The Casimir effect and one-dimensional scattering

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What is Quantum Field Theory? , Benasque, SPAIN, 2011

Sanctissimum est meminisse cui te dedeas.

Introduction

•Vacuum energy for arbitrary geometry: M. Bordag *et al* "New developments in the Casimir effect" (Phys.Rept. 353 (2001) 1-205) and "Advances in the Casimir effect" (Oxford Univ. Pr., 2009). E. Elizalde and A. Romeo J.Math.Phys. 30 (1989) 1133. O. Kenneth and I. Klich Phys.Rev.Lett. 97 (2006) 160401, and Phys. Rev. B 78, 014103 (2008). T. Emig, R.L. Jaffe *et al* J.Phys.A A41 (2008) 164001, Phys.Rev. D77 (2008) 025005, Phys.Rev.Lett. 99 (2007) 170403. D. V. Vassilevich Phys.Rept. 388 (2003) 279-360.

•Experimental results:J.N. Munday, F. Capasso, and V Adrian Parsegian, "Measured long-range repulsive Casimir-Lifshitz forces", Nature 457, 170-173 (2009).

•Boundary conditions: M. Asorey, A. Ibort and G. Marmo, Int.J.Mod.Phys. A20 (2005) 1001-1026.

The scalar Casimir effect

One real field.

- Fluctuations of 1D scalar fields on classical backgrounds

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}U(x)\Phi^2(x,t) \ , \quad \lim_{x \pm \infty} U(x) = 0 \ , \ \int_{-\infty}^{\infty} dx\, U(x) < +\infty$$

$$\Phi(t,x) = \int_{\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\phi_\omega(x) \ , \quad -\phi''_\omega(x) + U(x)\phi_\omega(x) = \omega^2\phi_\omega(x)$$

$$\left(-\omega^2 - d^2/dx^2 + U(x)\right) G_\omega^{(U)}(x,x') = \delta(x - x')$$

- Fluctuation vacuum energy.

$$E_V = \sum \omega - \sum \omega_0 = \sum_{j=1}^{N} \omega_j + \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi} k \left[\frac{d\delta_+}{dk} + \frac{d\delta_-}{dk}\right] \ , \quad \rho_{S_0} = \frac{L}{2\pi}$$

$$\rho_S(k) - \rho_{S_0} = \frac{1}{4\pi}\left[\frac{d\delta_+}{dk} + \frac{d\delta_-}{dk}\right] = \frac{1}{\pi}\int_{-L}^{L} dx\, \mathrm{Im}\left[G_\xi^{(U)}(x,x) - G_\xi^{(0)}(x,x)\right], \ \xi = i\omega$$

The scalar Casimir effect

Two real fields

- Two real scalar fields fluctuations with SUSY QM Fourier components:

$$\mathcal{L} = \frac{1}{2}\partial\Phi^\dagger_\mu \partial^\mu \Phi - \frac{1}{2}\Phi^\dagger \mathbf{U}\Phi; \quad \Phi \equiv \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix};$$

$$\mathbf{U} \equiv \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}; \quad \mathbf{Q} = i\begin{pmatrix} 0 & \frac{d}{dx} + \frac{dW}{dx} \\ 0 & 0 \end{pmatrix};$$

$$\{\mathbf{Q}, \mathbf{Q}^\dagger\} = \mathbf{H} = \begin{pmatrix} -\frac{d^2}{dx^2} + U_+ & 0 \\ 0 & -\frac{d^2}{dx^2} + U_- \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

$$U_\pm(x) = \pm\frac{d^2W}{dx^2} + \left(\frac{dW}{dx}\right)^2; \; H_\pm \phi_\omega^\pm(x) = \omega^2 \phi_\omega^\pm(x)$$

- Fluctuation vacuum energy.

$$\begin{aligned} E_V &= \sum \left(\omega^{(+)} + \omega^{(-)}\right) - 2\sum \omega_0 = 2\sum_{j=1}^{N} \omega_j \\ &+ \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi} k \frac{d}{dk}\left[\delta_+^{(+)} + \delta_-^{(+)} + \delta_+^{(-)} + \delta_-^{(-)}\right] \end{aligned}$$

TGTG formula for vacuum energy

- Compact objects in one dimension.

$$U(x) = U_1(x) + U_2(x)$$

$U_i(x)$ smooth functions with disjoint compact supports on the real line.

