Bogoliubov theory of disordered Bose-Einstein condensates

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Abstract

The interplay of interaction, disorder, and Bose statistics is a long standing problem of condensed matter physics, known as the ”dirty boson problem”. Here, we present a Bogoliubov theory for disordered Bose-Einstein condensates, i.e., the bosonic field operator is split into the (mean field) condensate and (quantum) fluctuations. The mean-field part consists in solving the Gross-Pitaevskii equation describing the deformed condensate wave function. The condensate, in turn, determines the Hamiltonian for the quantum fluctuations.

Diagonalizing this Bogoliubov Hamiltonian is a difficult task. As it is not desirable anyway to solve the problem for a particular realization of disorder, we resort to disorder perturbation theory in terms of Green functions and compute quantities like the disorder-averaged sound velocity or the mean free path of Bogoliubov excitations.

Beyond that, the Bogoliubov theory is used to count the number of particles that are excited out of the condensate, even at zero temperature. This depletion of the condensate is shown to remain small in presence of disorder, which validates a posteriori the Bogoliubov ansatz.

References:

C. Gaul & C.A. Müller, Phys. Rev. A, 83, 063629 (2011)

C.A. Müller & C. Gaul, New J. Phys. 14 075025 (2012)

Bogoliubov theory of disordered Bose-Einstein condensates

Outline

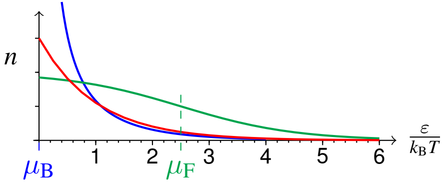
- Bose statistics + Interaction + Disorder
  $\rightarrow$ "Dirty Boson Problem"
- Experiments with ultracold quantum gases
- How is Bose-Einstein condensation affected by disorder?
  - How to define the condensate in presence of inhomogeneity?
  - Fraction of non-condensed particles
- How are the elementary excitations affected by disorder?

Bose statistics

- Classical:

- Bosons: indistinguishable, symmetric wf.: $\hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle$
- Indistinguishable bosons tend to cluster

- $n_\mathrm{B}(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon - \mu_\mathrm{B}}{k_\mathrm{B} T}\right) - 1}$



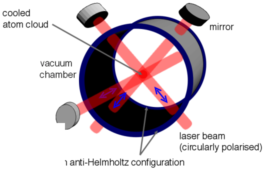
- Fermi: $n_\mathrm{F}(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon - \mu_\mathrm{F}}{k_\mathrm{B} T}\right) + 1}$
- Boltzmann: $n(\varepsilon) \propto e^{-\frac{\varepsilon}{k_\mathrm{B} T}}$

- n_B diverges for $\varepsilon \to \mu_\mathrm{B} \Rightarrow$ "Bose-Einstein" condensation (BEC)
  If: thermal de-Broglie wavelength $\sim$ average particle distance

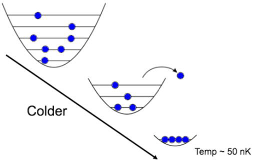
BEC in experiments

How to reach critical phase space density

- Magneto-optical trapping
- Doppler cooling
- Evaporative cooling

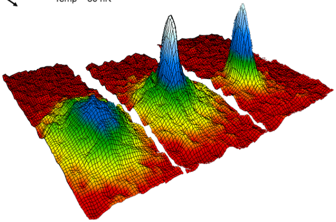


[wikipedia.org]



How to see it

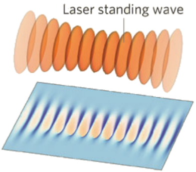
- Time-of-flight imaging
  $\rightarrow$ momentum-space density:
  $\rightarrow$ macroscopic occupation
  of single-particle orbital $\Phi(\boldsymbol{r})$



[www.colorado.edu/physics/2000/bec/]

