The Casimir effect and its mathematical implications

Klaus Kirsten

Baylor University

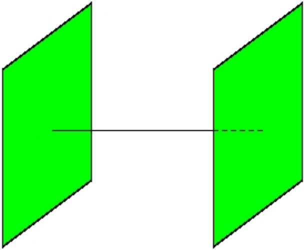
Benasque, July 08-14, 2012

(Supported by the NSF under PHY-0757791 and by the DFG)

Outline

Introduction

- Casimir's discovery



$$\mathbf{E}_{tan}(t,\vec{r})\,|_{\mathcal{F}} = \mathbf{B}_{nor}(t,\vec{r})|_{\mathcal{F}} = 0$$

- Casimir energy (per unit area) $E(a) = -\frac{\pi^2}{720}\frac{\hbar c}{a^3}$

- Casimir force (per unit area) $F(a) = -\frac{d}{da}E(a) = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}$

H.B.G. Casimir, Proc. K. Ned. Akad. Web B **51** (1948) 793-795

Introduction

- Many aspects of the Casimir energy/force have been considered:
  - influence of the shape of the boundary
  - influence of the boundary condition imposed
  - influence of non-trivial topology
  - influence of external parameters

- Calculations are often plagued by divergencies:
  - identify corresponding situations
  - formulate relevant quantities in terms of finite expressions

E. Elizalde, Ten Physical Applications of Spectral Zeta Functions, Springer, Berlin, 1995
K.A. Milton, The Casimir Effect, World Scientific, New Jersey, 2001.
M. Bordag, G.L. Klimchitskaya, U. Mohideen, V.M. Mostepanenko, Advances in the Casimir Effect, Oxford University Press, Oxford, 2009

Different formulations of the Casimir energy

- Action for a non-interacting massive scalar field

$$S[\varphi] = \int d^4x \mathcal{L}(x) = \int d^4x \left(\frac{1}{2} \partial^\nu \varphi \partial_\nu \varphi - \frac{m^2}{2} \varphi^2 + \Upsilon \varphi \right)$$

- Corresponding field equation

$$(\Box + m^2)\varphi(x) = 0$$

- Canonical energy-momentum tensor

$$T_{\mu\nu}(x) = \partial_\mu \varphi(x) \partial_\nu \varphi(x) - g_{\mu\nu} \mathcal{L}(x)$$

Different formulations of the Casimir energy

- Positive- and negative-frequency solutions

$$\varphi_J^{(+)}(t,\vec{r}) = \frac{1}{\sqrt{2\omega_J}} e^{-i\omega_J t}\Phi_J(\vec{r}), \qquad \varphi_J^{(-)}(t,\vec{r}) = \left[\varphi_J^{(+)}(t,\vec{r})\right]^*$$

where

$$-\Delta\Phi_J(\vec{r}) = \Lambda_J\Phi_J(\vec{r}), \quad \Phi_J(\vec{r})|_{\mathcal{F}} = 0,$$
$$\Lambda_J \equiv \omega_J^2 - m^2$$

- Field operator

$$\varphi(x) = \sum_J \left[\varphi_J^{(+)}(x)a_J + \varphi_J^{(-)}(x)a_J^+\right]$$

Different formulations of the Casimir energy

- Vacuum energy density

$$
\begin{aligned}
\langle 0 \,|\, T_{00}(x) |\, 0 \rangle &= \frac{1}{2} \left\langle 0 \left| \left[\sum_{\mu=0}^{3} \left(\frac{\partial \varphi}{\partial x^{\mu}} \right)^2 + m^2 \varphi^2 \right] \right| 0 \right\rangle \\
&= \sum_J \frac{1}{4\omega_J} \left[(\omega_J^2 + m^2) \Phi_J(\vec{r}) \Phi_J^*(\vec{r}) + \sum_{k=1}^{3} \frac{\partial \Phi_J(\vec{r})}{\partial x^k} \frac{\partial \Phi_J^*(\vec{r})}{\partial x^k} \right]
\end{aligned}
$$

- Total vacuum energy

$$
E_0 = \int_V d\vec{r} \, \langle 0 \,|\, T_{00}(x) |\, 0 \rangle = \frac{1}{2} \sum_J \omega_J
$$

- Green's function

$$(\Box_x + m^2) G(x,x') = \delta^4(x-x')$$

- Eigenmode expansion

$$\begin{aligned} G(x,x') &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_J \frac{\Phi_J(\vec{r})\Phi_J^*(\vec{r}')}{-\omega^2+\omega_J^2-i\epsilon} e^{-i\omega(t-t')} \\ &= i \sum_J \frac{1}{2\omega_J} e^{-i\omega_J|t-t'|} \Phi_J(\vec{r})\Phi_J^*(\vec{r}') \end{aligned}$$

- Field operator products

$$\begin{aligned} T\varphi(x)\varphi(x') &= \theta(t-t')\varphi(x)\varphi(x') + \theta(t'-t)\varphi(x')\varphi(x) \\ \langle 0 \left| \varphi(x)\varphi(x') \right| 0 \rangle &= \sum_J \frac{1}{2\omega_J} e^{-i\omega_J(t-t')} \Phi_J(\vec{r})\Phi_J^*(\vec{r}') \\ i \langle 0 \left| T\varphi(x)\varphi(x') \right| 0 \rangle &= G(x,x') \end{aligned}$$

- Vacuum energy in terms of Green's function

$$\begin{aligned} \langle 0\,|\,T_{00}(x)|\,0\rangle &= \frac{1}{2}\left\langle 0\left|\left[\sum_{\mu=0}^{3}\left(\frac{\partial\varphi}{\partial x^\mu}\right)^2 + m^2\varphi^2\right]\right|0\right\rangle \\ &\equiv -\frac{i}{2}\left(\sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\mu} + m^2\right)G(x,x')\bigg|_{x'=x} \end{aligned}$$

- Global vacuum energy

$$E_0 = i\int_V d\vec{r}\,\frac{\partial^2 G(x,x')}{\partial x_0^2}\bigg|_{x'=x}$$

- Path-integral approach

$$
\begin{aligned}
Z[\Upsilon] &= C \int D\varphi e^{iS[\varphi]} \\
E_0 &= \frac{i}{T} \ln Z[0]
\end{aligned}
$$

- Computation of the path-integral: finite-dimensional analogue

$$
\int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}(x,\tilde{\mathcal{K}}x)+(x,h)} = (2\pi)^{n/2}(det\tilde{\mathcal{K}})^{-1/2} e^{\frac{1}{2}(h,\tilde{\mathcal{K}}^{-1}h)}
$$

