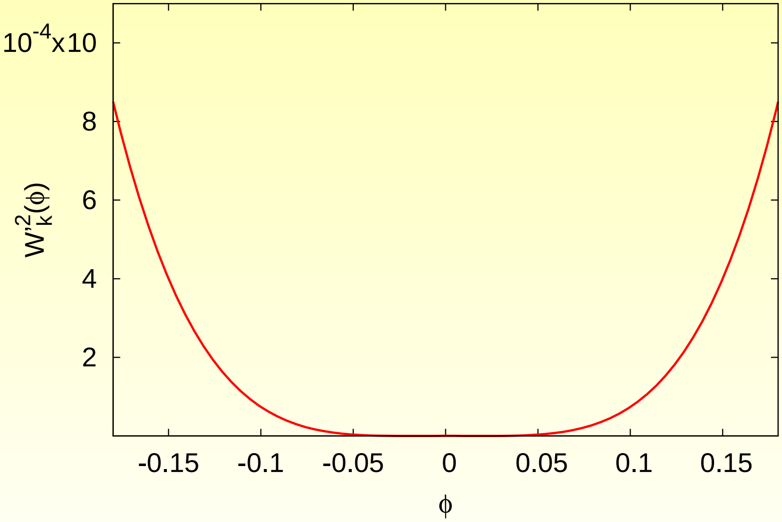
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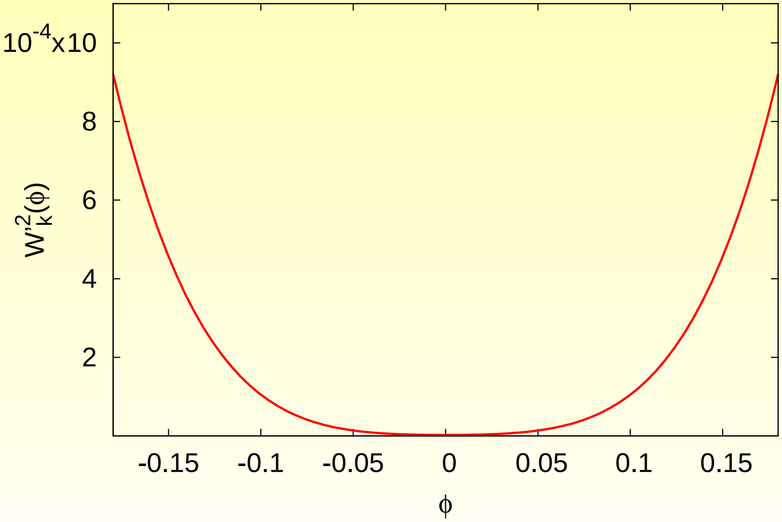
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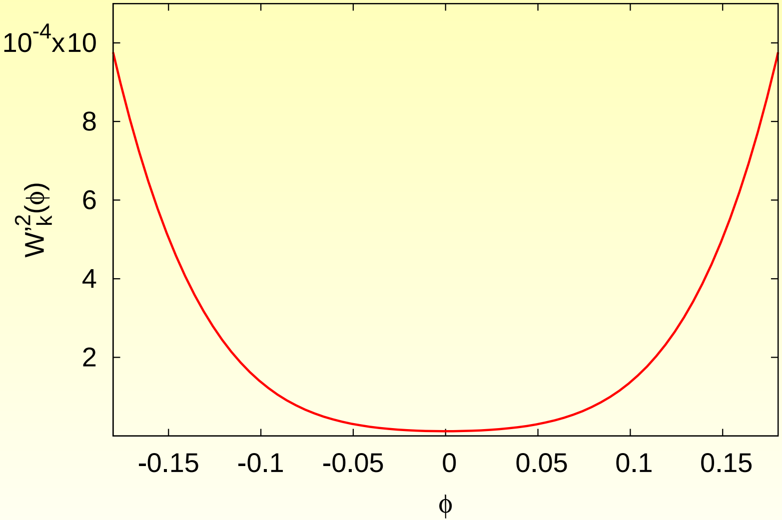

