Self-Avoiding Walks and Field Theory: Rigorous Renormalization Group Analysis

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What is Quantum Field Theory
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Outline

Based on :

1. [BGM]: Brydges, Mitter and Guadagni: J.Stat.Phys (2004) 115: 415-449

2. [BM] : Brydges and Mitter [paper in preparation]

3. [MS] : Mitter and Scoppola: J.Stat.Phys (2008) 133: 921-1011

4. [BS] : Brydges and Slade: Lecture in International Congress of Mathematicians (2010) and papers in preparation.

Quantum field theory has developed through its interaction with many fields. Besides elementary paricle physics, statistical mechanics, and, in particular, the theory of critical phenomena have played a very important role in the development of Quantum Field Theory. This stimulated the development of Wilson's renormalization group ideas. Random walks and their interactions provide examples of critical phenomena. Field theories arise naturally. By analyzing these theories rigorously we hope to learn something more about both interacting random walks as well as rigorous renormalization group analysis.

Consider two classes of walks on the lattice $\mathbb{Z}^d$: those with nearest neighbour jumps (simple walks) and those with long range jumps (Lévy walks). An interaction is then added to make them self repelling. The strength of the interaction is kept weak. They are called respectively: (weakly) SAWs and SALWs.

The Green's function of a weakly self avoiding walk on the lattice $\mathbb{Z}^d$ can be represented as the two point correlation function of a supersymmetric field theory. In this talk I will discuss the critical limit of a class of weakly self avoiding walks via rigorous renormalization group analysis.

The Green's function of a random walk from x to y is the expected sum of all walks to go from x to y. For a continuous time walk this is in terms of the transition probability

$$\mathcal{G}(x,y) = \int_0^\infty dt\, p_t(x,y)$$

The end to end distance for a walk starting at the origin is $E(|x_t|)$. The root mean square distance is $\sqrt{E(|x_t|^2)}$.

SAWs: Rigorous results:

$d > 4$: Critical SAWs are diffusive (Brydges-Spencer, [CMP 1984]). Proof by expansion methods.

$d = 4$ is the critical dimension. Brydges-Slade (in preparation) use supersymmetric representation plus rigorous RG methods to prove for critical continuous time SAWs canonical behaviour for the Green's function

$$\mathcal{G}(x - y) \sim \ const.|x - y|^{-2}$$

This is canonical behaviour in $d = 4$

In d=4, perturbative RG calculations say

$$E(x_t^2) \sim \ const.\ t\ log^{\frac{1}{4}} t$$

This has been proved for the hierachical s.a. walk (Brydges-Imbrie) but it has not been yet proved rigorously in general. It is part of the program of Brydges and Slade.

In d=3 there are no rigorous results, analogous to the situation for the d=3 Ising model. However ϵ expansion in the $n = 0$ limit of the n-vector model (Le Guillou, Zinn-Justin) and direct Monte Carlo studies (Madras-Sokal) indicate non-trivial scaling

$$E(x_t^2) \sim \ const.\ t^{2\nu}$$

with $\nu \neq \frac{1}{2}$.

SALWs:

What is the upper critical dimension? There are two parameters in play: The dimension d and the (Lévy Khintchine) parameter α with $0 < \alpha \leq 2$. $\alpha = 2$ corresponds to simple random walk. For SALWs the upper critical dimension (mean field theory) is $d = 2\alpha$. This means that $\epsilon = 2\alpha - d$ can be a small parameter analogous to the ϵ in Wilson and Fisher's ϵ expansion.

SALWs: The the idea that $\epsilon = 2\alpha - d > 0$ can taken as a small parameter has been exploited by Mitter and Scoppola ([MS] -2008). They take $d = 3$ and obtain rigorous results towards the critical Green's function of a class of continuous time weakly self avoiding Lévy walks (long range jumps) (SALW). They use the supersymmetric representation and recently developed rigorous RG methds. Low orders in perturbation theory in ϵ plus control of remainder.

