Level Spacing Distribution in Open Chaotic Systems:a Generalization of Wigner´s Surmise

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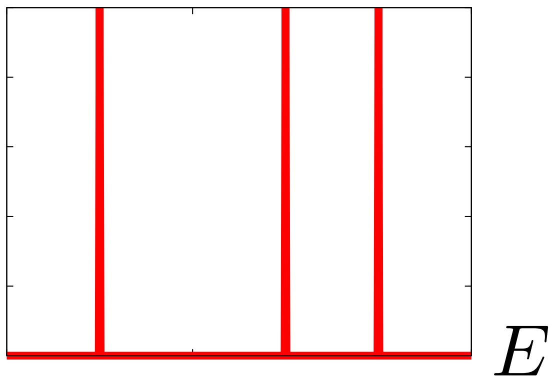
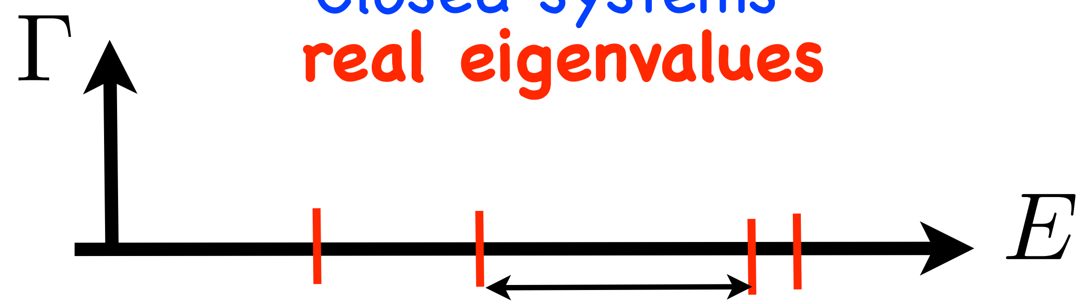
Level spacings for closed and open systems

Spectra



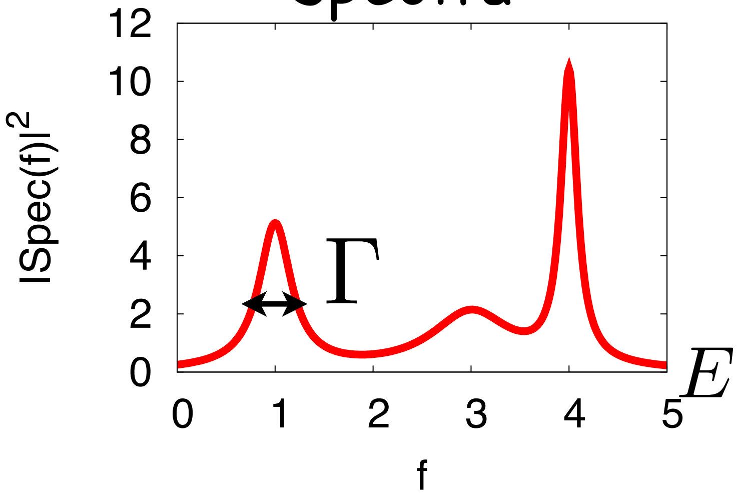
Closed systems

real eigenvalues

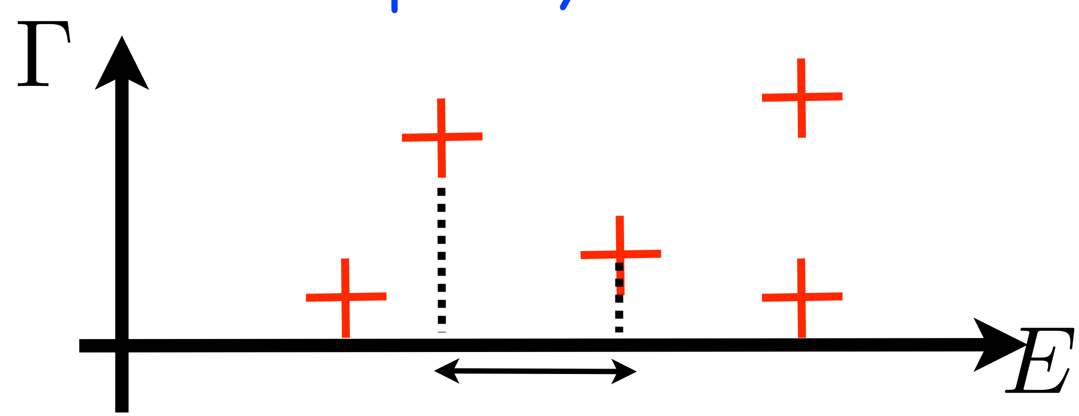


$$s_n = E_{n+1} - E_n$$

Spectra



open systems



$$s_n = E_{n+1} - E_n$$

The resonances lie in the complex plane

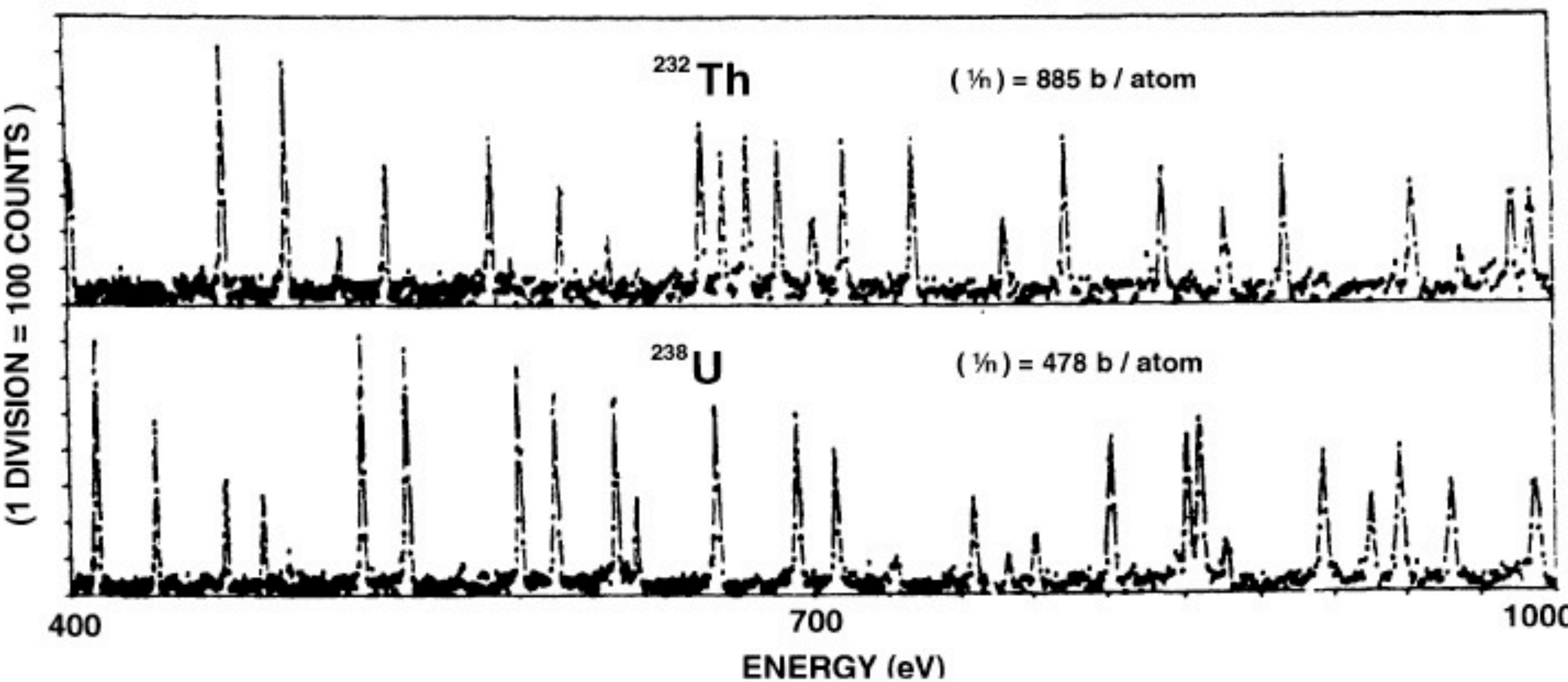


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

Mehta, "Random Matrices"