- Replacement

$$
(x,\tilde{\mathcal{K}}x) -> \ (\varphi,\tilde{\mathcal{K}}\varphi) = \int d^4x \varphi(x)(\tilde{K}\varphi)(x)
$$

- Note

$$
iS[\varphi] = -\frac{i}{2}\int d^4x\ \varphi(x)(\Box + m^2)\varphi(x) + i\int d^4x\ \Upsilon\varphi(x)
$$

- Here

$$h = i\Upsilon, \quad \tilde{\mathcal{K}} = i\mathcal{K} \equiv i(\Box + m^2), \quad K(x,x') = \delta(x-x')(\Box_{x'} + m^2),$$

- Generating functional

$$\begin{aligned} Z[\Upsilon] &= C(\det \tilde{\mathcal{K}})^{-1/2} e^{\frac{1}{2}(h,\tilde{\mathcal{K}}^{-1}h)} \\ &= C(\det \mathcal{K})^{-1/2} \exp\left[\frac{i}{2}\int d^4x d^4x' \Upsilon(x) K^{-1}(x,x')\Upsilon(x')\right], \\ & K^{-1}(x,x') = G(x,x') \end{aligned}$$

- Vacuum energy

$$E_0 = \frac{i}{T}\ln(\det \mathcal{K})^{-1/2} = -\frac{i}{2T}\mathrm{Tr}\ln \mathcal{K}$$

- Imposing boundary conditions in path-integrals

$$Z[\Upsilon] = C \int D\varphi \prod_{x\in\mathcal{F}} \delta(\varphi(x)) e^{iS[\varphi]}$$

- Analogy ($x = u(\eta)$, $x_0 = \eta_0$, $\vec{r} = \vec{u}(\eta_1, \eta_2)$)

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \implies$$

$$\prod_{x\in\mathcal{F}} \delta(\varphi(x)) = \int Db\ e^{i\int d\mu(\eta) b(\eta)\varphi(u(\eta))}$$

Perform φ-integration

$$\prod_{x\in\mathcal{F}} \delta(\varphi(x)) e^{iS[\varphi]} \longrightarrow$$

$$-\frac{i}{2}\int d^4x\ \varphi(x)(\Box + m^2)\varphi(x) + i\int d^4x \Upsilon(x)\varphi(x)$$

$$+i\int d\mu(\eta)\ b(\eta)\varphi(u(\eta))$$

- Introducing additional integration ($H(\eta, x) = \delta^4(x - u(\eta))$)

$$\int d\mu(\eta) b(\eta) \varphi(u(\eta)) = \int d^4x \int d\mu(\eta)\ b(\eta) H(\eta, x) \varphi(x)$$

- Exponent then reads

$$-\frac{i}{2} \int d^4x\ \varphi(x)(\Box + m^2)\varphi(x) + i \int d^4x \left[\Upsilon(x) + \int d\mu(\eta) b(\eta) H(\eta, x) \right] \varphi(x)$$

$$\Longrightarrow h = i \left[\Upsilon(x) + \int d\mu(\eta) b(\eta) H(\eta, x) \right]$$

- Relevant term for b-integration

$$\int d^4x \int d^4x' \int d\mu(\eta) b(\eta) H(\eta, x) K^{-1}(x, x') \int d\mu(\eta') b(\eta') H(\eta', x')$$

$$= \int d\mu(\eta) \int d\mu(\eta') b(\eta) G(u(\eta), u(\eta')) b(\eta')$$

- Identify relevant operator

$$\tilde{K}(\eta, \eta') = G(u(\eta), u(\eta'))$$

- Generating functional

$$Z[\Upsilon] = C(det K)^{-1/2}(det\tilde{K})^{-1/2}\exp\left[\frac{i}{2}\int d^4x d^4x' \Upsilon(x)\ {}^{\mathcal{F}}G(x,x')\Upsilon(x')\right]$$

- Vacuum energy

$$E_0 = -\frac{i}{2T}\mathrm{Tr}\ln\tilde{K} = \frac{i}{T}\ln(\det\tilde{K})^{-1/2}$$

M. Bordag, D. Robaschik and E. Wieczorek, Ann. Phys. **165** (1985) 192-213

Regularizations of the Casimir energy

- Zeta function regularization

$$E_0 = \frac{1}{2} \sum_J \omega_J \quad \rightarrow \quad E_0(s) = \frac{\mu^{2s}}{2} \sum_J \omega_J^{1-2s} = \frac{\mu^{2s}}{2} \zeta\left(s - \frac{1}{2}\right)$$

- We introduced the zeta function

$$\zeta(s) = \sum_J \omega_J^{-2s} \qquad \text{for} \quad \Re s > \frac{d}{2}$$

where

$$(-\Delta + m^2)\Phi_J(\vec{r}) = \omega_J^2 \Phi_J(\vec{r}), \quad \Phi_J(\vec{r})\big|_{\mathcal{F}} = 0$$

Dowker and Critchley, Hawking, Actor, Blau, Bordag, Elizalde, KK, Svaiter, Visser, Wipf, ...

Regularizations of the Casimir energy

- Point splitting regularization

$$E_0 = i \int_V d\vec{r} \frac{\partial^2 G(x,x')}{\partial x_0^2} \Bigg|_{x'=x} \to E_0(\epsilon) = i \int_V d\vec{r} \frac{\partial^2 G(x,x')}{\partial x_0^2} \Bigg|_{x'=x+\epsilon}$$

- Frequency-cutoff regularization

$$E_0 = \frac{1}{2} \sum_J \omega_J \quad \to \quad E_0(\delta) = \frac{1}{2} \sum_J \omega_J e^{-\delta \omega_J^2}$$

Asorey, Beneventano, Brevik, Fulling, Milton, Muñoz Castañeda, Santangelo, ...