SALW: [MS] give a theorem on a global RG trajectory for the critical supersymmetric lattice field theory in $\mathbb{Z}^3$. The trajectory for the supermeasure is uniformly bounded on all RG scales and defines a non-Gaussian field theory. Non-Gaussian fixed point in underlying continuum theory. From this plus some more work one can prove

$$\mathcal{G}(x-y) \sim const.|x-y|^{-(3-\alpha)}$$

If x_t^α is the continuous time SALW starting at he origin, define the critical exponent ν by

$$E(|x_t^\alpha|) \sim const.t^\nu$$

ν has not been established rigorously. However there is an $\epsilon = 2\alpha - d$ expansion prediction for our case:

$$\nu = \nu_L\left(1 + \frac{\epsilon}{6} + O(\epsilon^2)\right)$$

$\nu_L = \frac{1}{\alpha}$ is the exponent of the Lévy walk.
Thus we expect a non-trivial critical exponent. Proving this is a challenging problem.

Let $x_t^\alpha \in \mathbb{Z}^d$, $0 < \alpha \leq 2$, be a continuous time Lévy process (called a Lévy walk). This is a compound Poisson process with i.i.d jumps. Jump distribution depends on α. For $\alpha = 2$ we have simple random walks: nearest neighbour jumps. For $\alpha < 2$ the jump distributions have long range tails.

The characteristic function is given by the Lévy Khintchine formula

$$E(e^{i(k,x_t^{(\alpha)})}) = e^{(2d)^{\alpha/2}\, t\, \psi_\alpha(k)}$$

$$\psi_\alpha(k) = -(1 - \frac{1}{d}\sum_{j=1}^{d} \cos\ k_j))^{\alpha/2}$$

$$E(e^{i(k,x_t^{(\alpha)})}) = e^{-t(-\hat{\Delta})(k)^{\alpha/2}}$$

For $\alpha = 2$ the characteristic function is that of a simple random walk. But for $0 < \alpha < 2$ it can be shown that the transition probability

$$P_t(x, y) \sim \ const.\ |x - y|^{-(d+\alpha)}$$

This means that the variance is infinite but for $\alpha > 1$ the mean distance is finite.

For $d = 3$ the walk is transient and the Green's function is

$$C(x-y) = \int_0^\infty dt\, P_t^{(\alpha)}(x,y) = (-\Delta)^{-\alpha/2}(x-y)$$

$$\sim |x-y|^{-(3-\alpha)}$$

when $|x-y| \to \infty$. We will choose $\alpha = \frac{3+\epsilon}{2}$, with $0 < \epsilon \leq 1$.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite subset. The local time spent by the walk at $x \in \Lambda$ (upto time T) is

$$\tau_T(x) =: \int_0^T ds\, \delta(x_s^\alpha - x)$$

where δ is the lattice delta function. Let dx be the counting measure. Define

$$\tau_T(\Lambda) = \int_\Lambda dx\, \tau_T(x)$$

$$\tau_T^2(\Lambda) = \int_\Lambda dx\, \tau_T(x)^2 = \int_0^T \int_0^T ds\, dt\, \delta^d(x_t - x_s)\, I_{x_t, x_s \in \Lambda}$$

This is a measure of the self intersection.

We define the expectation for a weakly self-avoiding Lévy process by the law

$$E^{(s.a.)}_{g,T,\Lambda}(f(x^\alpha_t)) = Z^{-1}\, E\left(f(x^\alpha_t)\, e^{-\frac{g^2}{2}\tau^2_T(\Lambda)}\right)$$

Z is a normalization factor.

The exponential factor makes the walk tend to repel itself (weakly) in Λ for finite g.

The Green's function is defined by

$$\mathcal{G}^{\mu}(x,y) = \lim_{\Lambda \uparrow \mathbb{Z}^3} \mathcal{G}^{\mu}_{\Lambda}(x,y)$$

where

$$\mathcal{G}^{\mu}_{\Lambda}(x,y) = \int_0^{\infty} dT \; E_x(e^{-\frac{g^2}{2}\tau^2_T(\Lambda) - \mu\tau_T(\Lambda)} I_{x^{\alpha}_T = y})$$

This limit exists for a suitable choice $\mu = \mu_c = h(g)$, the critical mass which is related to the critical killing rate.