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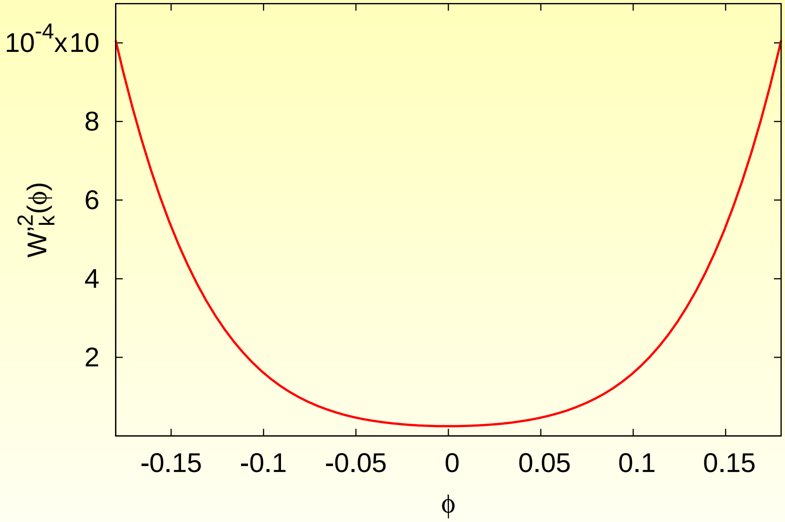
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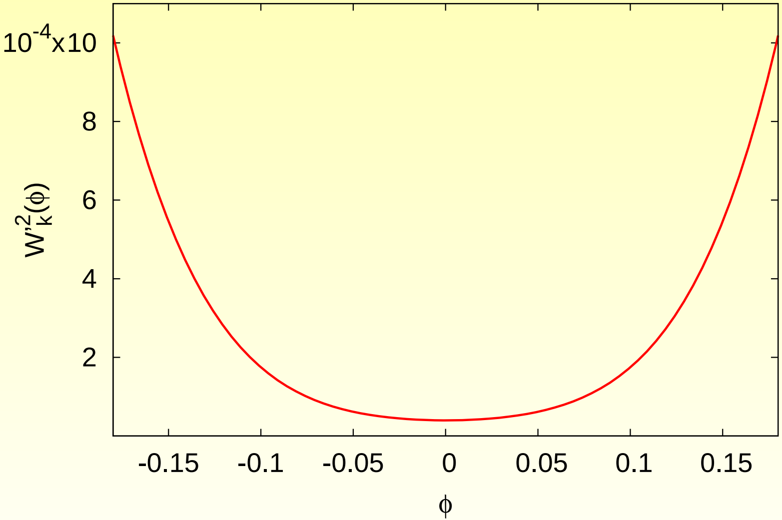

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Fixed point structure in 2 dimensions

- dimensionless quantities $t = \log(k/\Lambda)$, $W_k(\phi) = k w_t(\phi)$

$$\partial_t w_t(\phi) + w_t(\phi) = -\frac{1}{4\pi} \frac{w_t''(\phi)}{1 + w_t''(\phi)^2}$$

- nonlinear PDE
- even $w_t' \Rightarrow$ susy-breaking possible, Taylor series

$$w_t'(\phi) = \lambda_t(\phi^2 - a_t^2) + b_{4,t}\phi^4 + b_{6,t}\phi^6 + \dots$$

$\Longrightarrow \infty$-system of coupled ODE's

- fixed points: $\partial_t w_* = 0$

$\Longrightarrow \infty$-system of algebraic equations

fixed points continued

- linearization: $a_*/1/\sqrt{2\pi}$ always unstable
- keep terms up to $b_{2n,t}\phi^{2n} \Longrightarrow 2n$ non-Gaussian fixed points

$$\pm\left(\lambda_p^*, b_{4,p}^*, \ldots, b_{2n,p}^*\right), \quad p = 1, \ldots, n$$