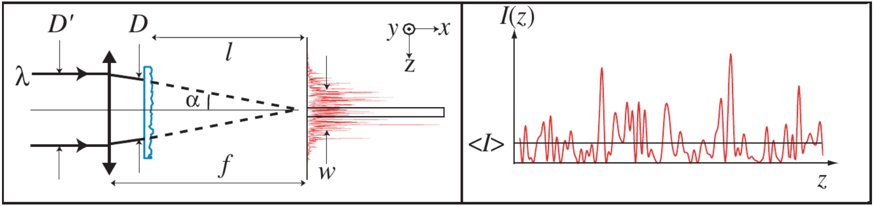
Optical potentials

Optical lattices:



[Greiner *et al.*, nature (2008)]

Speckle disorder



Well-known statistics: $\overline{V_{\mathbf{k}} V_{-\mathbf{k}'}} = \delta_{\mathbf{k}\mathbf{k}'} R(k)$

[Clément et al., New J. Phys., **8**, 165 (2006)]

Penrose-Onsager criterion

- Starting point: bosonic many-body Hamiltonian

$$E[\hat{\Psi}, \hat{\Psi}^\dagger] = \int \mathrm{d}^d r \, \hat{\Psi}^\dagger(\boldsymbol{r}) \left[\frac{-\hbar^2}{2m} \nabla^2 + V(\boldsymbol{r}) + \frac{g}{2} \hat{\Psi}^\dagger(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}) - \mu \right] \hat{\Psi}(\boldsymbol{r})$$

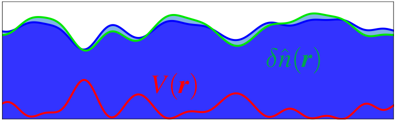
- One-body denity matrix (OBDM): $\langle \hat{\Psi}^\dagger(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}') \rangle$
- BEC: many particles occupy condensate orbital
- Penrose & Onsager (1956): $\int \mathrm{d}^d r' \underbrace{\langle \hat{\Psi}^\dagger(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}') \rangle}_{\text{OBDM}} \Phi(\boldsymbol{r}') = N_\mathrm{c} \Phi(\boldsymbol{r})$
  - Condensate $\Phi(\boldsymbol{r})$
  - Number of condensed particles $N_\mathrm{c} \gg 1$

Mean field and Bogoliubov theory

Condensate and quantum fluctuations

$$\hat{\Psi}(\boldsymbol{r}) = \Phi(\boldsymbol{r}) + \delta\hat{\psi}(\boldsymbol{r}, t)$$

$|\Phi(\boldsymbol{r})|^2$

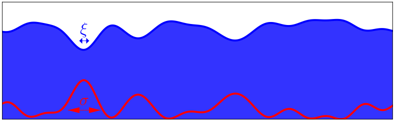


Meanfield: Minimize $E[\Phi] \rightarrow$ Gross-Pitaevskii equation

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + g|\Phi(\boldsymbol{r})|^2 + V(\boldsymbol{r}) - \mu\right]\Phi(\boldsymbol{r}) = 0$$

$$\left.\begin{array}{ll} \text{interaction:} & g\overline{|\Phi(\boldsymbol{r})|^2} = g n_c \\ \text{kinetic energy:} & \hbar^2 k^2/2m = \hbar^2/2m\xi^2 \end{array}\right\} \Rightarrow \xi^2 = \hbar^2/(2mgn_c)$$