Outline

Random Matrix Theory & distribution of level spacing

- Notion of complexity in Nuclear Physics and wave physics
- Theory of random Matrices
- Level spacing distribution (Wigner surmise)

Level Spacing Distribution for open systems: 1 channel

- Description via Effective Hamiltonian
- Analytical formula for Distribution for 1 channel
- Effect of coupling to environment on the level distribution

Level Spacing distribution for open systems: M channel

- Numerical simmulations with Random Matrices
- Comparison with experimental data

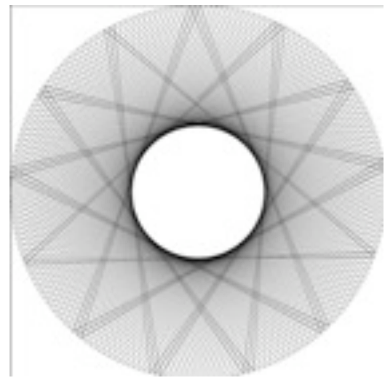
complexity in Nuclear Physics & wave chaotic systems

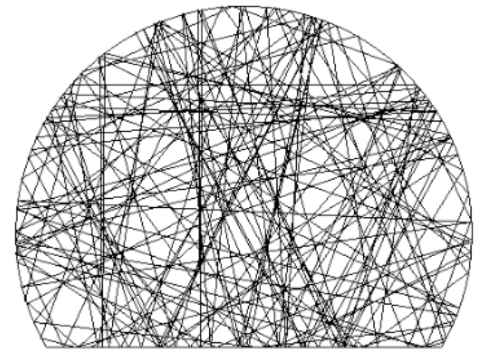
Nuclear Spectroscopy

$$^{A}_{Z}X + n \rightarrow \, ^{A+1}_{Z}X$$

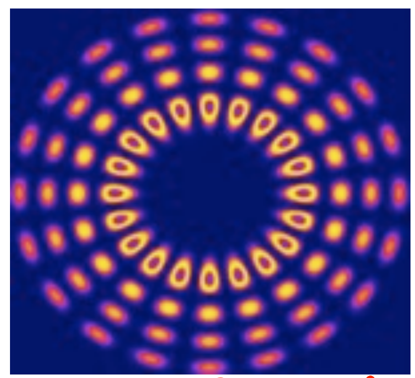
Complexity due to interaction of many degrees of freedom

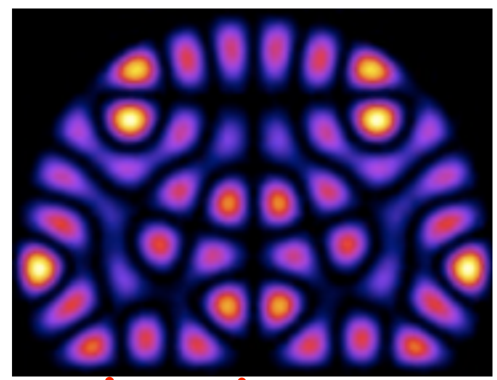
Classical (ray) chaos





Wave counterpart





Complexity due to deterministic chaos

Description of generical statistical properties of these systems?

THEORY OF RANDOM MATRICES

Wigner´s Idea: in Nuclear physics replace deterministic hamiltonian

by random matrix with the same invariance properties

H Hamiltonian $(N \times N)$ with $P(H) \propto \exp(-A\mathrm{Tr}H^2)$

The 3 WIGNER ENSEMBLES

Gaussian Orthogonal Ensemble (GOE) $H = H^T$

Time reversal symmetry

Gaussian Unitary Ensemble (GUE) $H = H^\dagger$

Broken Time reversal Symmetry

Gaussian Symplectic Ensemble (GSE) $H = H^S$

Spin 1/2 with Time Reversal Symmetry

Joint Distribution of Energies

$$P(\{E_n\}) \propto \prod_{n>m} |E_n - E_m|^\beta \exp\Big(-A\sum_n E_n^2\Big)$$

β Wigner´s Index

$\beta = 1$ for GOE $\quad \beta = 2$ for GUE $\quad \beta = 4$ for GSE

Distribution of level spacings $\quad s_n = E_{n+1} - E_n$

Approximation to 2 levels: WIGNER´S DISTRIBUTION

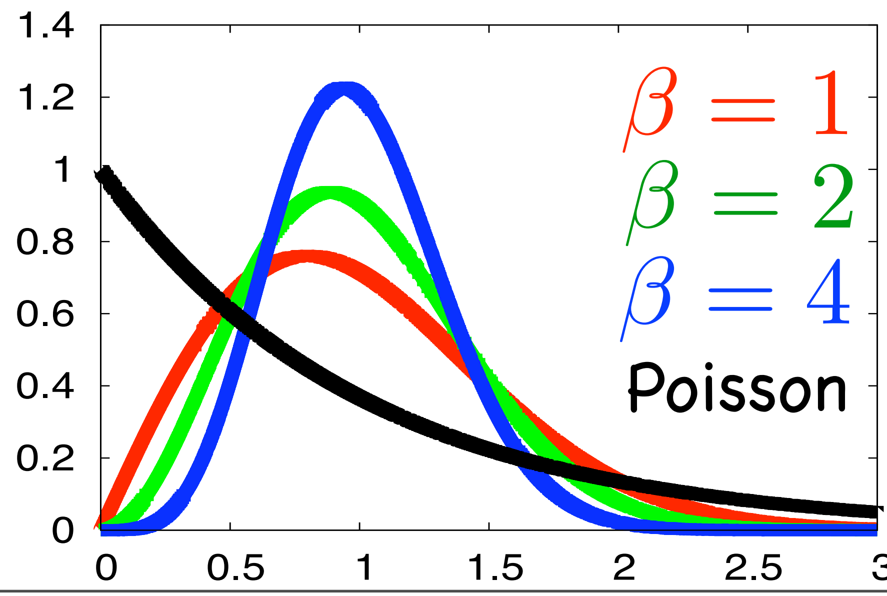
$$P^\beta_{\rm Wig}(s) \propto s^\beta e^{-As^2/2}$$

E. Wigner (1951)

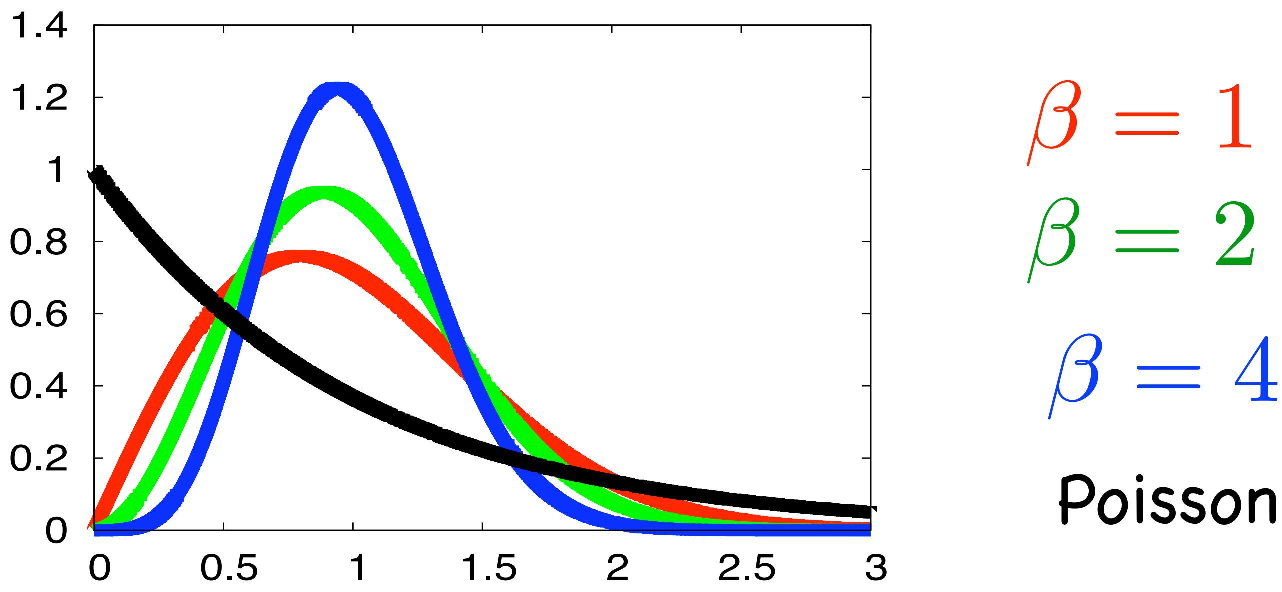
Main Features:

- -Correlated spectra
- -gaussian distribution
- **-Level Repulsion**

$$P^\beta_{\rm Wig}(s)_{s\sim 0} \simeq s^\beta$$



in 1980s Wigner´s idea was applied to Chaotic and regular systems, known as "Bohigas, Giannoni, Schmidt Conjecture". Also Casati, Vivaldi, Guarneri.



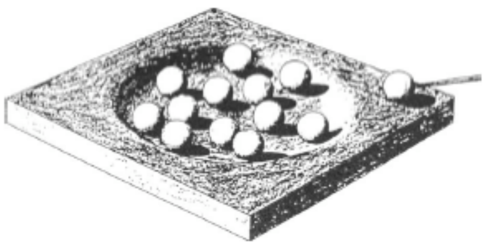
Wigner distribution

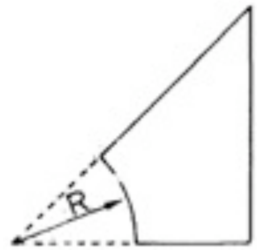
$$P^{\beta}_{\text{Wig}}(s) \propto s^{\beta} e^{-As^2/2}$$

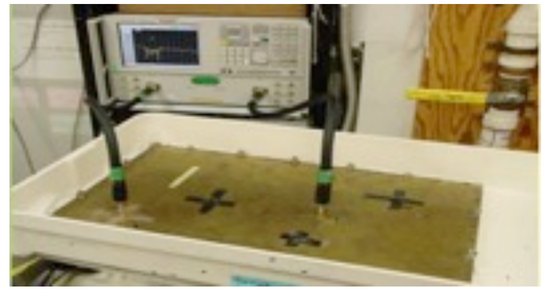
Nuclear Physics U^{238} at low energies

Quantum particle in infinite potential well

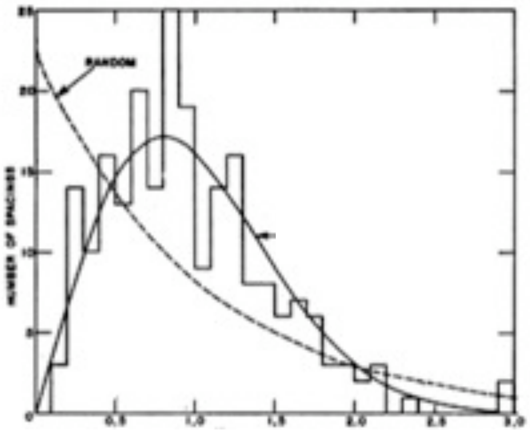
Micro-wave cavity with high quality factor



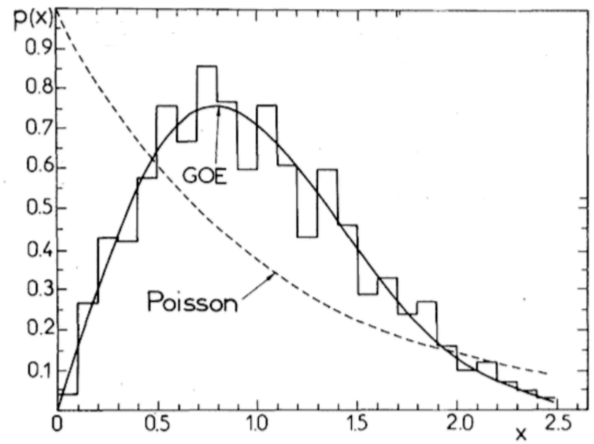




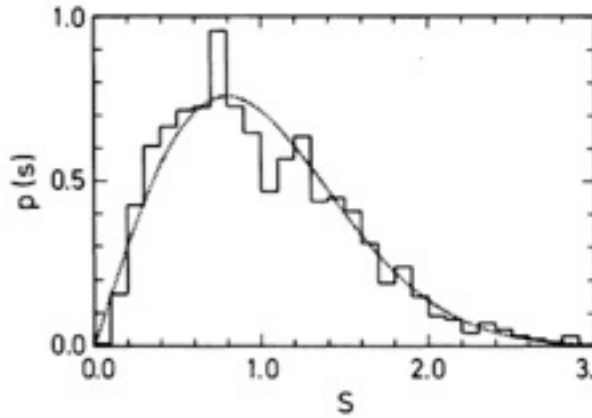
Garg et al. (1964)



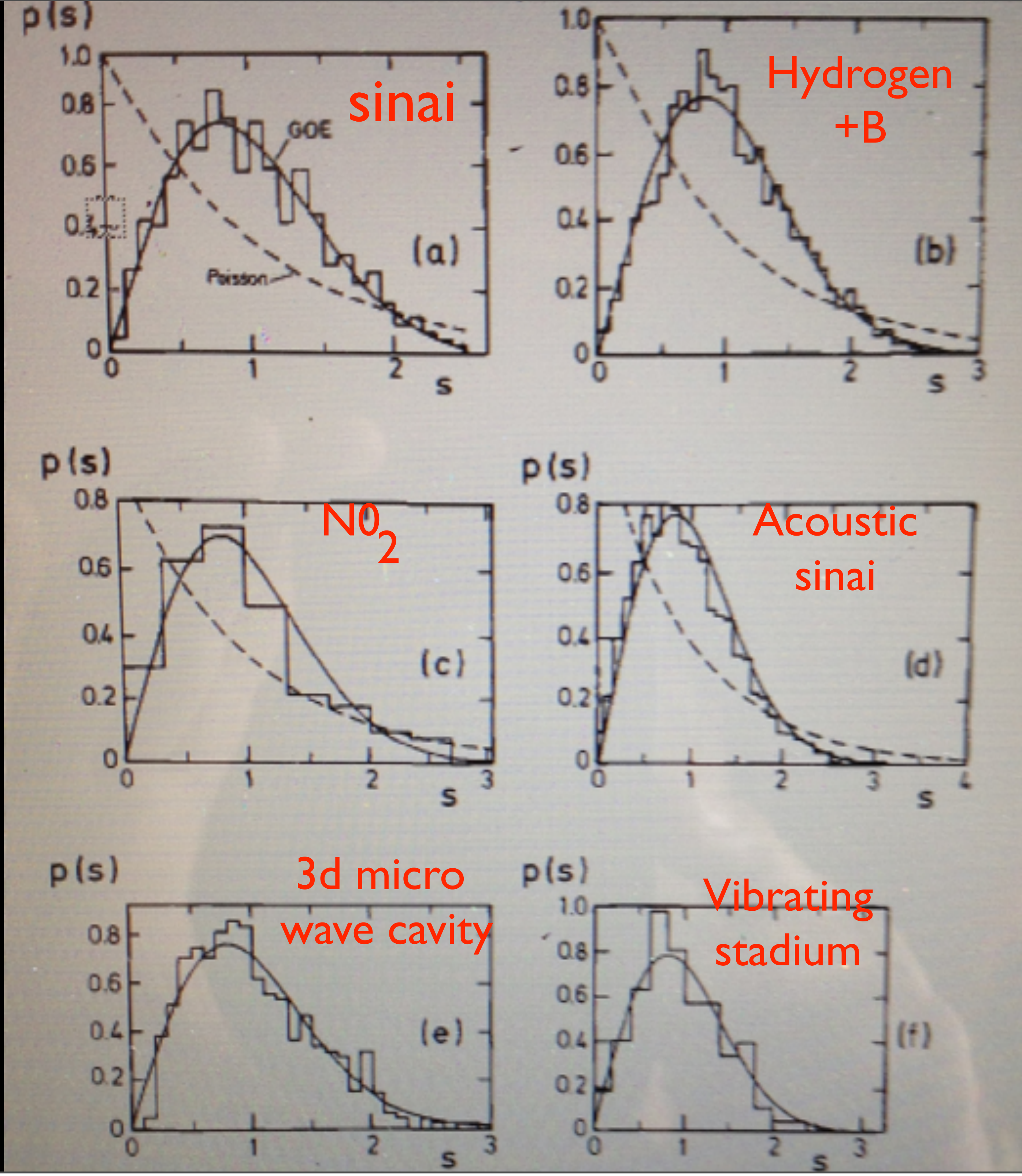
Bohigas et al. (1984)