FIELD THEORY REPRESENTATION:

Let ϕ_1, ϕ_2 be independent identically distributed Gaussian random fields in $\mathbb{Z}^3$ with covariance $\frac{1}{2}C$, and C is the Green's function of the walk.

$\phi = \phi_1 + i\phi_2$ and $\bar{\phi}$ its complex conjugate.

$\psi, \bar{\psi}$ are Grassmann fields (scalar fermions) of degree 1 and -1 respectively.

$$\Phi = (\phi, \psi),\ \bar{\Phi} = (\bar{\phi}, \bar{\psi})$$

Inner product:

$$(\Phi, \Phi) = \Phi\bar{\Phi} = \phi\bar{\phi} + \psi\bar{\psi}$$

Let $\Lambda \subset \mathbb{Z}^3$ be a finite subset. Define

$$V(\Lambda, \Phi, g, \mu) = g \int_\Lambda dx (\Phi\bar{\Phi})^2(x) + \mu \int_\Lambda dx \Phi\bar{\Phi}(x)$$

where the coupling constant $g_0 > 0$ and dx is the counting measure in $\mathbf{Z}^3$. Define the $|\Lambda| \times |\Lambda|$ matrix C_Λ by

$$C_\Lambda(x, y) = C(x - y) : \ x, y \in \Lambda$$

C_Λ is a symmetric, positive definite matrix.

Then our field theory in finite volume Λ is defined by the *supermeasure*

$$d\mu_\Lambda(\Phi) = d\mu_{C_\Lambda}(\Phi) e^{-V_0(\Lambda,\Phi,g,\mu)}$$

where $d\mu_{C_\Lambda}(\Phi)$ is the Gaussian supermeasure

$$d\mu_{C_\Lambda}(\Phi) = \prod_{x\in\Lambda} d\Phi(x)\; e^{-(\Phi, C_\Lambda^{-1}\bar{\Phi})_{L^2(\Lambda)}}$$

$$d\Phi(x) = d\phi_1(x) d\phi_2(x) d\psi(x) d\bar{\psi}(x)$$

Integration over the Grassmann fields is Berezin integration and $d\mu_\Lambda(\Phi)$ is interpreted as a linear functional on the Grassman algebra Ω_Λ (generated by the $\psi, \bar{\psi}$ over the ring of functions which are functionals of the $\phi, \bar{\phi}$). Determinants have cancelled out.

An important fact is that the potential $V_0(\Lambda, \Phi)$ is supersymmetric. Here by *supersymmetry* we mean invariance under the transformation $\mathcal{Q}$ defined on the fields as follows

$$\mathcal{Q}\phi = \psi, \quad \mathcal{Q}\bar{\phi} = -\bar{\psi}$$

$$\mathcal{Q}\psi = \phi, \quad \mathcal{Q}\bar{\psi} = \bar{\phi}$$

This induces in a natural way a supersymmetry transformation $\mathcal{Q}$ on functionals of fields. $\mathcal{Q}$ is nilpotent on gauge invariant functionals F

$$\mathcal{Q}^2 F = 0$$

The supermeasure μ_Λ is $\mathcal{Q}$ invariant. For any functional $F(\Phi)$

$$\mu_\Lambda(\mathcal{Q}F) = 0$$

If in addition $\mathcal{Q}F = 0$ then

$$\mu_\Lambda(F) = F(0)$$

.

As a consequence we have that the supermeasure $d\mu_\Lambda(\Phi)$ is normalized :

$$\int d\mu_\Lambda(\Phi)\, 1 = 1$$

No vacuum energy will be generated.