- The T operator associated to a potential $U(x)$.

$$G_\omega^{(U)}(x,x') = G_\omega^{(0)}(x,x') - \int dx_1 dx_2 G_\omega^{(0)}(x,x_1) T_\omega^{(U)}(x_1,x_2) G_\omega^{(0)}(x_2,x'),$$

- *TGTG* formula for the vacuum interaction energy.

$$\begin{aligned} E_0^{\rm int} &= -\frac{i}{2}\int \frac{d\omega}{2\pi} \operatorname{tr}\ln\left(\mathbf{1} - \mathcal{M}_\omega\right) \\ \mathcal{M}_\omega &= \mathcal{G}_\omega^{(0)} \mathcal{T}_\omega^{(1)} \mathcal{G}_\omega^{(0)} \mathcal{T}_\omega^{(2)} \\ M_\omega(x,x') &= \int dx_1 dx_2 dx_3 G_\omega^{(0)}(x,x_1) T_\omega^{(1)}(x_1,x_2) G_\omega^{(0)}(x_2,x_3) T_\omega^{(2)}(x_3,x') \end{aligned}$$

Kenneth and Klich Phys. Rev. B 78, 014103 (2008). Bordag *et al* “Advances in the Casimir effect” Oxford, UK: Oxford Univ. Pr. (2009).

Two double delta systems

- Two plates or two deltas: mimicking the Casimir effect

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left(\alpha\delta(x+a) + \beta\delta(x-a)\right)\Phi^2(x,t)$$

$$\left[-\frac{d^2}{dx^2} + \alpha\delta(x+a) + \beta\delta(x-a)\right]\phi_\omega(x) = \omega^2 \phi_\omega(x)$$

- Two plates or two deltas: SUSY interacting fluctuations:

$$\mathcal{L} = \frac{1}{2}\partial \Phi^\dagger_\mu \partial^\mu \Phi - \frac{1}{2}\Phi^\dagger \mathbf{U}\Phi; \quad W = \frac{1}{2}\left(\alpha\epsilon(x+a) + \beta\epsilon(x-a)\right)$$

$$U_\pm(x) = \pm(\alpha\delta(x+a) + \beta\delta(x-a)) + \frac{1}{2}\alpha\beta\epsilon(x+a)\epsilon(x-a) + \frac{\alpha^2+\beta^2}{4}$$

2-δ potential: scattering I.

- The potential

$$U(x) = \alpha\delta(x+a) + \beta\delta(x-a)$$

- Scattering zones: Zone II : $x < -a$, Zone I : $-a < x < a$, Zone III : $x > a$

- Delta conditions: continuity of ψ and finite step discontinuity of ψ'

$$\psi(\pm a_<) = \psi(\pm a_>) \ , \quad \psi'(\pm a_<) - \psi'(\pm a_>) = \lim_{\delta \to 0} \int_{\pm a - \delta}^{\pm a + \delta} dx\, U(x)\psi(x)$$

$$\psi'(-a_<) - \psi'(-a_>) = \alpha\psi(-a) \ , \quad \psi'(a_<) - \psi'(a_>) = \beta\psi(a)$$

- Scattering states (right-to-left and left-to-right), $\forall k \in \mathbb{R}^+$

$$\psi_k^r(x) = \begin{cases} e^{-ikx}\rho_r + e^{ikx} & , \quad x \in \mathrm{II} \\ A_r e^{ikx} + B_r e^{-ikx} & , \quad x \in \mathrm{I} \\ e^{ikx}\sigma_r & , \quad x \in \mathrm{III} \end{cases} \qquad \psi_k^l(x) = \begin{cases} e^{-ikx}\sigma_l & , \quad x \in \mathrm{II} \\ A_l e^{ikx} + B_l e^{-ikx} & , \quad x \in \mathrm{I} \\ e^{ikx}\rho_l + e^{-ikx} & , \quad x \in \mathrm{III} \end{cases}$$

2-δ potential: scattering II.