H.-J. Stöckmann (1990)



WIGNER Distribution describes very well the spectral statistics for Complex systems WEAKLY coupled to Environment



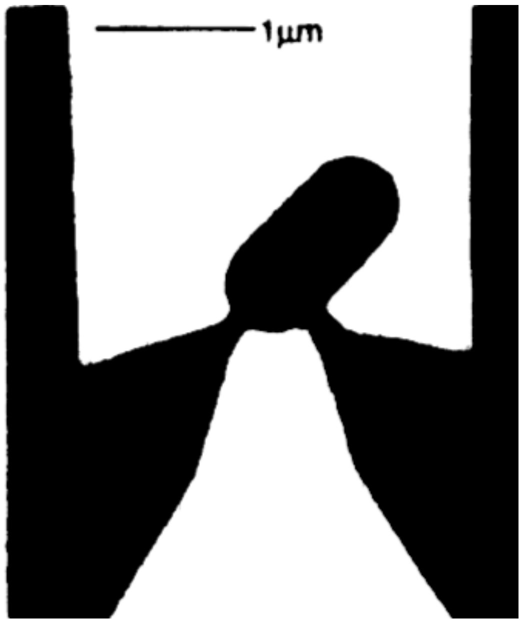
Stöckmann, “Quantum Chaos”, 1999

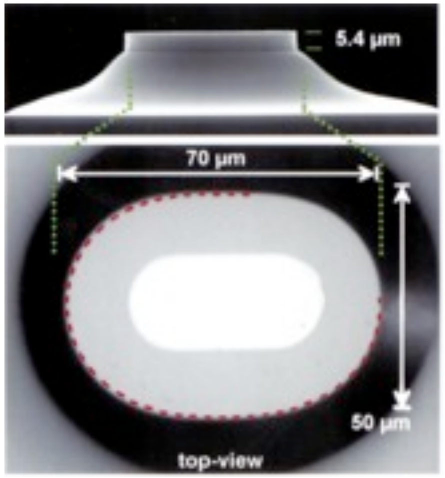
Many systems are actually open

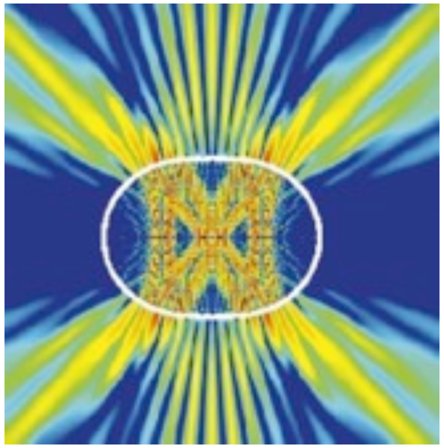
Micro-cavity laser
reflexion losses

concert hall
various sources of
absorption

Quantum dot connected
to leads



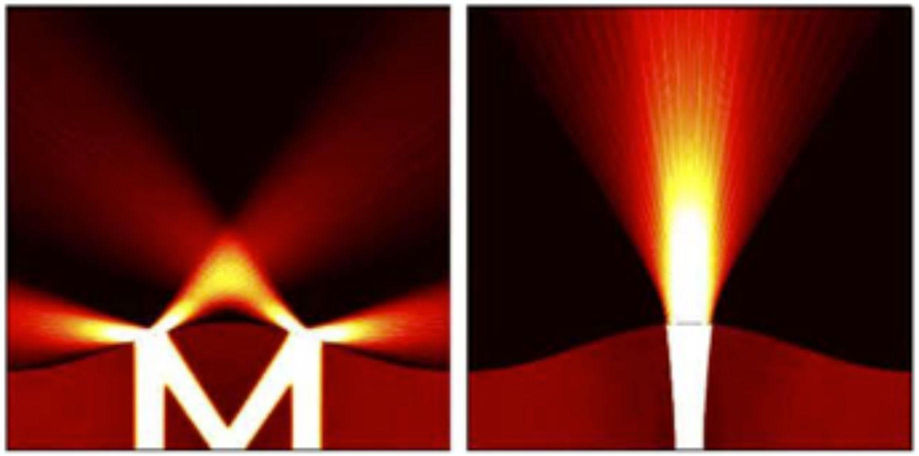




Coupling to environment must be taken into account

Another example:

Chaotic cavities connected to leads

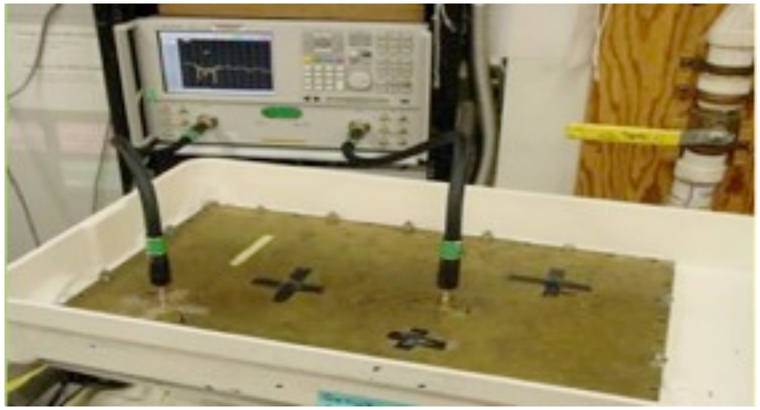


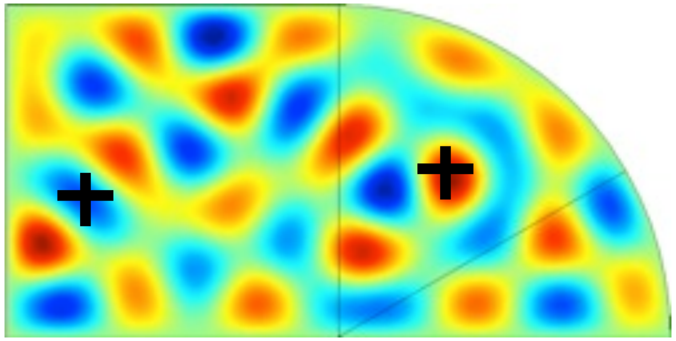
Microlaser design based on wave chaos,
J.A.Méndez-Bermúdez, G.A.L.A, Kuhl,
Stöckmann, 2005

Formalism of the Diffusion matrix

Coupling to environment modeled by Channels

Microwave cavities





Savin Legrand Mortessagne EPL 2006

measuring process

M_a point-antennas

$$M_p = M_a$$

Ohmic Losses

Boundary losses

$$M_l = \left[\frac{P}{\lambda/2}\right]$$

Surface losses

$$M_s = \left[\frac{S}{(\lambda/2)^2}\right]$$

All these mechanisms of losses can be modeled by channels coupling cavity to environment, characterized by coupling strength