The McKane-Parisi-Sourlas representation says that:

$$\tilde{\mathcal{G}}^{\mu}_{\Lambda}(x,y) = \int d\mu_{C_\Lambda}(\Phi) e^{-V(\Lambda,\Phi)} \bar{\phi}(x)\phi(y)$$

Proof: (Brydges, Evans and Imbrie and more recently Brydges, Imbrie, Slade).
We want to analyze the supermeasure

$$d\mu_{C_\Lambda}(\Phi) e^{-V(\Lambda,\Phi)}$$

by lattice renormalization group transformations. We will generate a sequence of measures (the RG trajectory) living in smaller and smaller cubes in finer and finer lattices.

A function $C(x, y)$ is said to have finite range L if

$$C(x, y) = 0 : |x - y| \geq L$$

Consider case where C is translation invariant. We are interested in positive definite functions (distributions) C which have expansions

$$C = \sum C_n$$

where each C_n is positive definite, finite range L^n and smooth. Green's functions of self adjoint, second order elliptic operators defined by Dirichlet forms in the continuum or on the lattice have such expansions (Brydges, Guadagni and Mitter, JSP 2004). Fractional powers thereof have such expansions.

Let $\mathcal{X} = \mathbb{R}^d$ or $\mathbb{Z}^d$. Suppose $\phi : \mathcal{X} \to \mathbb{R}$ is a Gaussian random variable distributed according to the covariance C. We have an expansion:

$$\phi = \sum_{n \geq 0} \zeta_n$$

where the ζ_n are independent Gaussian random variables distributed according to C_n. The ζ_n have finite range correlations:

$$E(\zeta_n(x)\zeta_n(y)) = 0 \ : \ |x - y| \geq L^n$$

Let $\Lambda \subset \mathcal{X}$ is a large cube. Let $F(\phi, \Lambda))$ be an L^1 function (on probablity space). Typically to begin with $F = exp - V$ where V is a local functional.

Suppose we want to calculate the expectation $E(F(\phi, \Lambda))$. We write this as a multiple expectation with respect to all the ζ_n, $n = 1, 2, ...,$ and carry out the expectations over each n starting with $n = 1$ sequentially. At each step we also perform a rescaling. Together we have a RG step. These steps generalize to super expectations.

Problem: Each such expectation is over fields in a very large region. We have to decouple distant parts to proceed efficiently.

Suppose we have performed $n-1$ RG steps. At this stage we have the fluctuation field ζ_n and the unintegrated fields $\phi = \sum_{j \geq n+1} \zeta_j$. Let F_{n-1} denote the evolved random function.

Pave $\mathcal{X}$ with unit blocks. Λ has the induced paving. Polymers are connected subsets of blocks. A polymer activity is a functional of fields over polymers.

The evolved random function $F_{n-1}(\zeta_n, \phi)$ can be written as a sum of products of activities of disjoint polymers with the spaces in between filled by local functionals of fields independent of ζ_n. Suppose the polymers are sufficiently disjoint. Then the ζ_n expectation factorizes by the finite range property. We are left to study the expectation over a small region. No cluster expansion is necessary.

Lattice RG transformations will be based on finite range expansion of the covariance C, [BGM 2004]. This is an alternative to Kadanoff- Wilson block spin RG.
Let L be a large triadic integer (3^p). Let $\delta_n = L^{-n}$. Let $d_s = \frac{(3-\alpha)}{2}$. We have a sequence of compatible lattices $(\delta_n \mathbb{Z})^3$ and positive definite functions $\Gamma_n : (\delta_n \mathbb{Z})^3 \to \mathbb{R}$ of finite range $\frac{L}{2}$ such that for all $x, y \in (\mathbb{Z})^3$

$$C(x-y) = \sum_{n \geq 0} L^{-2nd_s} \Gamma_n\Big(\frac{x-y}{L^n}\Big)$$

The series converges in $L^\infty(\mathbb{Z}^3)$

Regularity properties

The fluctuation covariances Γ_n have good regularity properties uniform in the lattice scale n. For example

$$\|\partial^m_{\delta_n}\Gamma_n\|_{L^\infty((\delta_n\mathbb{Z})^3)} \leq c_{L,m}$$

Moreover the sequence Γ_n converges exponentially fast to a smooth positive definite continuum function Γ_* of finite range $\frac{L}{2}$ in the following sense: For all $n \geq l \geq 0$, with l fixed we have

$$\|\partial^m_{\delta_n}\Gamma_n - \partial^m_c\Gamma_*\|_{L^\infty((\delta_l\mathbb{Z})^3)} \leq c_{L,m}L^{-qn}$$

for some $q > 0$.