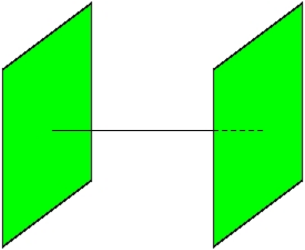
Examples for Casimir energy calculations

Typical situations one encounters:

- eigenvalue spectrum explicitly known
- eigenfunctions (but not the spectrum) known
- none of the above, but sufficiently high symmetry of the configuration
- other

Parallel plates in $\mathbb{R}^3$

- Parallel plates in $\mathbb{R}^3$



- Eigenvalue equation:

$$-\Delta u_\ell(x,y,z) = \omega_\ell^2 u_\ell(x,y,z), \qquad u_\ell(x,y,0) = u_\ell(x,y,a) = 0.$$

- Eigenfunctions:

$$u_{\vec{k},n}(x,y,z) = e^{ik_x x + ik_y y} \sin\left(\frac{n\pi z}{a}\right)$$
$$(k_x, k_y) \in \mathbb{R}^2, \quad n \in \mathbb{N}.$$

Parallel plates in $\mathbb{R}^3$

- eigenvalues

$$\omega^2_{\vec{k},n} = \vec{k}^2 + \left(\frac{\pi n}{a}\right)^2, \qquad \vec{k} \in \mathbb{R}^2, \quad n \in \mathbb{N}$$

- zeta function density

$$\zeta(s) \;=\; \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d^2k \sum_{n=1}^{\infty} \left[\vec{k}^2 + \left(\frac{\pi n}{a}\right)^2\right]^{-s} = \frac{1}{4\pi} \frac{1}{s-1} \left(\frac{\pi}{a}\right)^{2-2s} \zeta_R(2s-2)$$

- Casimir energy density

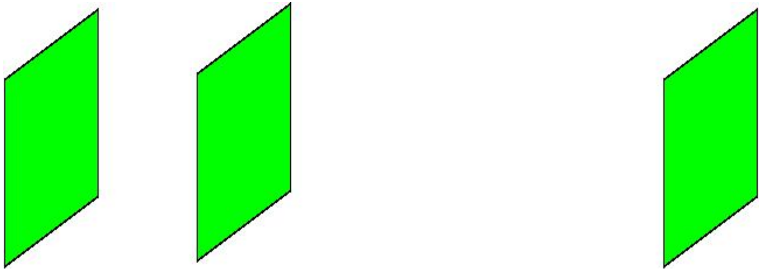
$$E_{Cas} = -\frac{\pi^2}{1440 a^3}$$

- Casimir force

$$F_{Cas} = -\frac{d}{da} E_{Cas} = -\frac{\pi^2}{480 a^4}$$

Parallel plates in $\mathbb{R}^3$

- contribution from the right half-plane



$$F_{Cas} = \frac{\pi^2}{480(L-a)^4} \quad \overset{L\to\infty}{\to} \quad 0$$

No contribution from the right half-plane

Torus

- Let $\mathcal{M} = S^1 \times \cdots S^1$. Consider

$$(-\Delta + m^2)\phi_{\vec{n}}(x_1, ..., x_d) = \omega_{\vec{n}}^2 \phi_{\vec{n}}(x_1, ..., x_d)$$

- Then

$$\begin{aligned} \phi_{\vec{n}}(x_1, ..., x_d) &= A \exp\left\{ \frac{2\pi n_1}{L_1} x_1 + ... + \frac{2\pi n_d}{L_d} x_d \right\} \\ \omega_{\vec{n}}^2 &= \left(\frac{2\pi n_1}{L_1} \right)^2 + ... + \left(\frac{2\pi n_d}{L_d} \right)^2 + m^2 \end{aligned}$$

Torus

- Relevant zeta function $[r_i = (2\pi/L_i)^2]$

$$\zeta(s) = \sum_{\vec{n}\in\mathbb{Z}^d} \left(r_1 n_1^2 + \ldots + r_d n_d^2 + m^2\right)^{-s} = \zeta_E\left(s, m^2|\vec{r}\right) \quad \text{for} \quad \Re s > \frac{d}{2}$$

- extremely useful results

$$\frac{1}{\lambda^s} = \frac{1}{\Gamma(s)} \int_0^\infty dt\ t^{s-1} e^{-\lambda t}$$

$$\sum_{n=-\infty}^{\infty} e^{-\left(\frac{\pi n}{a}\right)^2 t} = \frac{a}{\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{a^2 n^2}{t}}$$

$$\int_0^\infty dt\ t^{-\nu-1} e^{-ct-\frac{b}{t}} = 2\left(\frac{c}{b}\right)^{\nu/2} K_\nu(2\sqrt{cb})$$

P. Epstein, Math. Ann. **56** (1903) 615-644; **63** (1907) 205-216

Torus

- Analytical continuation of the Epstein zeta function

$$\zeta_E(s,c|\vec{r}) = \frac{\pi^{d/2}}{\sqrt{r_1\cdots r_d}}\frac{\Gamma\left(s-\frac{d}{2}\right)}{\Gamma(s)}\ c^{\frac{d}{2}-s}+\frac{2\pi^s c^{\frac{d-2s}{4}}}{\Gamma(s)\sqrt{r_1\cdots r_d}}$$
$$\times\sum_{\vec{n}\in\mathbb{Z}^d/\vec{0}}\left[\frac{n_1^2}{r_1}+\ldots+\frac{n_d^2}{r_d}\right]^{\frac{1}{2}\left(s-\frac{d}{2}\right)}K_{\frac{d}{2}-s}\left(2\pi\sqrt{c}\left(\frac{n_1^2}{r_1}+\ldots+\frac{n_d^2}{r_d}\right)^{1/2}\right)$$

Actor, Dowker, Elizalde, KK, Svaiter, ...
J. Ambjorn and S. Wolfram, Ann. Phys. **147** (1983) 1-32

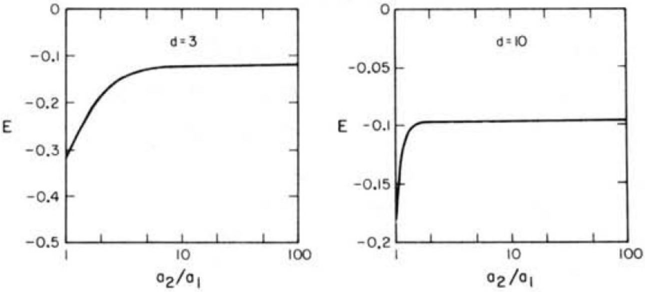


FIG. 3.2. Casimir energy divided by volume for a massless scalar field in d space dimensions constrained to be periodic with periods a_1 and a_2 along two orthogonal directions.