Statistics of Resonances: Effective Hamiltonian

$$\mathcal{H}_{eff} = H - \frac{i}{2} V V^{\dagger}$$

H NxN hamiltonian

V NxM Coupling matrix

V_n^j coupling of nth level to jth channel

M channels model the various types of Coupling to environment

NON hermitian

$$\mathcal{H}_{eff}\psi_n = \mathcal{E}_n \psi_n \qquad \text{where} \qquad \mathcal{E}_n = E_n - i\Gamma_n/2$$

RANDOM MATRIX THEORY

H Wigner Random Matrix $\qquad P(H) \propto \exp(-A\mathrm{Tr}H^2)$

V_n^c random independent gaussian variables

$$\langle V_n^c (V_{n'}^{c'})^* \rangle = (1/\eta)\delta_{nn'}\delta^{cc'} \quad , \quad (1/\eta)$$

Coupling Strength

Goal: Level distribution. Single channel

Starting point:

H.-J. Stöckmann and P. Seba (1998) J. Phys. A: Math. Gen. **31** (1998)

The joint energy distribution function for the Hamiltonian $H = H_0 - iWW^+$ for the one-channel case

inconvenient determinant factor is absent. One then has

$$P(E_{nR}, E_{nI}) = \prod_{n,m} |E_n - E_m^*|^{\frac{\beta-2}{2}} \prod_{n>m} |E_n - E_m|^2$$

$$\times \exp\left[-A\left(\sum_n (E_{nR})^2 - \sum_m (E_{nI})^2 + \left(\sum_m E_{nI} \right)^2 \right) - a \sum_n E_{nI} \right].$$

Stöckmann & Šeba
J.P.A 199

Level distribution. Single channel

2 level Model

$$\mathcal{P}^{\beta}_{M=1}(E_1, E_2; \Gamma_1, \Gamma_2) \propto \left[\Gamma_1\Gamma_2\left[(E_1 - E_2)^2 + \frac{1}{4}(\Gamma_1 + \Gamma_2)\right]^2\right]^{\frac{\beta-2}{2}}$$

$$\left[(E_1 - E_2)^2 + \frac{1}{4}(\Gamma_1 - \Gamma_2)^2\right] \times \exp\left[-A\left(E_1^2 + E_2^2 + \frac{1}{2}\Gamma_1\Gamma_2\right) - \frac{\eta}{2}(\Gamma_1 + \Gamma_2)\right]$$

steps:

1- Perform calculation spacing distribution. N=2 level model

$$s = E_1 - E_2 \qquad\qquad z = E_1 + E_2$$

$$\mathcal{P}^{\beta}_{M=1}(E_1, E_2; \Gamma_1, \Gamma_2) \rightarrow \mathcal{P}^{\beta}_{M=1}(s,\ z, \Gamma_1, \Gamma_2)$$

2- Integrate over variables (z, Γ_1, Γ_2)

Spacing distribution for GOE

$$\mathcal{P}^{\beta=1}_{M=1}(s) = \frac{A\eta}{16} e^{-\frac{A}{2}s^2} \int_0^\infty dx \frac{1}{\sqrt{s^2+\frac{x^2}{4}}} e^{-\frac{A}{16}x^2-\frac{\eta}{2}x} \left[(8s^2+x^2) I_0\left(\frac{Ax^2}{16}\right) + x^2 I_1\left(\frac{Ax^2}{16}\right)\right]$$

Spacing distribution for GUE

$$\mathcal{P}^{\beta=2}_{M=1}(s) = \sqrt{\frac{A}{2\pi}}\eta^2 e^{-\frac{A}{2}s^2}\left[\exp\left(\frac{\eta^2}{2A}\right)\mathrm{E}_1\left(\frac{\eta^2}{2A}\right)s^2 + \frac{2}{\eta^2} - \frac{1}{A}\exp\left(\frac{\eta^2}{2A}\right)\mathrm{E}_1\left(\frac{\eta^2}{2A}\right)\right]$$

Spacing distribution for GSE

$$\mathcal{P}^{\beta=4}_{M=1}(s) = \sqrt{\frac{A}{2\pi}}\frac{\eta^4}{12} e^{-\frac{A}{2}s^2}\left[s^4\left(-2+\frac{\eta^2+2A}{A}\exp\left(\frac{\eta^2}{2A}\right)\mathrm{E}_1\left(\frac{\eta^2}{2A}\right)\right)\right.$$
$$+s^2\left(-2\frac{\eta^4+4\eta^2A-4A^2}{\eta^2A^2}+\frac{\eta^4+6\eta^2A}{A^3}\exp\left(\frac{\eta^2}{2A}\right)\mathrm{E}_1\left(\frac{\eta^2}{2A}\right)\right)$$
$$\left.+2\frac{\eta^6+7\eta^4A-4\eta^2A^2+12A^3}{\eta^4A^3}-\frac{\eta^4+9\eta^2A+6A^2}{A^4}\exp\left(\frac{\eta^2}{2A}\right)\mathrm{E}_1\left(\frac{\eta^2}{2A}\right)\right]$$

Spacing distributions for One Channel

$$\mathcal{P}^{\beta}_{M=1}(s) = f_\beta(s) e^{-As^2/2}$$

GOE: $f_{\beta=1}(s) \propto$

$$\int_0^\infty dx \frac{1}{\sqrt{s^2 + \frac{x^2}{4}}} e^{-\frac{A}{16}x^2 - \frac{\eta}{2}x} \left[(8s^2 + x^2) I_0\left(\frac{Ax^2}{16}\right) + x^2 I_1\left(\frac{Ax^2}{16}\right) \right]$$

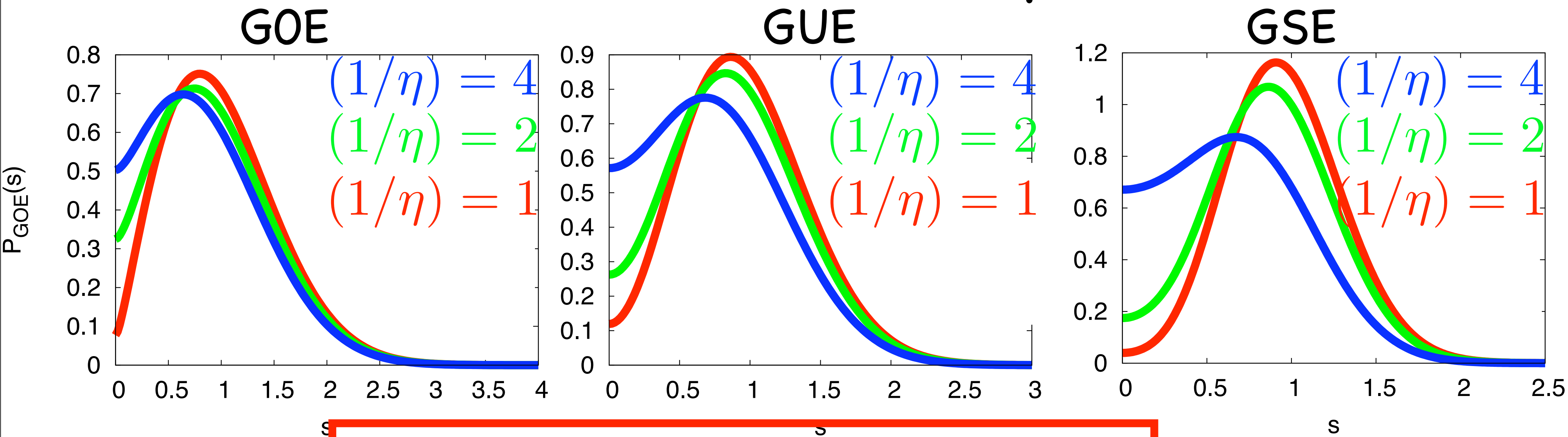
GUE: $f_{\beta=2} = a_2 s^2 + c_2$

GSE: $f_{\beta=4} = a_4 s^4 + b_4 s^2 + c_4$

$a_\beta,\ b_\beta,\ c_\beta$ depend on η

SUPPRESSION of LEVEL REPULSION: $f_\beta(s) \neq 0$

Les distributions des écarts pour un canal



$$\mathcal{P}^{\beta}_{M=1}(s) = f_\beta(s) e^{-As^2/2}$$

GOE:

$$f_{\beta=1}(s) \propto \int_0^\infty dx \frac{1}{\sqrt{s^2 + \frac{x^2}{4}}} e^{-\frac{A}{16}x^2 - \frac{\eta}{2}x} \left[(8s^2 + x^2) I_0\left(\frac{Ax^2}{16}\right) + x^2 I_1\left(\frac{Ax^2}{16}\right) \right]$$