Define by recursion the positive definite functions C_n on $(\delta_n \mathbb{Z})^3$

$$C_n(x) = \Gamma_n(x) + L^{-2d_s} C_{n+1}(\frac{x}{L})$$

The C_n have regularity properties uniformly in the lattice scale. They, and their lattice derivatives, converge to smooth continuum functions. Now we can define RG transformations. As usual this is the composition of fluctuation integration (next frame) with rescaling of fields. The scale transformation S_L is defined by $S_L \Phi(x) = L^{-d_s} \Phi(x/L)$.

Let $(-L^m/2, L^m/2) \subset \mathbb{R}^3$ and define $\Lambda_{m,n} = \Lambda_m \cap (\delta_n \mathbb{Z})^3$. Start with the density $\mathcal{Z}_0(\Lambda_{N,0}, \Phi) = exp(-V_0(\Lambda_{N,0}, \Phi))$ with initial parameters g_0, μ_0 in a large cube on the unit lattice. There is a sequence of RG transformations which gives the evolution of densities belonging to the Grassman algebra over the ring of bosonic fields on finer and finer lattices

$$T_{N-n,n} \; : \; \Omega^0(\Lambda_{N-n+1,n-1}) \to \Omega^0(\Lambda_{N-n,n})$$

defined by

$$\mathcal{Z}_n(\Lambda_{N-n,n}, \Phi) = \mu_{\Gamma_{n-1}} * \mathcal{Z}_{n-1}(\Lambda_{N-n+1,n-1}, S_L \Phi)$$

$$\int d\mu_{C_0} \mathcal{Z}_0(\Lambda_{N,0}, \Phi) = \int d\mu_{C_n} \mathcal{Z}_n(\Lambda_{N-n,n}, \Phi)$$

At any given step n of the sequence of RG transformations the densities will be given coordinates g_n, μ_n, K_n. Here g_n, μ_n are the evolved parameters of the local potential V_n. and K_n is a so called irrelevant (contracting) term characterized as a polymer activity. The density $\mathcal{Z}_n(\Lambda_{N-n,n}, \Phi)$ can be expressed in terms of these coordinates in a polymer gas representation.

Pave $\mathbb{R}^3$ with a disjoint union of open unit cubes which are integer translates of $(-\frac{1}{2}, \frac{1}{2})^3$. Take their intesection with a fine lattice. These are unit blocks of the fine lattice. A polymer $X \subset \Lambda_{N-n,n}$ is by default a *connected* union of unit blocks. Two disjoint polymers are separated by a distance ≥ 1. A polymer activity K_n is a map $(X, \Phi) \to \Omega^0(X)$ and

$$\mathcal{Z}_n(\Lambda_{N-n,n}) = \sum_{N \geq 0} \frac{1}{N!} e^{-V_n(X_c)} \sum_{X_1, .. X_N} \prod_{j=1}^{N} K_n(X_j)$$

The sum is over mutually disjoint connected polymers in $\Lambda_{N-n,n}$.

The density $\mathcal{Z}_n(\Lambda_{N-n,n}, \Phi + \xi)$ can also be written as

$$\mathcal{Z}_n(\Lambda_{N-n,n}) = \sum_{N \geq 0} \frac{1}{N!} e^{-\tilde{V}(Y_c)} \sum_{Y_1,..Y_N} \prod_{j=1}^{N} \mathcal{B}K(Y_j)$$

The sum is now over mutually disjoint connected L- polymers in $\Lambda_{N-n,n}$. $\mathcal{B}K$ is a non-linear functional of K_n, $\tilde{V}_n$ which depends on Φ, ξ. $\tilde{V}_n$ is a yet to be chosen local potential which depends only on Φ . Then the *fluctuation* map $S_L \mu_{\Gamma_n} *$ which integrates out the ξ *factorizes* over the product of polymer activities because of the finite range property of Γ_n since the connected L- polymers are separated by a distance $\geq L$. Thus the polymer representation is preserved after fluctuation integration.