Cubes

- Rectangle with Dirichlet boundary condition

$$\begin{aligned}
\phi_{n,k}(x,y) &= A\sin\left(\frac{n\pi x}{L_1}\right)\sin\left(\frac{k\pi y}{L_2}\right) \\
\lambda_{n,m} &= \left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{k\pi}{L_2}\right)^2 + m^2, \qquad n,k\in\mathbb{N} \\
\zeta(s) &= \sum_{n=1}^\infty\sum_{k=1}^\infty\left[\left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{k\pi}{L_2}\right)^2 + m^2\right]^{-s}
\end{aligned}$$

- Reorganizing summations $[r_i = (\pi/L_i)^2]$

$$\zeta(s) = \frac{1}{4}\zeta_E\left(s,m^2|\vec{r}\right) - \frac{1}{4}\zeta_E\left(s,m^2|r_1\right) - \frac{1}{4}\zeta_E\left(s,m^2|r_2\right) + \frac{1}{4}m^{-2s}$$

d	p	$E_{\varphi P}$	$E_{\varphi D}$	$E_{\varphi N}$
1	1	−0.13	−0.13	−0.13
2	1	−0.19	−0.024	−0.024
2	2	−0.71	+0.041	−0.22
3	1	−0.11	−0.0069	−0.0069
3	2	−0.31	+0.0048	−0.043
3	3	−0.81	−0.016	−0.29
4	1	−0.085	−0.0025	−0.0025
4	2	−0.19	+0.00081	−0.013
4	3	−0.39	−0.0016	−0.088
4	4	−0.85	+0.0061	−0.33
5	1	−0.065	−0.0010	−0.0010
5	2	−0.15	+0.00012	−0.0051
5	3	−0.27	−0.00031	−0.019
5	4	−0.48	+0.00050	−0.073
5	5	−0.95	−0.0025	−0.37

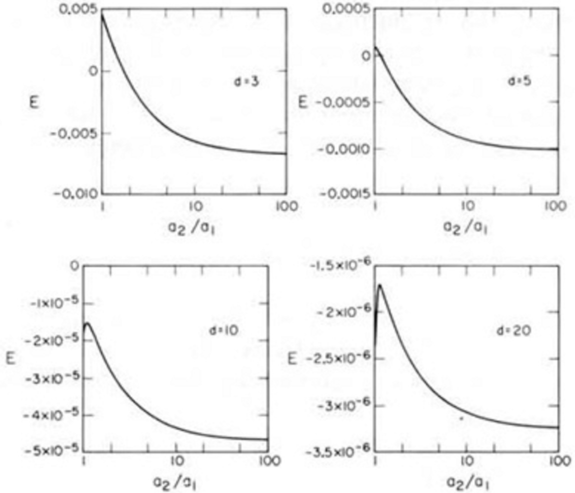


FIG. 3.4. Casimir energy divided by volume for a massless scalar field satisfying Dirichlet boundary conditions on the sides of a rectangular $a_1 \times a_2$ "tube" in d space dimensions.

Spheres

- Eigenvalue problem on the sphere of radius one

$$(-\Delta + \xi R)\mathbf{Y} = \lambda \mathbf{Y}, \qquad R = d(d-1)$$

- Choosing conformal coupling $\xi = \frac{1}{4}\ \frac{d-1}{d}$

$$\omega_\ell^2 = \left(\ell + \frac{d-1}{2}\right)^2, \quad \deg(\ell) = (2\ell + d - 1)\frac{(\ell + d - 2)}{\ell!(d-1)!}$$

- Relevant zeta function

$$\zeta(s) = \sum_{\ell=0}^{\infty} \frac{\deg(\ell)}{\left(\ell + \frac{d-1}{2}\right)^{2s}}$$

- Rewriting the degeneracy

$$\deg(\ell) = \binom{\ell+d-1}{d-1} + \binom{\ell+d-2}{d-1}$$

- Barnes zeta function

$$\begin{aligned}\zeta_{\mathcal{B}}(s,b) &= \sum_{\vec{m}=0}^{\infty} (b+m_1+\ldots+m_d)^{-s} \\ &= \sum_{\ell=0}^{\infty} \binom{\ell+d-1}{d-1} (\ell+b)^{-s}\end{aligned}$$

- Sphere zeta function in terms of Barnes zeta function

$$\zeta(s) = \zeta_{\mathcal{B}}\left(2s, \frac{d-1}{2}\right) + \zeta_{\mathcal{B}}\left(2s, \frac{d+1}{2}\right)$$

E.W. Barnes, Trans. Cambridge Philos. Soc. **19** (1903) 374-425.
J.S. Dowker, Commun. Math. Phys. **162** (1994) 633-648

- Example: $d = 2$, $\deg(\ell) = 2\ell + 1$

$$\zeta(s) = \sum_{\ell=0}^{\infty}(2\ell+1)\left(\ell+\frac{1}{2}\right)^{2s} = 2\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)^{2s-1} = 2\zeta_H\left(2s-1;\frac{1}{2}\right)$$

- Casimir energy

$$E_0 = 2\zeta_H\left(-2;\frac{1}{2}\right) = 0$$

- In fact, the Casimir energy vanishes for all even dimensions. Furthermore, e.g,

$$E_0^3 = \frac{1}{240}, \quad E_0^5 = -\frac{31}{60480}, \quad E_0^7 = \frac{289}{604800}, \ldots$$

- The Casimir energy is singular for other couplings.

B.P. Dolan and C. Nash, Commun. Math. Phys. **148** (1992) 139-153

Balls

- Basel problem (Pietro Mengoli in 1644, solved by Leonard Euler in 1735)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1.64493... = \frac{\pi^2}{6}$$

- "Proof:" From Calculus II

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \Longrightarrow \\ \frac{\sin x}{x} &= 1 - \frac{x^2}{6} + \frac{x^4}{5!} - ... \end{aligned}$$

- Also

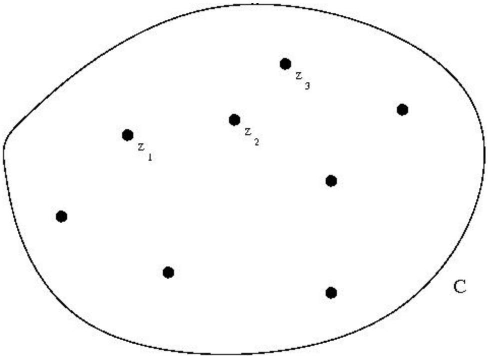
$$
\begin{aligned}
\frac{\sin x}{x} &= \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\times\dots\\
&= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\times\dots\\
&= -\frac{x^2}{\pi^2}\left(1+\frac{1}{4}+\frac{1}{9}+\dots\right)+\dots\\
&= -\frac{x^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}+\dots
\end{aligned}
$$

- Compare coefficients in powers of x^2 to see

$$
-\frac{1}{6} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \Longrightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}
$$

- More advanced proof (complex analysis): Residue Theorem
  Let C be a simple closed contour and let $f(z)$ be analytic on C and at all points inside C except for isolated singularities at $z_1, z_2, z_3, ..., z_\ell$. Then

$$\frac{1}{2\pi i}\int_C dz\ f(z) = \sum_{n=1}^{\ell} \mathrm{Res}\,[f(z), z_n]$$