GUE: $f_{\beta=2} = a_2 s^2 + c_2$ $\qquad a_\beta,\ b_\beta,\ c_\beta$ depend on η

GSE: $f_{\beta=4} = a_4 s^4 + b_4 s^2 + c_4$

The weak and strong coupling limits

Vanishing coupling limit $(1/\eta) \to 0$

$$\mathcal{P}^{\beta}_{M=1}(s) \to P^{\beta}_{Wig}(s)$$

The distribution for all classes tend to Wigner distributions

Infinite coupling limit $(1/\eta) \to \infty$

$$\mathcal{P}^{\beta}_{M=1}(s) \to \sqrt{\frac{2A}{\pi}} e^{-\frac{A}{2}s^2}$$

Distributions tend to a gaussian law:

Characteristic of a decorrelated spectra (2 levels).

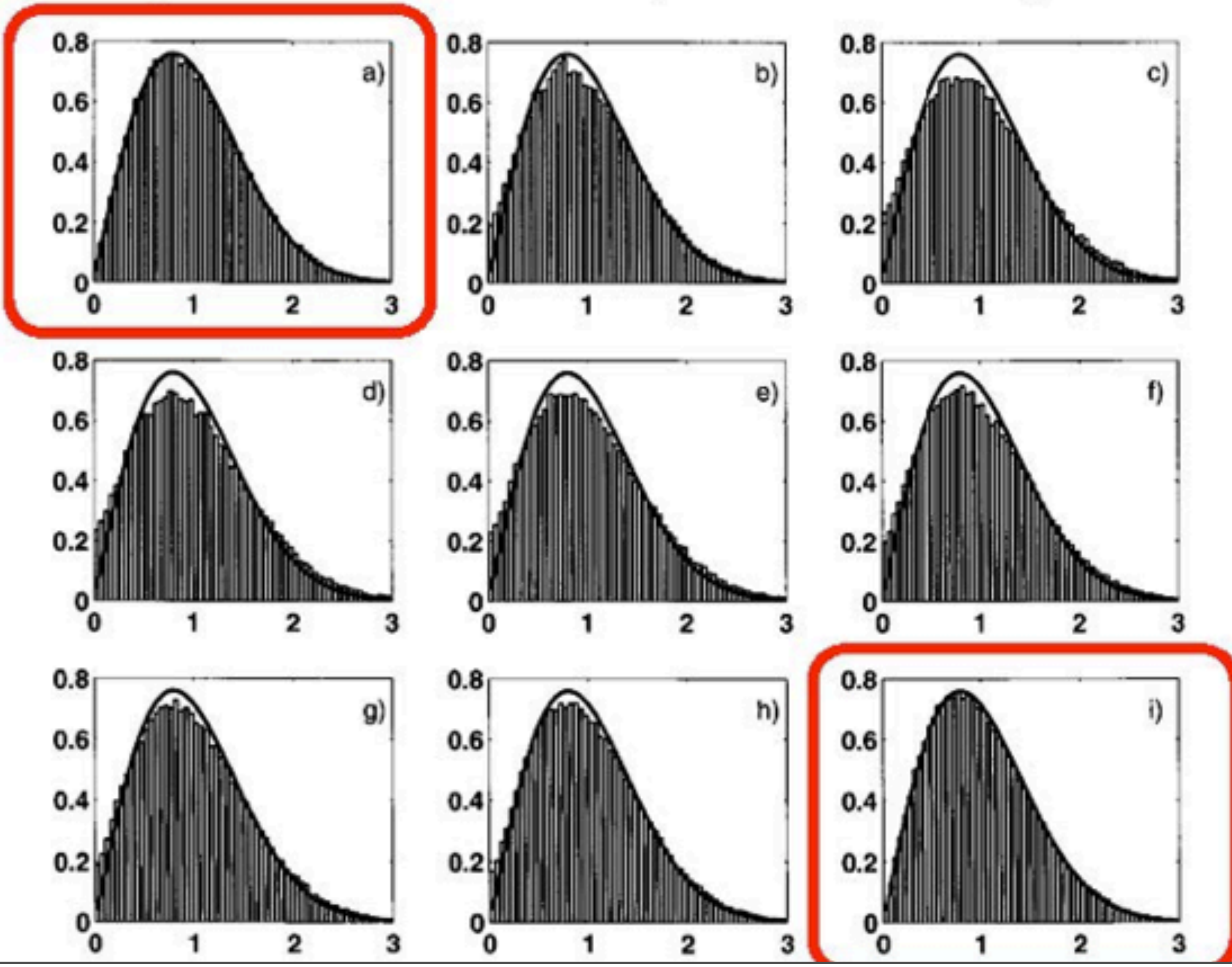
BUT N-level Exact *Stöckmann Seba JPA* **31** *3439 (1998)*

if $M \ll N$ then $\mathcal{P}^{\beta}_{M=1}(s) \to P^{\beta}_{Wig}(s)$

Spacings distribution for the one-channel case

$s_n = E_{n+1} - E_n$ H.-J. Stöckmann P. Seba JPA **31** 3439 (1998)

In the strong coupling regime
the spacing distribution correspond to the Wigner surmise



Questions

1- Is our 2-level model, single channel distribution a good approximation of a N level model? up to what regime of coupling strengths?

2- can the Strong coupling regime be described by some effective coupling strength ?

3- Can this ammended distribution correctly describe the histograms for any number of channels and any coupling strength?