The fluctuation inntegration plus rescaling has given a map

$$V_n \to \tilde{V}_{n,L} = S_L V_n, \quad K_n \to \mathcal{F} K_n$$

The image polymer activity lives on polymers in $\Lambda_{N-1-n,n+1}$. The polymers have become finer and the cube smaller.

We can profit from the arbitrariness in the choice of $\tilde{V}_n$ to subtract out the (localized) expanding parts of $\mathcal{F}K_n$, and absorb them in $\tilde{V}_{n,L}$ thus producing a flow of parameters. This subtraction operation on $\mathcal{F}K_n(X, \Phi)$ needs only to be be done for small sets X, because large sets provide contracting contributions. The new subtracted polymer activities have good contraction properties (irrelevant terms). Finally note that supersymmetry is preserved by these maps. No vacuum energy terms are produced by supersymmetry.

This procedure produces our final RG map f_{n+1}

$$f_{n+1,V}(V_n, K_n) = V_{n+1}, \quad f_{n+1,K}(V_n, K_n) = K_{n+1}$$

Using second order perturbation theory,

$$K_n = e^{-V_n} Q_n + R_n$$

Q_n is a second order contribution. It is form invariant and depends on g_n, μ_n and a non-local kernel w_n which converges fast to continuum kernel. We will ignore it for simplicity. R_n is a remainder. Let $u_n = (g_n, \mu_n, R_n)$ represent a point on the RG trajectory. The RG map produces a discrete flow:

$$u_{n+1} = f_{n+1}(u_n)$$

The flow map in components is:

$$g_{n+1} = f_{n+1,g}(u_n) = L^{\epsilon} g_n((1 - L^{\epsilon} a_n g_n) + \xi_n(u_n)$$

$$\mu_{n+1} = f_{n+1,\mu}(u_n) = L^{\frac{3+\epsilon}{2}} \mu_n - L^{2\epsilon} b_n g_n^2 + \rho_n(u_n)$$

$$R_{n+1} = f_{n+1,R}(u_n) =: U_{n+1}(u_n)$$

The coefficients a_n are positive and converge fast to a limit $a_c > 0$. We have an approximate flow $\bar{g}_n$ obtained by ignoring the remainder ξ_n and replacing a_n by its limiting value a_c. This approximate flow has an attractive fixed point $\bar{g} = O(\epsilon)$, for ϵ sfficiently small.

Let $\tilde{g}_n = g_n - \bar{g}$. Then $v_n = (\tilde{g}_n, \mu_n, R_n)$ are the new coordinates. Then

$$\tilde{g}_{n+1} = f_{n+1,g}(v_n) = (2 - L^{\epsilon})\tilde{g}_n + \tilde{\xi}_n(v_n)$$

$$\mu_{n+1} = f_{n+1,\mu}(v_n) = L^{\frac{3+\epsilon}{2}}\mu_n + \tilde{\rho}_n(v_n)$$

$$R_{n+1} = f_{n+1,R}(v_n) =: U_{n+1}(v_n)$$

are the new flow equations. $(2 - L^{\epsilon}) = (1 - O(logL)\epsilon$ is a contraction factor.

Banach spaces:

We will consider the RG action on polymer activities in the formal infinite volume limit. This makes sense because of the finite range property of fluctuation covariances.

We endow polymer activities K_n with a norm $\|\cdot\|_n$. This norm tends to a continuum norm as $n \to \infty$. This gives us a Banach space of Polymer activities at scale n. The norm measures:

1. Large ϕ field growth (large field regulator)
2. Partial derivatives in the ϕ, ψ, finite number in ϕ, and all in ψ.
3. Puts in a weight which says that large polymers contribute small amount (large set regulator).