$\xi \ll \sigma$



Effective Hamiltonian for quantum fluctuations

$$E[\Phi+\delta\hat{\psi}] \approx E[\Phi] + \underbrace{\frac{1}{2}\int \mathrm{d}^d r\, \mathrm{d}^d r' \left(\delta\hat{\psi}^\dagger(\boldsymbol{r}'), \delta\hat{\psi}(\boldsymbol{r}')\right) \mathcal{H}(\boldsymbol{r}',\boldsymbol{r}) \begin{pmatrix} \delta\hat{\psi}(\boldsymbol{r}) \\ \delta\hat{\psi}^\dagger(\boldsymbol{r}) \end{pmatrix}}_{\hat{H}}$$

$$\mathcal{H} = \delta(\boldsymbol{r}-\boldsymbol{r}') \left\{ \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) - \mu \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + g \begin{pmatrix} |\Phi(\boldsymbol{r})|^2 & \frac{1}{2}\Phi(\boldsymbol{r})^2 \\ \frac{1}{2}\Phi^*(\boldsymbol{r})^2 & |\Phi(\boldsymbol{r})|^2 \end{pmatrix} \right\}$$

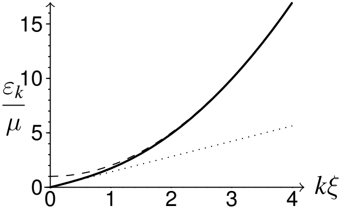
- In terms of density and phase: $\Phi(\boldsymbol{r}) + \hat{\psi}(\boldsymbol{r}) = e^{\mathrm{i}\delta\hat{\varphi}(\boldsymbol{r})}\sqrt{n_\mathrm{c} + \delta\hat{n}(\boldsymbol{r})}$
- Fourier- & Bogoliubov trafo: “bogolons”

$$\hat{\gamma}_{\boldsymbol{k}} = \delta\hat{n}_{\boldsymbol{k}}/(2a_k\sqrt{n_\mathrm{c}}) + \mathrm{i}a_k\sqrt{n_\mathrm{c}}\delta\hat{\varphi}_{\boldsymbol{k}} \qquad a_k = \sqrt{\varepsilon^0_k/\varepsilon_k}$$

$$\hat{H} = \sum_{\boldsymbol{k}} \varepsilon_k \hat{\Gamma}^\dagger_{\boldsymbol{k}}\hat{\Gamma}_{\boldsymbol{k}} + \sum_{\boldsymbol{k},\boldsymbol{k}'} \hat{\Gamma}^\dagger_{\boldsymbol{k}} \mathcal{V}_{\boldsymbol{k}\boldsymbol{k}'} \hat{\Gamma}_{\boldsymbol{k}'}, \qquad \hat{\Gamma}_{\boldsymbol{k}} = \begin{pmatrix} \hat{\gamma}_{\boldsymbol{k}} \\ \hat{\gamma}^\dagger_{-\boldsymbol{k}} \end{pmatrix}$$

Homogeneous Bogoliubov problem

$$\hat{H}^{(0)} = \sum_{\boldsymbol{k}} \varepsilon_k \hat{\gamma}_{\boldsymbol{k}}^\dagger \hat{\gamma}_{\boldsymbol{k}}$$



$[\xi^2 = \frac{\hbar^2}{2mgn_c}]$

- Bogoliubov dispersion relation

$$\varepsilon_k = \sqrt{\varepsilon_k^0(2gn_c + \varepsilon_k^0)}, \quad \varepsilon_k^0 = \frac{\hbar^2 k^2}{2m}$$

- Condensate depletion

$$\delta n^{(0)} = \frac{1}{L^d} \sum_{\boldsymbol{k}} \langle \delta\hat{\psi}_{\boldsymbol{k}}^\dagger \delta\hat{\psi}_{\boldsymbol{k}} \rangle = \frac{1}{L^d} \sum_{\boldsymbol{k}} v_k^2 \overset{(3D)}{=} \frac{1}{6\sqrt{2}\pi^2} \xi^{-3} \propto \xi^{-d}$$

$$\delta\hat{\psi}_{\boldsymbol{k}} = u_k \hat{\gamma}_{\boldsymbol{k}} + v_k \hat{\gamma}_{-\boldsymbol{k}}^\dagger$$

- Relative depletion $\frac{\delta n^{(0)}}{n_c} \overset{(3D)}{=} \frac{8}{3\sqrt{\pi}} \sqrt{n a_s^3}$