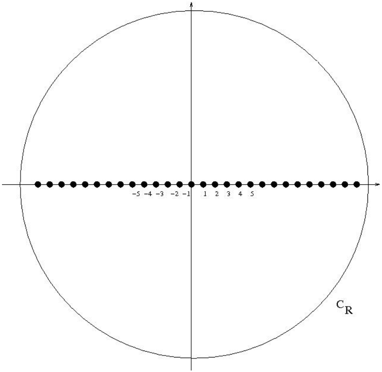


- Find suitable function:

$$F(z) = \sin \pi z$$

- Consider

$$\lim_{R\to\infty} \quad \frac{1}{2\pi i} \int_{C_R} dz \, \frac{1}{z^2} \, \frac{d}{dz} \ln F(z)$$



- Suitable contour integral

$$\lim_{R\to\infty} \quad \frac{1}{2\pi i} \int_{C_R} dz \, \frac{1}{z^2} \, \frac{d}{dz} \ln F(z)$$

$$= \lim_{R\to\infty} \frac{1}{2\pi i} \int_{C_R} dz \, \frac{1}{z^2} \pi \cot \pi z$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{\pi^2}{3} = 0$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

- Why bother?

$$\text{NSum}[1/n^2, \{n, 1, \textit{Infinity}\}] = 1.64493$$

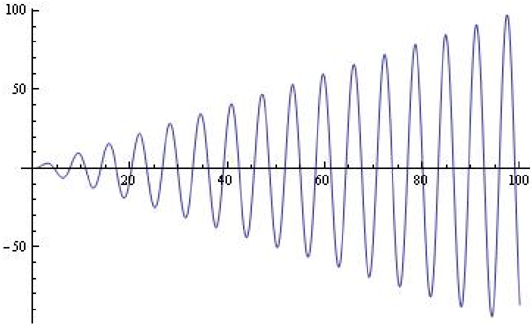
$$\text{Sum}[1/n^2, \{n, 1, \textit{Infinity}\}] = \frac{\pi^2}{6}$$

- Sum zeroes of other functions: Let x_n be the solutions of

$$\tan x = x \quad \Longrightarrow \quad \sin x - x \cos x = 0$$

.

- Find $\sum_{n=1}^{\infty} x_n^{-2}$



- Suitable contour integral: $F(z) = \sin z - z\cos z$

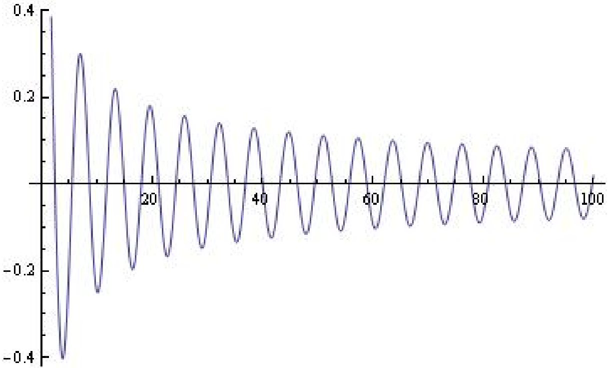
$$\lim_{R\to\infty} \quad \frac{1}{2\pi i}\int_{C_R} dz\ \frac{1}{z^2}\ \frac{d}{dz}\ln F(z)$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\int_{C_R} dz\ \frac{1}{z^2}\ \frac{z\sin z}{\sin z - z\cos z}$$

$$= 2\sum_{n=1}^{\infty}\frac{1}{x_n^2} - \frac{1}{5} = 0$$

$$\Longrightarrow \sum_{n=1}^{\infty}\frac{1}{x_n^2} = \frac{1}{10}$$

- Sum zeroes of other functions: Let $j_{0,n}$ be the zeroes of $J_0(x)$.
- Find $\sum_{n=1}^{\infty} j_{0,n}^{-2}$



- Suitable contour integral: $F(z) = J_0(z)$

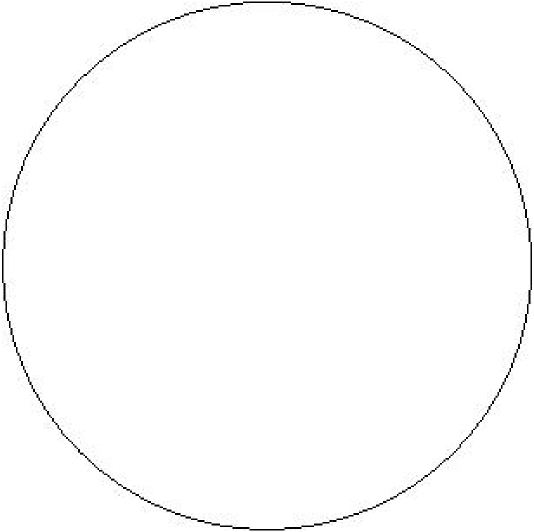
$$\lim_{R\to\infty} \quad \frac{1}{2\pi i} \int_{C_R} dz \; \frac{1}{z^2} \; \frac{d}{dz} \ln F(z)$$

$$= \lim_{R\to\infty} \frac{1}{2\pi i} \int_{C_R} dz \; \frac{1}{z^2} \; \frac{J_0'(z)}{J_0(z)}$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{j_{0,n}^2} - \frac{1}{2} = 0$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{j_{0,n}^2} = \frac{1}{4}$$

The spherical shell



- Laplacian for the two dimensional disk

$$\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]u_{m,n}(r,\varphi)=\omega_{m,n}^2 u_{m,n}(r,\varphi)$$

- Separation of variables:

$$u_{m,n}(r,\varphi)=J_{|m|}(\omega_{m,n}r)e^{im\varphi},\quad m\in\mathbb{Z}$$

- Impose boundary condition:

$$J_{|m|}(\omega_{m,n})=0$$

- Eigenvalues determined by

$$J_{|m|}(\omega_{m,n}) = 0$$

- Associated zeta function

$$\begin{aligned}\zeta(s) &= \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \omega_{m,n}^{-2s} \\ &= \sum_{m=-\infty}^{\infty} \lim_{R\to\infty} \frac{1}{2\pi i} \int_{C_R} dz\ z^{-2s} \frac{d}{dz} \ln J_{|m|}(z)\end{aligned}$$

- Techniques
  - deformation of the contour γ
  - Debye expansions
  - commutations of summation and integration
  - Mellin-Barnes integral representation