- Left-to-right (*diestro*) scattering amplitudes

$$\rho_r = -\frac{ie^{-2iak}\left(\beta e^{4iak}(2k - i\alpha) + \alpha(2k + i\beta)\right)}{\Delta(k;\alpha,\beta,a)},$$

$$\sigma_r = \frac{4k^2}{\Delta(k;\alpha,\beta,a)}, \; A_r = \frac{2k(2k+i\beta)}{\Delta(k;\alpha,\beta,a)}, \; B_r = -\frac{2ik\beta e^{2iak}}{\Delta(k;\alpha,\beta,a))}$$

- Right-to-left(*zurdo*) scattering amplitudes

$$\rho_l = -\frac{ie^{-2iak}\left(\alpha e^{4iak}(2k - i\beta) + \beta(2k + i\alpha)\right)}{\Delta(k;\alpha,\beta,a)},$$

$$\sigma_l = \frac{4k^2}{\Delta(k;\alpha,\beta,a)}, \; A_l = -\frac{2ik\alpha e^{2iak}}{\Delta(k;\alpha,\beta,a)}, \; B_l = \frac{2k(2k+i\alpha)}{\Delta(k;\alpha,\beta,a)},$$

- Denominator of scattering amplitudes:

$$\Delta(k;\alpha,\beta,a) = \alpha\beta\left(-1 + e^{4iak}\right) + 4k^2 + 2ik(\alpha+\beta)$$

- Phase shifts and spectral density

$$e^{2i\delta_\pm} = \sigma \pm \sqrt{\rho_l\rho_r} \quad , \quad \rho_S(k) = \frac{1}{2\pi}\frac{d(\delta_+ + \delta_-)}{dk} + \rho_{S_0}$$

2-δ potential: Bound states.

Study the roots of Δ in the positive imaginary axis in terms of $\Lambda = \alpha a$ and $\Gamma = \beta a$. 3 possibilities: no bound states, one bound state and two bound states.

•NO BOUND STATES (real vacuum energy):

- $\alpha, \beta > 0$
- $\alpha \cdot \beta < 0$ and $-2a < \frac{\alpha+\beta}{\alpha\beta} < 0$

• ONE BOUND STATE (complex vacuum energy):

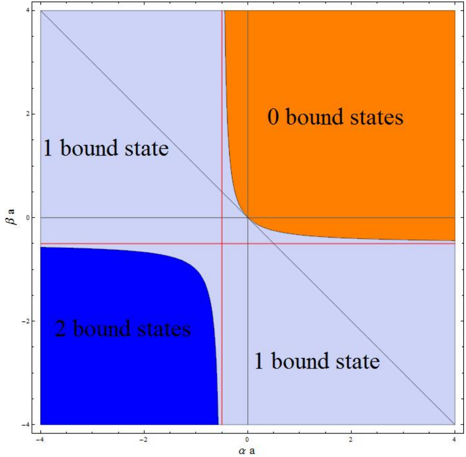
- $\alpha, \beta < 0$ and $\frac{\alpha+\beta}{\alpha\beta} < -2a$
- $\alpha + \beta < 0$ and $\alpha \cdot \beta < 0$
- $\alpha + \beta > 0$ and $-2a < \frac{\alpha+\beta}{\alpha\beta} < 0$

• TWO BOUND STATES (complex vacuum energy):

- $\alpha, \beta < 0$ and $-2a < \frac{\alpha+\beta}{\alpha\beta} < 0$

2-δ potential: Bound states II.