- ordering $\lambda_n^* > \lambda_{n-1}^* > \ldots \Longrightarrow$

  λ_n^* : 1 IR-unstable direction a_t^2

  λ_{n-1}^* : 2 IR-unstable directions

  λ_{n-2}^* : 3 IR-unstable directions

  $\vdots$

- root belonging to IR-stable fixed point $\lambda_n^* \stackrel{n\to\infty}{\longrightarrow} \lambda_{\text{crit}} = 0.9816$

λ^*	$\Re(\theta^I)$ of non-Gaussian fixed points, truncation at 2n=16							
$\pm.9816$	-1.54	-7.43	-18.3	-37.3	-68.9	-120	-204	-351
$\pm.8813$	6.16	-1.64	-9.82	-25.6	-52.5	-96.9	-170	-300
$\pm.7131$	21.4	4.37	-1.57	-11.1	-30.1	-63.3	-120	-223
$\pm.5152$	28.7	13.3	3.33	-1.39	-11.6	-32.8	-71.7	-145
$\pm.3158$	20.0	20.0	8.40	2.57	-1.14	-11.6	-34.3	-80.4
$\pm.1437$	11.2	11.2	8.63	5.19	1.95	$-.842$	-11.1	-35.7
$\pm.0322$	4.20	4.20	2.86	2.72	2.72	1.47	$-.540$	-10.5
$\pm.0003$	1.57	1.57	1.43	1.43	1.14	.542	.542	-0.221

- back to fixed point equation for $u(\phi) = w_*''(\phi)$:

$$(1-u^4)u'' = 2u'^2(3-u^2)u - (1-u^2)^3 4\pi u$$

- periodic solutions for $u'(0) < 2\lambda_{\rm crit}$
- previous polynomials converge to periodic solution

next-to leading-order flows

- wave function renormalization $\Longrightarrow \eta = -\partial_t \log Z_k^2$
- θ^0 critical exponent of relevant direction a_t^2 (related to W)
- new superscaling relation (exact in NLO)

$$\nu_w \equiv \frac{1}{\theta_0} = \frac{d-\eta}{2}$$

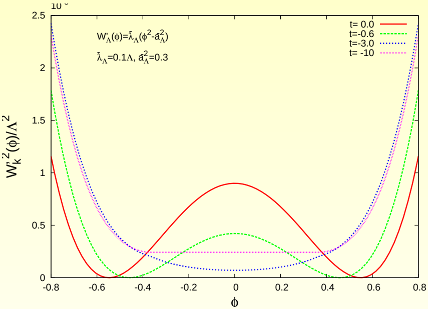
superscaling relation at maximally IR-stable fixed point (d=2)

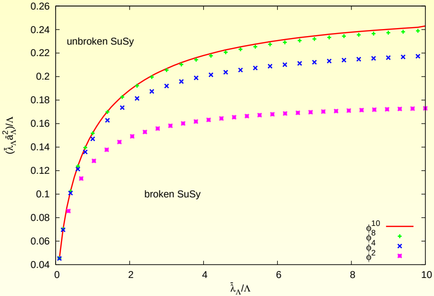
$2n$	2	4	6	8	10	12	14
η	0.3284	0.4194	0.4358	0.4386	0.4388	0.4387	0.4386
$1/\nu_W$	0.8358	0.7903	0.7821	0.7807	0.7806	0.78065	0.7807

- number of IR-unstable direction = number of nodes of u plus 1

Supersymmetry breaking

- supersymmetric phase: $\min_\phi V_{k=0}(\phi) = \min_\phi W'^2_{k=0}(\phi) = 0$
- susy broken: $W'_{k=0}(\phi)$ has no node