Construction of analytic continuation: add and subtract asymptotics

Example for clarification ($0 < a < 1$):

$$\sum_{\ell=1}^{\infty} \frac{1}{(\ell+a)^s}\Bigg|_{s=0} = \sum_{\ell=1}^{\infty} \frac{1}{\ell^s} \frac{1}{\left(1+\frac{a}{\ell}\right)^s}\Bigg|_{s=0}$$

$$= \sum_{\ell=1}^{\infty} \left[\frac{1}{\ell^s} \left(\frac{1}{\left(1+\frac{a}{\ell}\right)^s} - 1 + \frac{as}{\ell} \right) + \frac{1}{\ell^s} - \frac{as}{\ell^{s+1}} \right]_{s=0}$$

$$= \sum_{\ell=1}^{\infty} \left[\frac{1}{\ell^s} \left(\frac{1}{\left(1+\frac{a}{\ell}\right)^s} - 1 + \frac{as}{\ell} \right) \right]_{s=0} + \zeta_R(s)|_{s=0} - as\zeta_R(s+1)|_{s=0}$$

$$= 0 + \zeta_R(0) - a \, \mathrm{Res}\, \zeta_R(1) = -\frac{1}{2} - a$$

- Final answer for Casimir energy in three dimensions (el. magn. field)

$$E_0^{el.magn.} = \frac{1}{2}\zeta\left(-\frac{1}{2}\right) = \frac{0.04617}{a}$$

- Resulting repulsive Casimir force

$$F_{Cas} = -\frac{d}{da}E_0^{el.magn.} = \frac{0.04617}{a^2}$$

T.H. Boyer, Phys. Rev. **174** (1968) 1764-1774
K.A. Milton, Jr. L.L. De Raad and J. Schwinger, Ann. Phys. **115** (1978) 388-403
N.G. van Kampen, B.R.A. Nijboer and K. Schram, Phys. Lett. A **26** (1968) 307-308
Bordag, Elizalde, KK, Leseduarte, Romeo, ...

Background potential

- Spherically symmetric potential:

$$\widetilde{P} = -\Delta + M^2 + V(r) \qquad P^{free} = -\Delta + M^2$$

- Analysis of radial eigenvalue problem:

$$\psi_{p,\ell,m}(r,\Omega) = \frac{1}{r}\phi_{p,\ell}(r) Y_{lm}(\Omega) \quad \Rightarrow$$

$$\widetilde{P_\ell}\ \phi_{p,\ell}(r) := \left(-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + M^2 + V(r)\right)\phi_{p,\ell}(r) = p^2\phi_{p,\ell}(r)$$

- Regular solution:

$$\phi_{p,\ell}(r) \quad \sim \quad \hat{j}_\ell(pr) = \sqrt{\frac{\pi pr}{2}} J_{\ell+1/2}(pr) \quad \text{as } r \to 0$$

$$\phi_{p,\ell}(r) \quad \sim \quad \frac{i}{2}\left[f_\ell(p)\hat{h}_\ell^-(pr) - f_\ell^*(p)\hat{h}_\ell^+(pr)\right] \quad \text{as } r \to \infty$$

System in a finite ball of radius R:

Implicit eigenvalue equation:

$$f_\ell(p)\hat{h}_\ell^-(pR) - f_\ell^*(p)\hat{h}_\ell^+(pR) = 0$$

Zeta function analysis:

$$\begin{aligned}\zeta_{\widetilde{P}}(s) &= \sum_{\ell=0}^{\infty}(2\ell+1)\times \\ &\int_\gamma \frac{dk}{2\pi i}(k^2+M^2)^{-s}\frac{\partial}{\partial k}\ln\left[f_\ell(k)\,\hat{h}_\ell^-(kR) - f_\ell^*(k)\,\hat{h}_\ell^+(kR)\right] \\ &+\sum_{\ell=0}^{\infty}(2\ell+1)\sum_j (M^2-\kappa_{j,\ell}^2)^{-s}\end{aligned}$$

Minkowski space theory subtracted:

$$\zeta(s) = \sum_{\ell=0}^{\infty}(2\ell+1)\times$$

$$\int\limits_{\gamma} \frac{dk}{2\pi i}(k^2+M^2)^{-s}\frac{\partial}{\partial k}\ln\left[\frac{f_\ell(k)\hat{h}_\ell^-(kR) - f_\ell^*(k)\hat{h}_\ell^+(kR)}{\hat{h}_\ell^-(kR) - \hat{h}_\ell^+(kR)}\right]$$

$$+\sum_{\ell=0}^{\infty}(2\ell+1)\sum_j (M^2-\kappa_{j,\ell}^2)^{-s}$$

$$(R\to\infty) = \sum_{\ell=0}^{\infty}(2\ell+1)\int\limits_{\gamma}\frac{dk}{2\pi i}(k^2+M^2)^{-s}\frac{\partial}{\partial k}\ln f_\ell(k)$$

$$+\sum_{\ell=0}^{\infty}(2\ell+1)\sum_j (M^2-\kappa_{j,\ell}^2)^{-s}$$

- Zeta function in terms of Jost function

$$\zeta(s) \quad = \quad \frac{\sin \pi s}{\pi} \sum_{\ell=0}^{\infty} (2\ell+1) \int_{M}^{\infty} dk \, (k^2 - M^2)^{-s} \frac{\partial}{\partial k} \ln f_\ell(ik)$$

- Integral equation for the regular solution

$$\phi_{\ell,p}(r) = \frac{i}{2}\left(\hat{h}^-(pr) - \hat{h}^+(pr)\right) + \int_0^r dr' \mathcal{G}_{\ell,p}(r,r') V(r') \phi_{\ell,p}(r')$$

with the Green's function

$$\mathcal{G}_{\ell,p}(r,r') = \frac{1}{p}\left[\hat{h}_\ell^-(pr)\hat{h}^+(pr') - \hat{h}_\ell^+(pr)\hat{h}_\ell^-(pr')\right]$$

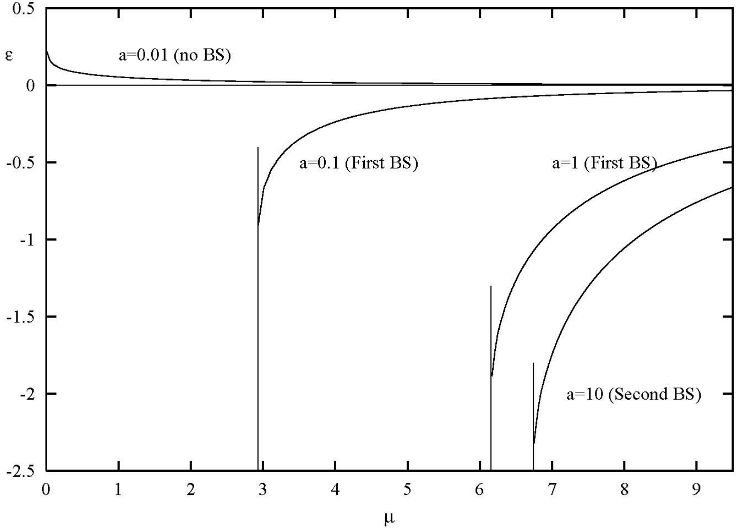
- Jost function

$$f_\ell(ip) = 1 + \int_0^\infty dr\ rV(r)\phi_{\ell,ip}K_{\ell+1/2}(pr)$$

- Asymptotics of the Jost function reduced to Bessel functions asymptotics.
- Example

$$V(r) = \lambda \frac{a^2(1-r^2)^2}{(a+r^2)^2}$$

Bordag, Dunne, Farhi, Graham, Hur, Jaffe, KK, Min, Quandt, Weigel, ...