Simulations numériques de matrices aléatoires

$$\mathcal{H}_{eff} = H - \frac{i}{2} V V^\dagger$$

H matrice aléatoire à la Wigner

V matrice de couplage $(N \times M)$

$$\langle V_n^c (V_{n'}^{c'})^* \rangle = (1/\eta) \delta_{nn'} \delta^{cc'}$$

Histogrammes réalisés avec 100 matrices de taille $N = 1000$

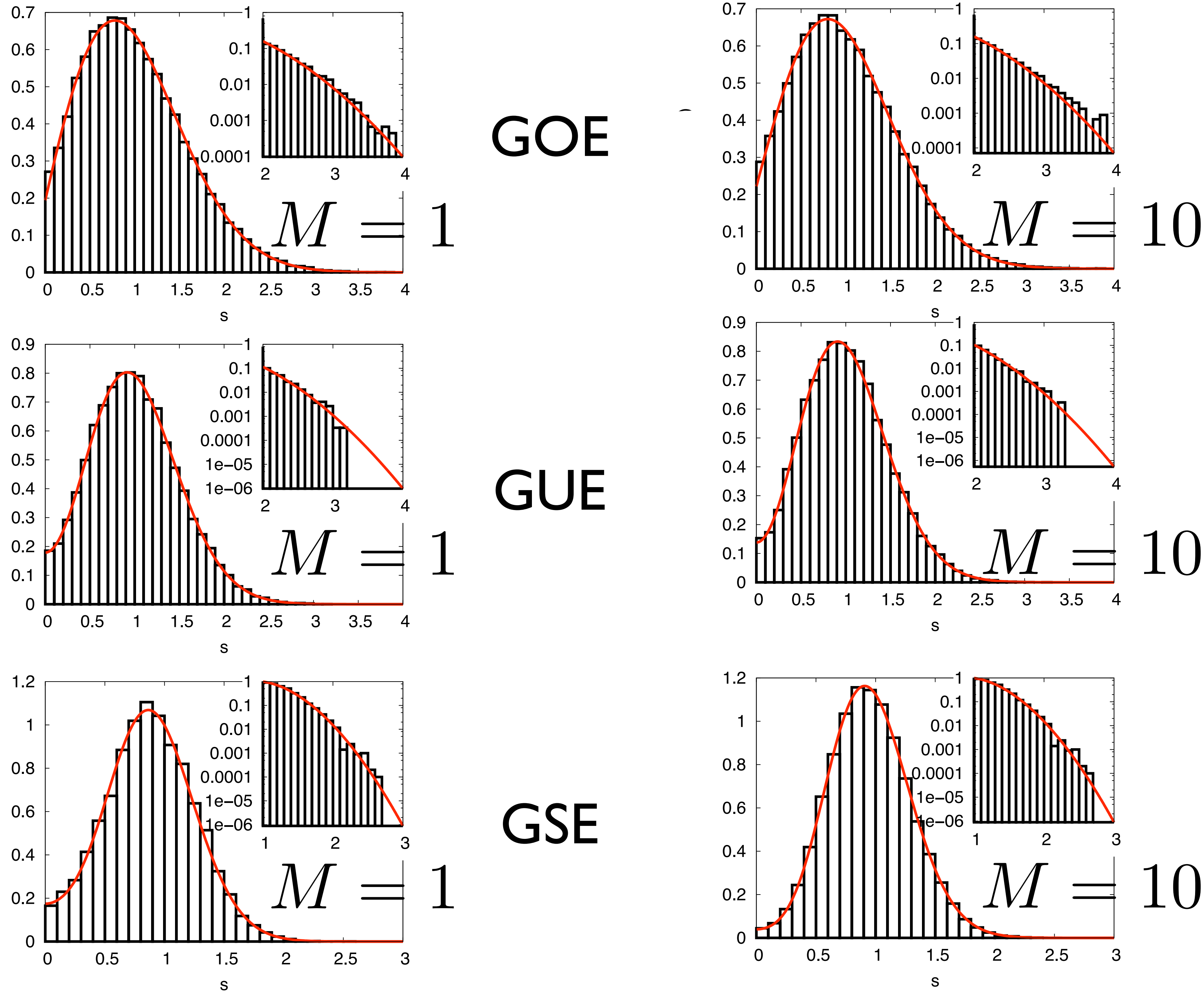
Nombre de canaux et force de couplage considérés:

$M = 1,\ 3,\ 5 \text{ et } 10$ $\qquad$ $\langle \Gamma \rangle = [0.1,\ 30] \Delta$

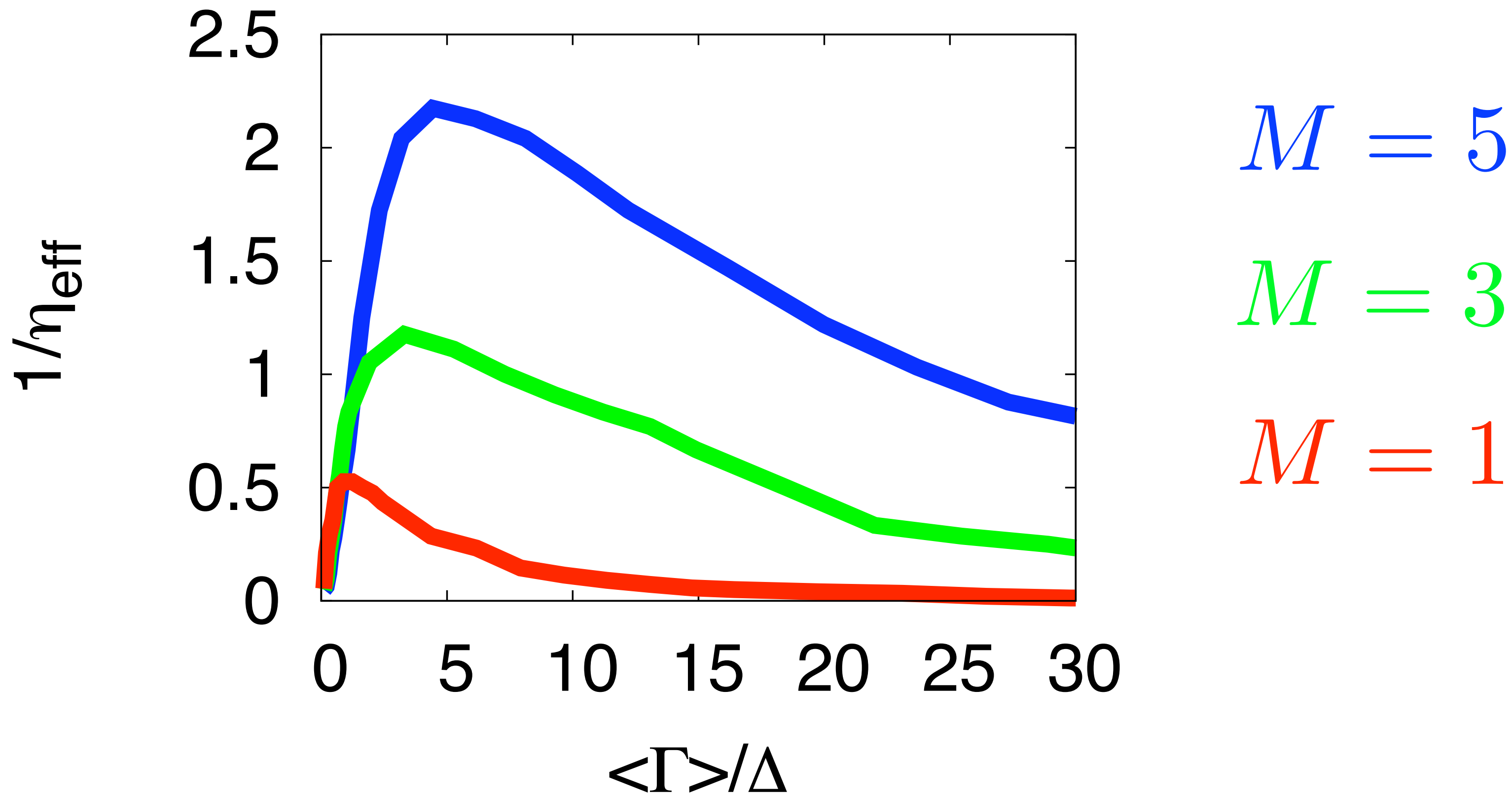
Pour $M > 1$ les histogrammes numériques sont ajustés en considérant $(1/\eta)$ comme un paramètre libre

Dans toute la gamme de couplage analysée, le niveau de confiance de la procédure d'ajustement est supérieur à 99.5%