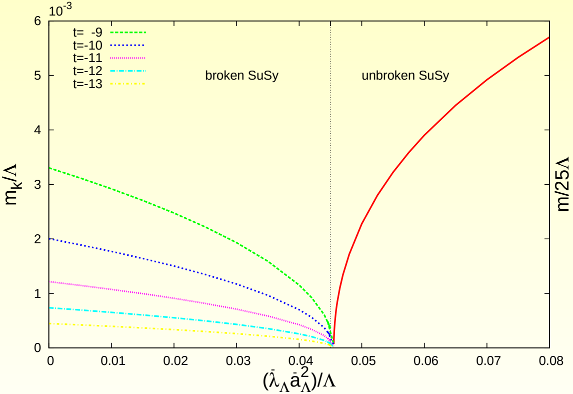




left: flow of a potential $V = W'^2$ with susy breaking, $W'_\Lambda(\phi) = \bar\lambda_\Lambda(\phi^2 - \bar a^2_\Lambda)$
right: phase diagram for couplings specified at Λ, different truncations.

masses of bosons and fermions

- supersymmetric phase: $Z_k^4 m_{k,\mathrm{boson}}^2 = W_k''^2(\chi_{\min}/Z_k) = Z_k^4 m_{k,\mathrm{fermion}}^2$
- broken phase: $Z_k^4 m_{k,\mathrm{boson}}^2 = W_k'(0) W_k'''(0) \sim k^{1+\eta/2}$



Wess-Zumino model in 3 dimensions

with J. Braun and F. Synatschke-Czerwonka

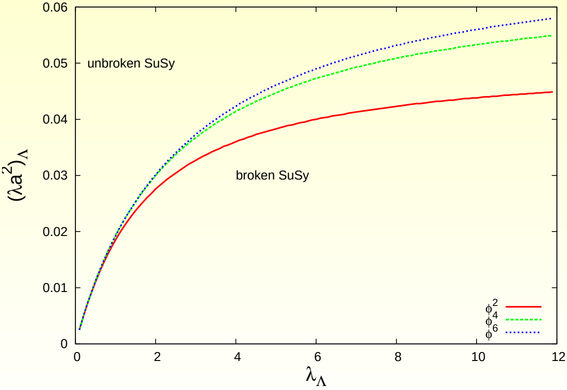
- one Wilson-Fisher fixed point
- a_t^2 defines the only IR-unstable direction
- LPA, polynomial expansion
- rapid convergence

Wilson-Fisher fixed point from polynomial expansion

$2n$	$\pm\lambda^*$	$\pm b_4^*$	$\pm b_6^*$	$\pm b_8^*$	$\pm b_{10}^*$	$\pm b_{12}^*$
4	1.546	2.305				
6	1.590	2.808	6.286			
8	1.595	2.873	7.150	13.41		
10	1.595	2.873	7.155	13.48	1.212	
12	1.595	2.870	7.118	12.90	-8.895	-183.3

$2n$	critical exponents for different truncations								
6	-0.799	-5.92	-20.9						
8	-0.767	-4.83	-14.4	-38.2					
10	-0.757	-4.35	-11.5	-26.9	-60.8				
12	-0.756	-4.16	-9.94	-21.4	-43.8	-89.0			
14	-0.756	-4.10	-9.13	-18.3	-35.1	-65.4	-123		
16	-0.756	-4.08	-8.72	-16.4	-29.9	-52.9	-91.9	-163	
18	-0.756	-4.08	-8.54	-15.2	-26.4	-45.0	-75.0	-124	-209