0.5
ε
0
-0.5
-1
-1.5
-2
-2.5
a=0.01 (no BS)
a=0.1 (First BS)
a=1 (First BS)
a=10 (Second BS)
0
1
2
3
4
5
6
7
8
9
μ

Renormalization of the Casimir energy

- Singularities in the frequency-cutoff

$$E_0(\delta) = \frac{1}{2}\sum_J \omega_J e^{-\delta\omega_J^2}$$

- Representation of exponentials

$$e^{-v} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\alpha\ \Gamma(\alpha) v^{-\alpha}, \qquad \Re c > 0$$

- This shows

$$\begin{aligned} E_0(\delta) &= \frac{1}{2}\sum_J \omega_J \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha)(\delta\omega_J^2)^{-\alpha} \\ &= \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} d\alpha\ \Gamma(\alpha)\delta^{-\alpha}\zeta\left(\alpha - \frac{1}{2}\right), \qquad \Re c > \frac{d+1}{2} \end{aligned}$$

Meromorphic structure of the zeta function

- Relation between the zeta function and the heat kernel

$$\zeta(s) = \sum_J (\omega_J^2)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt\ t^{s-1} \sum_J e^{-t\omega_J^2} := \frac{1}{\Gamma(s)} \int_0^\infty dt\ t^{s-1} K(t)$$

- Heat kernel

$$K(t) = \sum_J e^{-t\omega_J^2}$$

- General situation: $\mathcal{M}$, $\partial\mathcal{M}$, $g_{\mu\nu}$, $\mathcal{B}$

$$P = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E)$$

- Heat kernel: $\{\lambda_k, \varphi_k\}$

$$\begin{aligned} K(t) &= \int_{\mathcal{M}} dx\; K(t,x,x) \\ &= \sum_{k=0}^{\infty} e^{-t\lambda_k} \end{aligned}$$

- Localized heat kernel:

$$\begin{aligned} K(t,F) &= \int_{\mathcal{M}} dx\; F(x)\; K(t,x,x) \\ &\stackrel{t\to 0}{\sim} \sum_{n=0,1/2,1,\dots}^{\infty} a_n(F,P,\mathcal{B})\; t^{n-D/2} \end{aligned}$$

General form of the coefficients

- Dimensional consideration:

$$
\begin{aligned}
\lambda_k &: \text{length}^{-2} \\
e^{-t\lambda_k} &: [t] = \text{length}^2 \\
a_n \ t^{n-D/2} &: [a_n] = \text{length}^{D-2n}
\end{aligned}
$$

- Structure of the coefficients:

$$
\begin{aligned}
a_n(F,P,\mathcal{B}) &= \int\limits_{\mathcal{M}} dx \ c_n(x,F,P) \\
&+ \int\limits_{\partial\mathcal{M}} dy \ b_n(y,F,P,\mathcal{B})
\end{aligned}
$$

$$
\begin{aligned}
c_n(x,F,P) &: \text{length}^{-2n} \\
b_n(y,F,P,\mathcal{B}) &: \text{length}^{1-2n}
\end{aligned}
$$

- Building blocks:

$$E\ ,\ R\ ,\ R_{ij}\ ,\ R_{ijkl}\ :\qquad \text{length}^{-2}$$

$$K_{ab} = N_{a;b}\ :\qquad \text{length}^{-1}$$

contractions

covariant derivatives

- Structure of the coefficients:

$$\begin{aligned} a_n(F,P,\mathcal{B}) &= \int_{\mathcal{M}} dx\ c_n(x,F,P) \\ &+ \int_{\partial\mathcal{M}} dy\ b_n(y,F,P,\mathcal{B}) \end{aligned}$$

$$\begin{aligned} c_n(x,F,P) &: \quad \text{length}^{-2n} \\ b_n(y,F,P,\mathcal{B}) &: \quad \text{length}^{1-2n} \end{aligned}$$

$$b_{1/2} = (4\pi)^{-d/2}\, \delta\, F\, [\partial\mathcal{M}]$$

$$b_{1/2} = (4\pi)^{-d/2}\left(-\frac{1}{4}\right)F(1)[\partial\mathcal{M}]$$

$$b_1 = (4\pi)^{-D/2}\, 6^{-1}\, [\, d_0\, F\, K + d_1\, F_{;m}\,]\, [\partial\mathcal{M}]$$

$$b_1 = (4\pi)^{-D/2} 6^{-1}[2F(1)d + 3F'(1)][\partial\mathcal{M}]$$

$$\begin{aligned} b_{3/2} &= (4\pi)^{-d/2}\,\delta\, 96^{-1}\, \{\, F\,(c_0 E + c_1 R + c_2 R_{mm} + c_3 K^2 \\ &\quad + c_4 K_{ab} K^{ab}) + c_5 K F_{;m} + c_6 F_{;mm}\,\}\, [\partial\mathcal{M}] \end{aligned}$$

$$\begin{aligned} b_{3/2} &= (4\pi)^{-d/2}\delta 96^{-1}\left\{F(1)(7d^2 - 10d) + 30dF'(1)\right. \\ &\quad \left. + 24F''(1)\right\}[\partial\mathcal{M}] \end{aligned}$$

How to find the remaining universal multipliers?