- The plane $(\alpha \cdot a, \beta \cdot a)$ is divided in three zones by the hyperbola $\frac{\alpha+\beta}{\alpha\beta} = -2a$:



The ultra-strong limit I.

$\beta = \alpha \to \infty$

• For $\alpha = \beta$ ($A_r = B_l$ and $B_r = A_l$):

$$\Delta(k;\alpha,\alpha,a) = \alpha^2\left(-1+e^{4iak}\right) + 4k^2 + 4ik\alpha; \quad \Delta_2 \equiv \left(-1+e^{4iak}\right)$$

$$\rho = -\frac{i\alpha e^{-2iak}\left(e^{4iak}(2k-i\alpha)+\alpha(2k+i\alpha)\right)}{\Delta(k;\alpha,\alpha,a)},$$

$$\sigma = \frac{4k^2}{\Delta(k;\alpha,\alpha,a)},\ A_r = \frac{2k(2k+i\alpha)}{\Delta(k;\alpha,\alpha,a)},\ B_r = -\frac{2ik\alpha e^{2iak}}{\Delta(k;\alpha,\alpha,a)}$$

• Ultra-strong limit for arbitrary $k > 0$

$$\lim_{\alpha\to\infty}\rho = -e^{-2iak};\ \lim_{\alpha\to\infty}\sigma = \lim_{\alpha\to\infty}A_r = \lim_{\alpha\to\infty}B_r = 0$$

• There are no quantum fluctuations between plates in the ultra-strong limit for arbitrary $k > 0$.

The ultra-strong limit II.

Unitary QFT between plates

- Non-trivial ultra-strong limit:

$$\Delta_2(k,a) = 0 \Rightarrow k_n = \frac{\pi}{2a} n \quad , \quad n \in \mathbb{Z}^+$$

Δ_2 is the Dirichlet spectral function obtained by Asorey, Munoz-Castaneda *et al*.

- For $k_n \in \ker(\Delta_2)$ and $\alpha = \beta = \infty$:

$$A_r(k_n) = 1/2 = B_l(k_n) \quad , \quad B_r(k_n) = -\frac{e^{2iak_n}}{2} = A_l(k_n)$$

$$\Rightarrow \psi_n(x) \equiv \psi(x,k_n) = -e^{2iak_n}\psi_l(x,k_n) = \frac{1}{2}\left(e^{ik_n x} - e^{2iak_n}e^{-ik_n x}\right)$$

Dirichlet boundary conditions are satisfied by $\psi(x,k_n)$ for all $n \in \mathbb{Z}^+$.
$\psi_{2m+1} \sim \cos\left(k_{2m+1}x\right)$ and $\psi_{2m} \sim \sin\left(k_{2m}x\right)$, $m \in \mathbb{Z}^+$.

- Zeta function prescription for regularized vacuum energy:

$$E_d(s) = \frac{1}{2}\sum_{n=1}^{\infty}\left(n^2\pi^2/(2a)^2\right)^{-s} = \frac{1}{2}\left(\frac{\pi}{2a}\right)^{-2s}\zeta(2s)\,, \quad s \in \mathbb{C}.$$

Physical limit $s = -1$: $E_d = \frac{\pi}{4}\zeta(-1) = -\pi/(48a)$.

TGTG calculation I.

- Vacuum interaction between two deltas: first calculated by Bordag *et al* in , J. Phys. A 25, 4483 (1992). Direct calculation for identical deltas.