↑ dilute-gas parameter

[Lee, Huang & Yang (1957)]

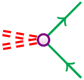
Bogolons in a disordered medium

- Hamiltonian

"Bogoliubov-Nambu spinor"

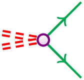
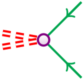
$$\hat{H} = \sum_{\boldsymbol{k}} \varepsilon_k \hat{\Gamma}^\dagger_{\boldsymbol{k}} \hat{\Gamma}_{\boldsymbol{k}} + \sum_{\boldsymbol{k},\boldsymbol{k}'} \hat{\Gamma}^\dagger_{\boldsymbol{k}} \mathcal{V}_{\boldsymbol{k}\boldsymbol{k}'} \hat{\Gamma}_{\boldsymbol{k}'}, \qquad \hat{\Gamma}_{\boldsymbol{k}} = \begin{pmatrix} \hat{\gamma}_{\boldsymbol{k}} \\ \hat{\gamma}^\dagger_{-\boldsymbol{k}} \end{pmatrix}$$

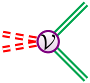
- Vertex $\mathcal{V} = \begin{pmatrix} W & Y \\ Y & W \end{pmatrix}$

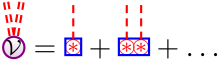
- $W_{\boldsymbol{k}\boldsymbol{p}} \hat{\gamma}^\dagger_{\boldsymbol{k}} \hat{\gamma}_{\boldsymbol{p}} \longrightarrow$  $\quad W^{(1)}_{\boldsymbol{k}\boldsymbol{p}} = w^{(1)}_{\boldsymbol{k}\boldsymbol{p}} V_{\boldsymbol{k}-\boldsymbol{p}}, \; W^{(2)}_{\boldsymbol{k}\boldsymbol{p}} = \dots$

- Anomalous scattering

$$Y_{\boldsymbol{k}',-\boldsymbol{k}} \hat{\gamma}^\dagger_{\boldsymbol{k}'} \hat{\gamma}^\dagger_{\boldsymbol{k}} + Y_{-\boldsymbol{k}',\boldsymbol{k}} \hat{\gamma}_{\boldsymbol{k}'} \hat{\gamma}_{\boldsymbol{k}} \longrightarrow$$

 + 

- Bogoliubov-Nambu vertex  ,



Disorder-averaged effective medium

How do Booliubov quasi-particles travel on average through the disordered medium?

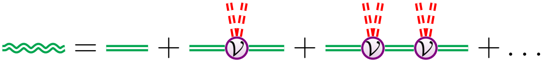
- Matrix-valued (retarded) Green function

$$\mathcal{G}_{\boldsymbol{k}\boldsymbol{k}'}(t) = \frac{\Theta(t)}{i\hbar} \langle [\hat{\Gamma}_{\boldsymbol{k}}(t), \hat{\Gamma}^{\dagger}_{\boldsymbol{k}'}(0)] \rangle ,$$

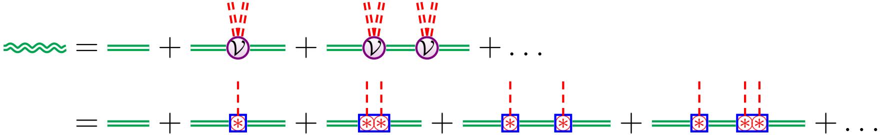
Contains dispersion relation: $[\mathcal{G}_0(\boldsymbol{k},\omega)]_{11} = [\hbar\omega - \varepsilon_k + i0^+]^{-1}$

- Expansion in terms scattering vertex $\mathcal{V} =$  and $\mathcal{G}_0 =$ ═══ :

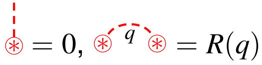
$$\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_0 \mathcal{V} \mathcal{G} \quad \Leftrightarrow$$