Confrontation simulations numériques / théorie



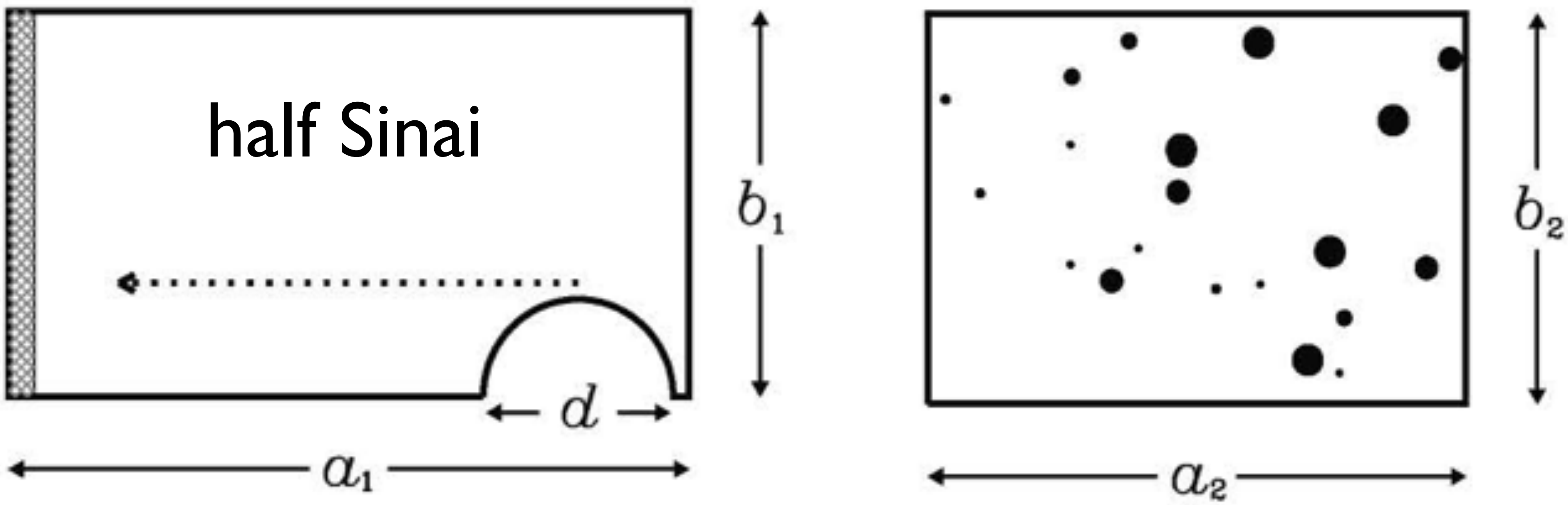
The effective coupling parameter as a function of the mean level width



Number of channels and coupling strengths:

$$M = 1,\ 3,\ 5 \text{ et } 10 \qquad \langle\Gamma\rangle = [0.1,\ 30]\Delta$$

Comparison with a micro-wave experiment



$$a_1 = 43cm, \quad b_1 = 23.7cm, \quad d = 12cm, \quad h = 78mm$$

1.0< Frequency< 19.4 GHz

Distribution of Reflection Coefficients in Absorbing Chaotic Microwave Cavities

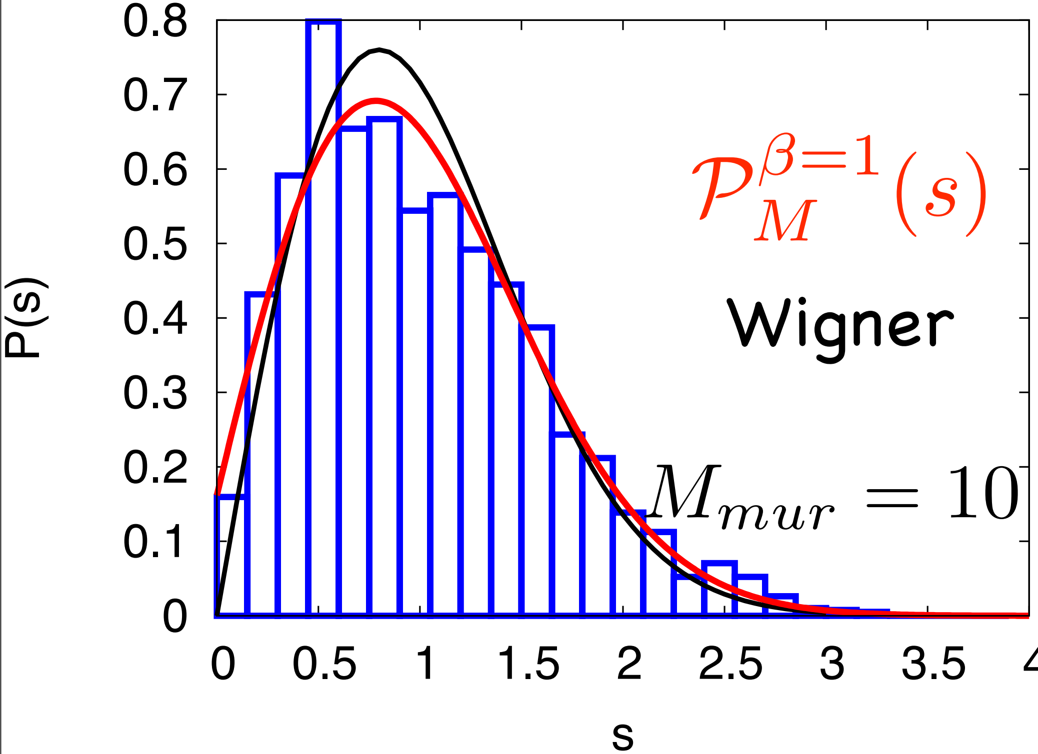
R. A. Méndez-Sánchez,[1] U. Kuhl,[2] M. Barth,[2] C. H. Lewenkopf,[3] and H.-J. Stöckmann[2] PRL, 2003, PRL 2005

Resonance Widths in Open Microwave Cavities Studied by Harmonic Inversion

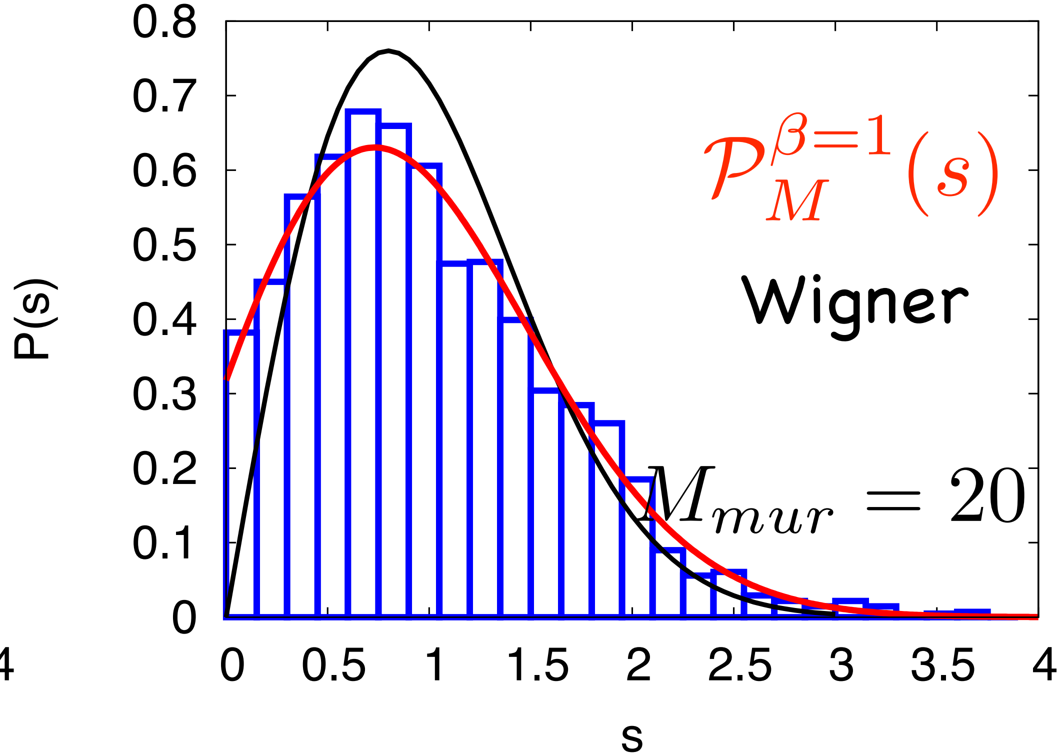
U. Kuhl et al. PRL ***100*** *254101 (2008)*

Histograms

Weak Coupling [1,6]GHz



Strong Coupling [15,16]GHz



Confidence level:

Wigner: less than 50%

$\mathcal{P}_M^{\beta=1}(s)$: 90% and 97%

Summary

Exact Analytical Expression (N=2) and Single Channel for Resonance Spacing Distribution for 3 universal classes

Introduced effective coupling. works well for level spacing distributions for 3 Wigner ensembles and for any number of channels (Generalization of Wigner´s surmise)

Our theoretical results supported by numerical simulations and by experimental data

Nearest Level Spacing Distributions In Open Chaotic Wave Systems: A generalization of the Wigner surmise, PRL, 2012

Working on

Distribution of distances in the complex plane and...