We introduce a second norm $|\cdot|_n$. This is the same as the previous norm except that we evaluate poymer activities at $\phi = 0$, and therefore no large field growth to be measured.

We measure the remainder R_n in a norm $|||\cdot|||_n$, where

$$|||R_n|||_n = max\{|R_n|,\ \bar{g}^2||R_n||\}$$

Define a Banach space E_n consisting of elements $v_n = (\tilde{g}_n, \mu_n, R_n)$ with norm

$$||v_n|| = max\{(\nu\bar{g})^{-1}|\tilde{g}_n|, \bar{g}^{-(2-\delta)}|\mu_n|, \bar{g}^{-(11/4-\eta)}|||R_n|||_n\}$$

where $\delta, \eta > 0$ are very small numbers and $0 < \nu < 1/2$.

Let $E_n(r) \subset E_n$ be an open ball of radius r centered at the origin. Let $v_n \in E_n(1)$. Then our first theorem says

$$|\xi_n(v_n)| \leq C_L \bar{g}^{11/4-\eta}, \quad |\rho_n(v_n)| \leq C_L \bar{g}^{11/4-\eta}$$

These are estimates for the error terms in the g_n, μ_n flow. Moreover $R_{n+1} = U_{n+1}(v_n)$ has the bound

$$|||U_{n+1}(v_n)|||_{n+1} \leq L^{-1/4} \bar{g}^{11/4-\eta}$$

On the right hand side we have a contraction factor. We also have Lipshitz continuity in the above norms with the same constants.

Existence of stable manifold

We consider a Banach **E** space of sequences $\mathbf{s} = \{v_n\}_{n \geq 0}$, each $v_n \in E_n$, supplied with the norm

$$||\mathbf{s}|| = \sup_{n \geq 0} ||v_n||$$

$\mathbf{E}(r) \subset \mathbf{E}$ is an open ball of radius r. Let $v_0 = (\tilde{g}_0, \mu_0, 0)$. The next theorem states that if $\tilde{g}_0$ is held in a sufficiently small open ball then there exists a Lipshitz continuous critical mass $\mu_c(\tilde{g}_0)$ such that the trajectory $u_{n+1} = f_{n+1}(u_n)$ is uniformly bounded in the norm on the space of sequences.

Main Theorem:

Let L be sufficiently large, ν be sufficiently small depending on L, then ϵ sufficiently small depending on L. Let $v_0 \in E_0(1/32)$. Let $\tilde{g}_0 \in U_0(r) \subset \mathbb{R}$ where $U_0(r)$ is an open ball of radius r sufficiently small. Then there is a Lipshitz continuous function $h : U_0(r) \to \mathbb{R}$ such that if $\mu_0 = h(\tilde{g}_0)$ then there is a sequence $\mathbf{s} = \{v_n\}_{n \geq 0}$ in $\mathbf{E}(1/4)$ satisfying $v_{n+1} = f_{n+1}(v_n)$ for all $n \geq 0$. The stable manifold is the graph $W_S = \{\tilde{g}_0, h(\tilde{g}_0)\}$.

Corollary: the theorem implies $v_n \in E_n(1/4)$ for all $n \geq 0$ and hence $|\tilde{g}_n| = |g_n - \bar{g}| \leq 1/4\nu\bar{g}$. Since $0 < \nu < 1/2$, it follows that g_n is non-vanishing at all scales.

Proof of the theorem is done in 2 steps inorder to circumvent lattice artifacts.

First we iterate the RG map a large (but finite) number of times. We come to a high scale (sufficiently fine lattice). We are then able to prove the existence of the stable manifold starting at this scale using a fixed point argument on the Banach space of sequences plus an argument of Schub. Exponential convergence of finite range fluctuation covariances to the continuum plays an important role.

Stable manifold has been established at a high scale. Now use the Banach space implicit function theorem to prove that there exists a unique unit lattice critical mass which is a C^1 function of $\tilde{g}_0$ in a small enough neigborhood such that after a finite number of iterations we arrive at the data of the stable manifold at the high scale.