- Euclidean propagator of a single delta at $x = 0$ ($k^2 = \xi^2$):

$$G_\xi^{(\alpha)}(x,y) = \begin{cases} G_\xi^{+}(x,y) = -\frac{e^{-k|x-y|}}{(2k)} - \frac{\alpha}{2k+\alpha}\frac{e^{-k(|x|+|y|)}}{2k}, & \mathrm{sgn}(x) = \mathrm{sgn}(y) \\ G_\xi^{-}(x,y) = \frac{e^{-k|x-y|}}{2k+\alpha}, & \mathrm{sgn}(x) \neq \mathrm{sgn}(y) \end{cases}$$

- T-operator of a single delta potential placed at $x = 0$:

$$T_\xi^{(\alpha)}(x,y) = \delta(x)\delta(y)\frac{2k\alpha}{2k+\alpha}$$

- One delta at $x = -a$ with weight α and another at $x = a$ with weight β:

$$\mathrm{tr}\ln(1 - \mathcal{M}_\xi^{(\alpha,\beta)}) = \ln\left(1 - \frac{\alpha\beta e^{-4ka}}{(2k+\alpha)(2k+\beta)}\right)$$

TGTG calculation II.

- Casimir energy and force

$$E_{\rm int}(\alpha,\beta;a) = \frac{1}{2\pi}\int_0^\infty dk \ln\left(1-\frac{\alpha\beta e^{-4ka}}{(2k+\alpha)(2k+\beta)}\right)$$

$$F_{\rm int}(\alpha,\beta;a) = -\frac{1}{2}\frac{dE_{\rm int}(a)}{da} = -\frac{\alpha\beta}{4\pi}\int_0^\infty dk \frac{4k}{e^{4ak}(2k+\alpha)(2k+\beta)-\alpha\beta}$$

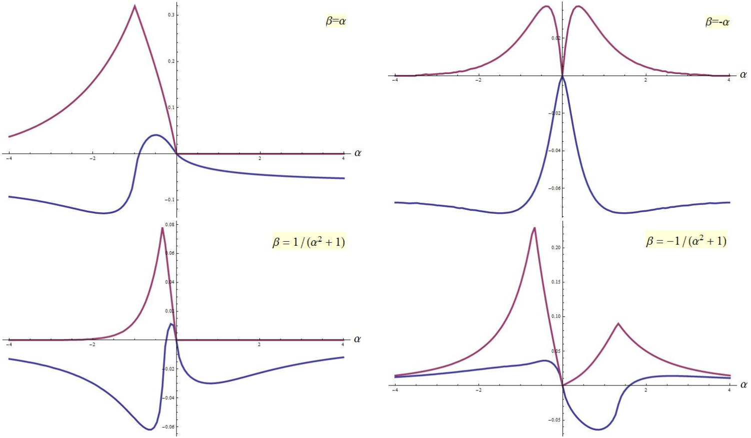
- Ultra-strong limit $\alpha=\beta=\infty$:

$$E_{\rm int}^\infty(a) = \frac{1}{2\pi}\int_0^\infty dk \ln\left(1-e^{-4ka}\right) = -\frac{\pi}{24}\frac{1}{2a}$$

$$F_{\rm int}^\infty(a) = -\frac{\alpha^2}{4\pi}\int_0^\infty dk \frac{4k}{e^{4ak}(2k+\alpha)^2-\alpha^2} = \frac{\pi}{24}\frac{1}{4a^2}$$

TGTG calculation III.

Some plots of the energy.



$\mathrm{re}(E_{\mathrm{int}}) = \bullet$; $\mathrm{im}(E_{\mathrm{int}}) = \bullet$

The SUSY $\delta - \delta$ spectrum.

Scattering amplitudes and bound states.

- Scattering amplitudes for the $U_+(x)$ problem:

$$\rho_r = \frac{e^{-2ia(k+q)}\left(e^{4iaq}(k+q-i\alpha)(k-q+i\beta)-(k-q-i\alpha)(k+q+i\beta)\right)}{\Delta}$$

$$\rho_l = \frac{e^{-2ia(k+q)}\left(e^{4iaq}(k-q+i\alpha)(k+q-i\beta)-(k+q+i\alpha)(k-q-i\beta)\right)}{\Delta}$$

$$\sigma_r = \sigma_l = -\frac{4kqe^{-2iak}}{\Delta}$$

$$\Delta = e^{2iaq}(k-q+i\alpha)(k-q+i\beta)-e^{-2iaq}(k+q+i\alpha)(k+q+i\beta)$$