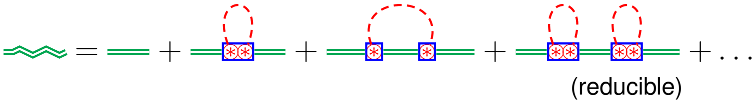


Computing the disorder-averaged Green function

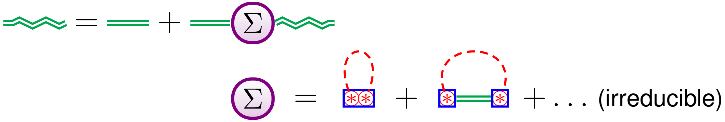


Disorder average:





Dyson equation: self-energy Σ

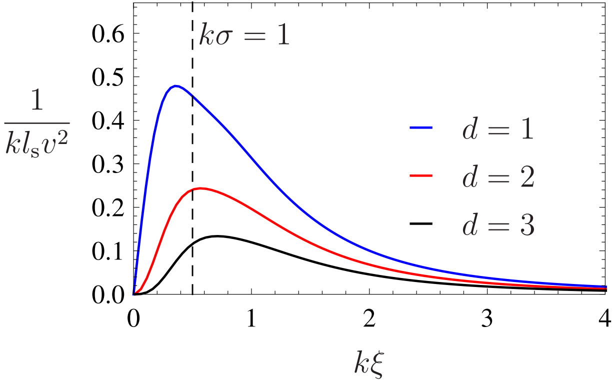


- Renormalized dispersion relation $\hbar\omega = \varepsilon_k + \Sigma_{11}^{(2)}(k,\omega)$
- ImΣ $\to$ finite mean free path

Mean free path

- E.g. Gaussian disorder $\overline{V_{\boldsymbol{q}} V_{-\boldsymbol{q}'}} = L^{-d} \delta_{\boldsymbol{q}\boldsymbol{q}'} \underbrace{(\sqrt{2\pi}\sigma)^d (vgn_c)^2 e^{-\frac{q^2\sigma^2}{2}}}_{R(q)}$

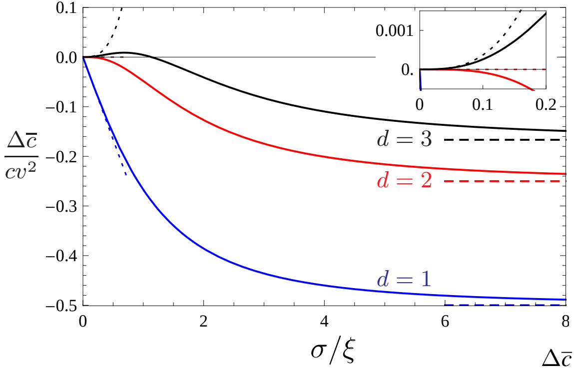
- Finite mean free path $(kl_s)^{-1} \propto |\mathrm{Im}\Sigma|$



- Related to *localization* of Bogoliubov quasiparticles
  [Lugan et al. PRA (2011)]

Disorder-renormalized speed of sound

ReΣ renormalizes sound velocity



$v_\delta^2 = R(0)/(gn_c\xi^d)$

$\Delta\bar{c}/c$	$\sigma \gg \xi$	$\sigma \ll \xi$
$d=1$	$-v^2/2$	$-\frac{3}{16\sqrt{2}}v_\delta^2$
$d=2$	$-v^2/4$	0
$d=3$	$-v^2/6$	$+\frac{5}{48\sqrt{2}\pi}v_\delta^2$ *

* [Giorgini et al., PRB 1994]