- phase diagram from parameter study of $W'_{k\to 0}$



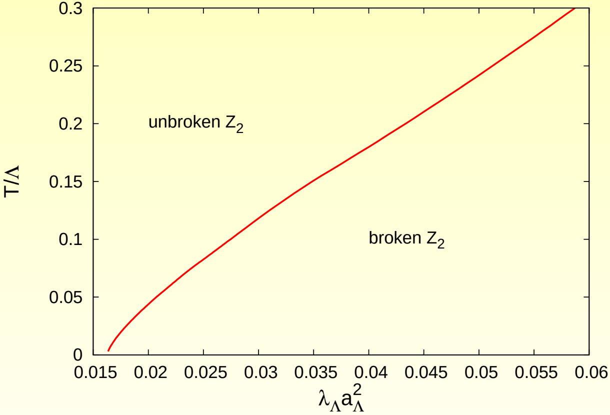
Finite temperature

- $\int dp_0 \longrightarrow$ summation over Matsubara frequencies
- sums can be calculated explicitly $\Longrightarrow$ two flow equations

$$\partial_k W_k'^{\text{bos}} = -\frac{k^2}{8\pi^2} W_k''' \frac{k^2 - W_k''^2}{(k^2 + W_k''^2)^2} \times F_{\text{bos}}(T,k)$$
$$\partial_k W_k'^{\text{ferm}} = -\frac{k^2}{8\pi^2} W_k''' \frac{k^2 - W_k''^2}{(k^2 + W_k''^2)^2} \times F_{\text{ferm}}(T,k)$$

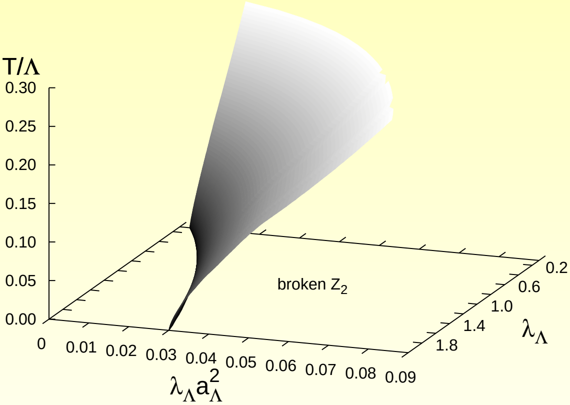
- susy breaking by thermal fluctuations
- $T = 0$: susy broken $\longleftrightarrow$ $\mathbb{Z}_2$ unbroken
- study $\mathbb{Z}_2$ breaking at finite T

Phase diagram



finite-temperature phase diagram for fixed $\lambda_\Lambda = 0.8$

Phase diagram, continued



finite-temperature phase diagram

Linear sigma-models

- O(N)-invariant supersymmetric action
- rescaled dimensionless field $\varrho \propto \vec{\phi} \cdot \vec{\phi}$
- rescaled superpotential $w(\varrho) \propto W(\varrho/N)$
- RGE: contribution from Goldstone modes and radial mode

$$\partial_t w - \rho w' + 2w = -\frac{(1-\frac{1}{N})w'}{1+w'^2} - \frac{\frac{1}{N}(w' + 2\rho w'')}{1+(w'+2\rho w'')^2}$$

- large-N limit: radial mode decouples $\Longrightarrow$

$$\partial_t u + \partial_\rho u \left[1 - \rho - u^2 \frac{3+u^2}{(1+u^2)^2}\right] = -u \qquad (u = w')$$

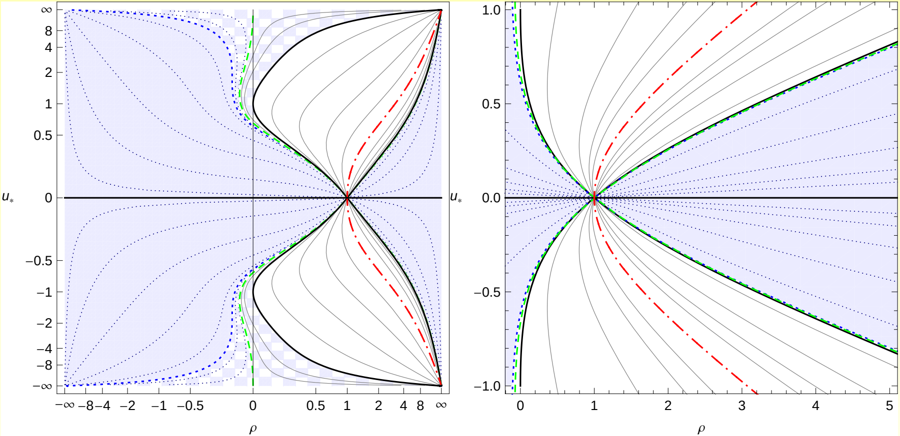
- methods of characteristics $\Longrightarrow$ exact solution