$E=\omega^2=q^2+(\alpha-\beta)^2=k^2+(\alpha+\beta)^2$. $U_-(x)\Rightarrow(\alpha,\beta)\mapsto(-\alpha,-\beta)$

- Identical deltas: $\alpha=\beta$. Bound states given by ($k=i\mu$)

$$\cot(qa)-\tan(qa)=\frac{q}{\alpha+\mu}-\frac{\alpha+\mu}{q}$$

- Ultra-strong limit: $\alpha=\beta\to\infty$. Non zero zero quantum fluctuations between plates

$\Leftrightarrow \quad \sin(2aq)=0$

The single δ step I.

Scattering amplitudes and propagator.

- Single δ step potential ($s\delta$ potential) centered at $x = 0$:

$$U(x) = U_{s\delta}(x; \alpha, s) = \alpha\delta(x) + s^2\left(1 - \theta(x)\right)$$

Scattering amplitudes and Wronskian

$$\sigma_r = \frac{2q}{\Delta_{s\delta}};\ \sigma_l = \frac{2k}{\Delta_{s\delta}} \quad ; \quad \rho_r = -1 + \sigma_r;\ \rho_l = -1 + \sigma_l;$$

$$W(\psi_r, \psi_l) = \frac{4ikq}{\Delta_{s\delta}} \quad ; \quad \Delta_{s\delta} \equiv k + q + i\alpha$$

k and q are the momenta in the zones $x > 0$ and $q < 0$: $E = \omega^2 = k^2 = q^2 + s^2$

- Reduced propagator:

$$G_\omega^{++}(x, y) = -\frac{e^{ik|x-y|}}{2ik} - \frac{e^{ik(x+y)}}{2ik}\left(-1 + \frac{2k}{\Delta_{s\delta}}\right)$$

$$G_\omega^{--}(x, y) = -\frac{e^{iq|x-y|}}{2iq} - \frac{e^{iq(x+y)}}{2iq}\left(-1 + \frac{2q}{\Delta_{s\delta}}\right)$$

$$G_\omega^{+-}(x, y) = i\frac{e^{i(kx-qy)}}{\Delta_{s\delta}};\ G_\omega^{-+}(x, y) = i\frac{e^{i(ky-qx)}}{\Delta_{s\delta}}$$

The single δ step I.

T-operator.

- For 1 dimensional systems $T_\omega(x,y) = V(x)\delta(x-y) - V(x)G_\omega(x,y)V(y)$. Calculation for the $s\delta$ potential:

$$T_\omega^{++}(x,y) = \left(1 + \frac{i\alpha^2}{\Delta_{s\delta}}\right)\delta(x)\delta(y);$$

$$T_\omega^{--}(x,y) = \left(1 + \frac{i\alpha^2}{\Delta_{s\delta}}\right)\delta(x)\delta(y) + \frac{i\alpha s^2}{\Delta_{s\delta}}\left(e^{-iqy}\delta(x) + e^{-iqx}\delta(y)\right);$$

$$T_\omega^{+-}(x,y) = \frac{i\alpha^2}{\Delta_{s\delta}}\delta(x)\delta(y) + \frac{i\alpha s^2}{\Delta_{s\delta}}e^{-iqy}\delta(x);$$

$$T_\omega^{-+}(x,y) = \frac{i\alpha^2}{\Delta_{s\delta}}\delta(x)\delta(y) + \frac{i\alpha s^2}{\Delta_{s\delta}}e^{-iqx}\delta(y)$$

- Kernel of operator $\mathcal{M}_\omega$:

$$M_\omega(x,y) = \int dz N_\omega^{(1)}(x,z)N_\omega^{(2)}(z,y); \quad N_\omega(x,y) = \int dz G_\omega^{(0)}(x,z)T\omega(z,y)$$

The single δ step II.