Momentum distribution of fluctuations

- To compute: $\delta n_{\boldsymbol{k}} = \langle \delta\hat{\psi}_{\boldsymbol{k}}^\dagger \delta\hat{\psi}_{\boldsymbol{k}} \rangle$
- We have: Hamiltonian for $\hat{\gamma}_{\boldsymbol{k}} = \delta\hat{n}_{\boldsymbol{k}}/(2a_k\sqrt{n_c}) + \mathrm{i}a_k\sqrt{n_c}\delta\hat{\varphi}_{\boldsymbol{k}}$

$$\delta\hat{\psi}(\boldsymbol{r}) = \delta\hat{n}(\boldsymbol{r})/[2\Phi(\boldsymbol{r})] + \mathrm{i}\Phi(\boldsymbol{r})\delta\hat{\varphi}(\boldsymbol{r})$$

- Transformation $\delta\hat{\psi}_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} \left(u_{\boldsymbol{kp}}\hat{\gamma}_{\boldsymbol{p}} - v_{\boldsymbol{kp}}\hat{\gamma}_{-\boldsymbol{p}}^\dagger \right)$, with

$$u_{\boldsymbol{kp}} = \frac{1}{2\sqrt{N_c}}\left[a_p^{-1}\Phi_{\boldsymbol{k}-\boldsymbol{p}} + a_p\check{\Phi}_{\boldsymbol{k}-\boldsymbol{p}}\right], \qquad \check{\Phi}_{\boldsymbol{k}} = [n_c/\Phi(\boldsymbol{r})]_{\boldsymbol{k}}$$

$$v_{\boldsymbol{kp}} = \frac{1}{2\sqrt{N_c}}\left[a_p^{-1}\Phi_{\boldsymbol{k}-\boldsymbol{p}} - a_p\check{\Phi}_{\boldsymbol{k}-\boldsymbol{p}}\right]$$

$$\delta n_{\boldsymbol{k}} = \sum_{\boldsymbol{p},\boldsymbol{p}'}\left\{\delta_{\boldsymbol{pp}'}|v_{\boldsymbol{kp}}|^2 + \left(u_{\boldsymbol{kp}}^* u_{\boldsymbol{kp}'} + v_{\boldsymbol{kp}}^* v_{\boldsymbol{kp}'}\right)\langle\hat{\gamma}_{\boldsymbol{p}}^\dagger\hat{\gamma}_{\boldsymbol{p}'}\rangle - \left(u_{\boldsymbol{kp}}^* v_{\boldsymbol{kp}'}\langle\hat{\gamma}_{\boldsymbol{p}}^\dagger\hat{\gamma}_{-\boldsymbol{p}'}^\dagger\rangle + c.c.\right)\right\}$$

$T = 0$: only due to inhomogeneity $V(\boldsymbol{r})$

homogeneous quantum depletion

Momentum distribution of fluctuations

Pick second-order terms of

$$\delta n_{\boldsymbol{k}} = \sum_{\boldsymbol{p},\boldsymbol{p}'} \Big\{ \delta_{\boldsymbol{p}\boldsymbol{p}'} |v_{\boldsymbol{kp}}|^2 + \big(u^*_{\boldsymbol{kp}} u_{\boldsymbol{kp}'} + v^*_{\boldsymbol{kp}} v_{\boldsymbol{kp}'}\big) \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle - \big(u^*_{\boldsymbol{kp}} v_{\boldsymbol{kp}'} \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}^\dagger_{-\boldsymbol{p}'} \rangle + c.c.\big) \Big\}$$

- $\langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle = \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle^{(0)} + \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle^{(1)} + \langle \hat{\gamma}^\dagger_{\boldsymbol{p}} \hat{\gamma}_{\boldsymbol{p}'} \rangle^{(2)} + \dots$
- $u_{\boldsymbol{kp}} = u^{(0)}_{\boldsymbol{kp}} + u^{(1)}_{\boldsymbol{kp}} + u^{(2)}_{\boldsymbol{kp}} + \dots$

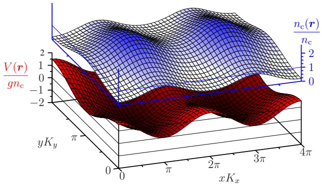
$$\Rightarrow \quad \delta n^{(2)}_{\boldsymbol{k}} = \sum_{\boldsymbol{p}} M^{(2)}_{\boldsymbol{kp}} |V_{\boldsymbol{k}-\boldsymbol{p}}|^2$$