$$\frac{\rho-1}{u} - F(u) = G(ue^t) \quad , \quad F(u) = \frac{u}{1+u^2} + 2\arctan u$$

- $u(\rho)$ at cutoff $\Lambda \Longrightarrow G(ue^t)$
- fixed point solutions depend on one real parameter c

$$\rho = 1 + H(u_*) + c\,u_*, \quad H(u_*) = u_*\,F(u_*)$$

- c exact marginal coupling
- one-parameter family of u_*
- two families: global vs. non-global



- 1-parameter family of fixed point solutions: one marginal coupling $\sim 1/c$

  weak coupling: $c \to \infty \Rightarrow u_* = 0$ horizontal line
  intermediate coupling: two fixed point solution
  strong coupling: $c \to 0 \Rightarrow$ not globally defined

eigenperturbation

- flow in vicinity of fixed points

$$u(t,\rho) = u_*(\rho) + \delta u(t,\rho)$$

- exact solution of linearized flow equation

$$\delta u(t,u) = C\, e^{\theta t}\, u_*^{\theta+1}\, u_*' \,.$$

- regularity at $\rho = 1 \Longrightarrow$ all critical exponents

$$\theta = -1, 0, 1, 2, 3, \cdots$$

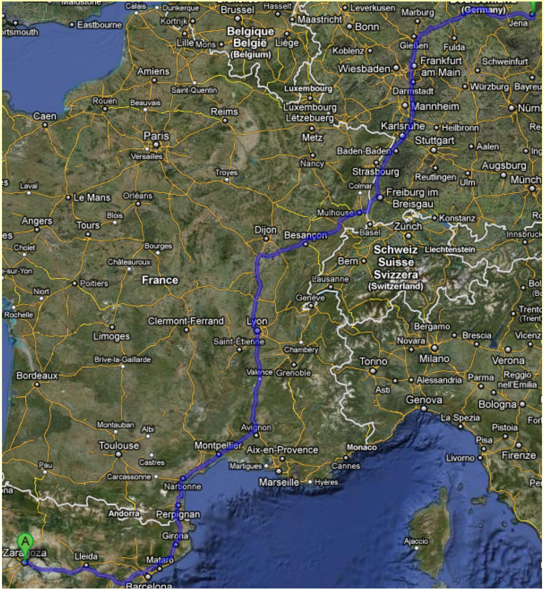
- 4 distinct massive and massless phases
- $N \to \infty$ limit not smooth

Litim, Mastaler, Synatschke, W.: arXiv:1107.3011

summary and next

- constructed manifest supersymmetric FRG
- masses and coupling in infrared
- supersymmetry breaking, phase transitions
- critical phenomena: fixed points, universal exponents
- non-renormalization theorems
- exact solution for W_k in $O(N \to \infty)$ sigma model
- supersymmetric $O(n)$ and $CP(n)$ models
  first lattice results for supersymmetric $CP(1)$ — R. Flore, D. Körner, C. Wozar
  flow equations: how does spectrum vary with θ — R. Flore
- supersymmetric gauge theories
  first lattice-results in $d = 1, 2, 3$ — B. Wellegehausen
  study of flow equations — F. Synatschke-Czerwonka, M. Mastaler

On October 20th: Congratulations, Manolo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