N-operator.

- N-operator can be computed:

$$
\begin{aligned}
N_\omega^{++}(x,y) &= -\frac{1}{2ik}\left[\frac{1}{2}+\frac{i\alpha^2}{\Delta_{s\delta}}-\frac{\alpha s^2}{(k+q)\Delta_{s\delta}}\right] e^{ikx}\delta(y)\\
N_\omega^{--}(x,y) &= -\frac{1}{2ik}\left[\left(\frac{1}{2}+\frac{i\alpha^2}{\Delta_{s\delta}}-\frac{\alpha s^2}{(q-k)\Delta_{s\delta}}\right) e^{-ikx}\right.\\
&+ \left.\frac{2k\alpha s^2}{(q^2-k^2)\Delta_{s\delta}} e^{-iqx}\right]\delta(y)-\frac{\alpha s^2}{2k\Delta_{s\delta}} e^{ikx}e^{-iqy}\\
N_\omega^{+-}(x,y) &= -\frac{1}{2ik}\left[\frac{1}{2}+\frac{i\alpha^2}{\Delta_{s\delta}}-\frac{\alpha s^2}{(k+q)\Delta_{s\delta}}\right] e^{ikx}\delta(y)\\
&- \frac{\alpha s^2}{2k\Delta_{s\delta}} e^{ikx}e^{-iqy}\\
N_\omega^{-+}(x,y) &= -\frac{1}{2ik}\left[\left(\frac{1}{2}+\frac{i\alpha^2}{\Delta_{s\delta}}-\frac{\alpha s^2}{(q-k)\Delta_{s\delta}}\right) e^{-ikx}\right.\\
&+ \left.\frac{2k\alpha s^2}{(q^2-k^2)\Delta_{s\delta}} e^{-iqx}\right]\delta(y)
\end{aligned}
$$

SUSY $\delta - \delta$ vacuum energy.

Calculation prescription.

- Prescription for vacuum energy calculation between two delta steps:

$$U(x) = U_{s\delta}(x+a;\alpha,s) + U_{s\delta}(-(x-a);\beta,s)$$

- For each delta step the N operator is easily obtained:

$$N_\omega^{(\alpha)}(x,y) = N_\omega^{s\delta}(x+a,y+a;\alpha,s) \quad ; \quad N_\omega^{(\beta)}(x,y) = N_\omega^{s\delta}(-x+a,-y+a;\beta,s)$$

- Operator $\mathcal{M}_\omega$ and power traces are computed using the general expression for the kernel:

$$M_\omega(x,y;s,a,\alpha,\beta) = \int dz\, N_\omega^{s\delta}(x+a,z+a;\alpha,s) N_\omega^{s\delta}(-z+a,-y+a;\beta,s)$$

- First order approximation to the vacuum energy (euclidean):

$$E_0^{\text{int}} \simeq \frac{1}{2}\int_s^\infty \frac{d\xi}{2\pi}\int dx\, M_{i\xi}(x,x;s,a,\alpha,\beta) + O\left(M_{i\xi}^2\right)$$

- SUSY restriction for $\alpha = \beta$: $s^2 = \alpha^2$.

Conclusions and outlook.

- New paths for obtaining quantum field theories in bounded domains has been explored.
- Potentials concentrated on hypersurfaces allow to use the *TGTG* method to compute the vacuum energy.
- Integral expressions for the vacuum energy have been obtained for arbitrary values of the delta weights in the double delta potential. For the case of the SUSY double delta the vacuum energy can easily be obtained from the calculations done.

- Which boundary conditions can be implemented using potentials concentrated on hypersurfaces? $\Rightarrow \delta + \delta'$ systems.
- Extension of the *TGTG* to curved backgrounds: double δ in a kink background (double δ in a Posch-Teller background).

MANOLO, ANIVERSARIA DIES FELIX!