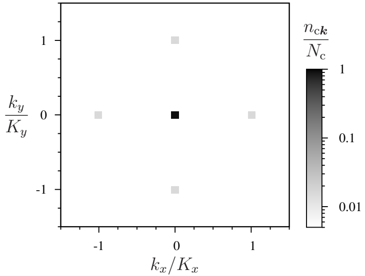
↑ a “monstrous” envelope function

Momentum distribution in a 2D lattice

$$V(\boldsymbol{r}) = \sum_j V_j \cos(\boldsymbol{K}_j \cdot \boldsymbol{r})$$

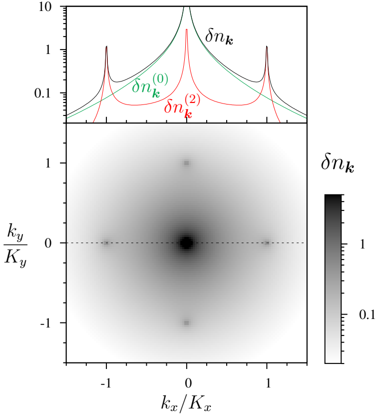
Condensate deformation $|\Phi_k|^2$





Quantum fluctuations $\delta n_{\boldsymbol{k}}$

- "Quantum depletion" $\delta n^{(0)}$
- "Potential depletion" $\delta n^{(2)}$



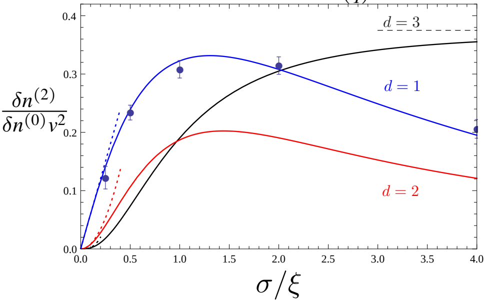
Condensate depletion due to Gaussian disorder

$$\delta n^{(2)} = L^{-d} \sum_{\boldsymbol{k}\boldsymbol{p}} M^{(2)}_{\boldsymbol{k}\boldsymbol{p}} \overline{|V_{\boldsymbol{k}-\boldsymbol{p}}|^2}$$

$$\overline{|V_{\boldsymbol{q}}|^2} = L^{-d} \underbrace{(\sqrt{2\pi}\sigma)^d (vgn_c)^2 e^{-\frac{q^2\sigma^2}{2}}}_{R(q)}$$

$\frac{\delta n^{(2)}}{\delta n^{(0)}}$	$\sigma \gg \xi$	$\sigma \ll \xi$
$d=1$	$-\frac{1}{8}v^2$	$0.245 v_\delta^2$
$d=2$	0	$0.135 v_\delta^2$
$d=3$	$\frac{3}{8}v^2$	$0.160 v_\delta^2$

$$v_\delta^2 = R(0)/(gn_c\xi^d)$$



- $\sigma \ll \xi$: depletion correction scales with $v_\delta^2 \propto R(0)$
- $\sigma \gg \xi$: depletion coincides with local density approximation

$$\frac{\delta n^{(2)}_{\mathrm{TF}}}{\delta n^{(0)}} = \frac{d(d-2)v^2}{8}$$

Take-home messages

- ✓ Hamiltonian for quantum excitations on top of deformed condensate
- ✓ Diagrammatic disorder perturbation theory
  - Mean free path
  - Renormalized speed of sound
- ✓ Calculation of the potential-induced condensate depletion
  - Depletion remains small ⇒ validates Bogoliubov ansatz

References

C. Gaul and C. A. Müller, Phys. Rev. A, 83, 063629 (2011)
C. A. Müller and C. Gaul, New J. Phys. 14 075025 (2012)

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