- Consider product manifolds

$$\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2, \quad \partial\mathcal{M}_2 = \emptyset, \quad P_1 = \Delta_1, \quad P_2 = -\Delta_2 + E(x_2)$$

$$a_{3/2}(1, P, \mathcal{B}) = a_{3/2}(1, P_1, \mathcal{B}) a_0(1, P_2) + a_{1/2}(1, P_1, \mathcal{B}) a_1(1, P_2)$$

- Note

$$R(\mathcal{M}_1 \times \mathcal{M}_2) = R(\mathcal{M}_1) + R(\mathcal{M}_2)$$

$$\delta\ 96^{-1}(c_0 E + c_1 R(\mathcal{M}_2)) = \delta\ 6^{-1}(6E + R(\mathcal{M}_2))$$

$$\Longrightarrow c_0 = 96 \qquad c_1 = 16$$

- Conformal transformations

$$\begin{aligned} P(\epsilon) &= e^{-2\epsilon F} P, \qquad \mathcal{B}(\epsilon) = e^{-\epsilon F}\mathcal{B} \\ \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_n(1, P(\epsilon), \mathcal{B}(\epsilon)) &= (D-2n) a_n(F, P, \mathcal{B}) \end{aligned}$$

- How to find heat kernel coefficients for $P(\epsilon)$?

$$g^{ij}(\epsilon) = e^{-2\epsilon F} g^{ij}, \qquad \Delta = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right)$$

- Coefficient of $F_{;mm}$

$$\frac{1}{2}(D-2)c_0 - 2(D-1)c_1 - (D-1)c_2 - (D-3)c_6 = 0$$
$$\Longrightarrow c_2 = -8$$

P.B. Gilkey, Asymptotic formulae in spectral geometry, Chapman& Hall/CRC, Boca Raton, FL, 2003
KK, Spectral functions in mathematics and physics, Chapman& Hall/CRC, Boca Raton, FL, 2002
D.V. Vassilevich, Phys. Rept. **388** (2003) 279-260

- Relation between the zeta function and the heat kernel

$$\begin{aligned}\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}K(t)\\ &\approx \frac{1}{\Gamma(s)}\sum_{n=0,1/2,1,\dots}^{\infty} a_n(1,P,\mathcal{B})\int_0^1 t^{s-1+n-\frac{D}{2}}\\ &\approx \frac{1}{\Gamma(s)}\sum_{n=0,1/2,1,\dots}^{\infty}\frac{a_n(1,P,\mathcal{B})}{s+n-\frac{D}{2}}\end{aligned}$$

- Meromorphic structure of the zeta function

$$\text{Res}\ \left(\zeta(s)\Gamma(s)\right)\big|_{s=\frac{D}{2}-n} = a_n(1,P,\mathcal{B})$$

$$\text{Res}\ \zeta(z) = \frac{a_{\frac{D}{2}-z}(1,P,\mathcal{B})}{\Gamma(z)} \qquad \zeta(-q) = (-1)^q q!\, a_{\frac{D}{2}+q}(1,P,\mathcal{B})$$

- Earlier we derived

$$E_0(\delta) = \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} d\alpha\ \Gamma(\alpha)\delta^{-\alpha}\zeta\left(\alpha - \frac{1}{2}\right)$$

- Regularized Casimir energy in the frequency-cutoff

$$\begin{aligned} E_0(\delta) &= \frac{1}{\sqrt{\pi}} a_0 \delta^{-2} + \frac{\sqrt{\pi}}{4} a_{1/2}\delta^{-3/2} + \frac{1}{2\sqrt{\pi}} a_1 \delta^{-1} \\ &+ \frac{1}{4\sqrt{\pi}} a_2 \left[\gamma + \ln \delta\right] + \frac{1}{2} \mathrm{FP}\ \zeta\left(-\frac{1}{2}\right) + \mathcal{O}(\delta) \end{aligned}$$

- Regularized Casimir energy in the zeta function scheme

$$E_0 = \frac{1}{2}\sum_J \omega_J \quad \to \quad E_0(s) = \frac{\mu^{2s}}{2}\sum_J \omega_J^{1-2s} = \frac{\mu^{2s}}{2}\zeta_P\left(s - \frac{1}{2}\right)$$

$$E_0(s) = -\frac{1}{4\sqrt{\pi}} a_2 \left[\frac{1}{s} + \ln \mu^2\right] + \frac{1}{2}\mathrm{FP}\ \zeta\left(-\frac{1}{2}\right) + \mathcal{O}(s)$$

- Massive scalar field

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\ t^{s-1} e^{-m^2 t} K(t)$$
$$m \overset{\sim}{\to} \infty \quad \frac{1}{\Gamma(s)} \sum_{n=0,1/2,1,\dots}^{\infty} a_n \frac{\Gamma\left(s+n-\frac{3}{2}\right)}{m^{2\left(s+n-\frac{3}{2}\right)}}$$

- In particular

$$E_0\left(-\frac{1}{2}+\epsilon\right) = -\frac{m^4}{8\sqrt{\pi}} a_0 \left(\frac{1}{s} - \frac{1}{2} + \ln\left[\frac{4\mu^2}{m^2}\right]\right)$$
$$-\frac{m^3}{3} a_{1/2} - \frac{m^2}{4\sqrt{\pi}} a_1 \left(\frac{1}{2} - 1 + \ln\left[\frac{4\mu^2}{m^2}\right]\right)$$
$$+\frac{1}{2} m a_{3/2} - \frac{1}{4\sqrt{\pi}} a_2 \left(\frac{1}{s} - 2 + \ln\left[\frac{4\mu^2}{m^2}\right]\right) + \mathcal{O}\left(\frac{1}{m}\right) + \mathcal{O}(\epsilon)$$

- Normalization condition

$$\lim_{m\to\infty} E_0^{ren} = 0$$

Configurations with finite Casimir energy/force

- Piston configurations
- Electromagnetic field
- Separate bodies

- Relevant heat kernel coefficients

$$
\begin{aligned}
a_0^{\pm} &= (4\pi)^{-D/2}[\mathcal{M}] \\
a_{1/2}^{\pm} &= \pm(4\pi)^{(D-1)/2}\frac{1}{4}[\partial\mathcal{M}] \\
a_1^{\pm} &= (4\pi)^{-D/2}E[\mathcal{M}] + (4\pi)^{-D/2}6^{-1}\left(2K+12S\right)[\partial\mathcal{M}] \\
a_{3/2}^{\pm} &= \pm\frac{1}{384(4\pi)^{(D-1)/2}}\left(96E+\begin{pmatrix}13\\7\end{pmatrix}K^2+\begin{pmatrix}2\\-10\end{pmatrix}K_{ab}K^{ab}\right. \\
&\qquad \left.+96SK+192S^2\right)[\partial\mathcal{M}] \\
a_2^{\pm} &= (4\pi)^{-D/2}360^{-1}\left(60\Delta E+180E^2\right)[\mathcal{M}] \\
&+ (4\pi)^{-D/2}360^{-1}\left(\begin{pmatrix}-240\\120\end{pmatrix}E_{;m}+24K_{:a}^{a}+120EK+\begin{pmatrix}40/3\\40/21\end{pmatrix}K^3\right. \\
&+ \begin{pmatrix}8\\-88/7\end{pmatrix}K_{ab}K^{ab}K+\begin{pmatrix}32/3\\320/21\end{pmatrix}K_{ab}K_c^{b}K^{ac}+720SE+144SK^2 \\
&+ \left.48SK_{ab}K^{ab}+480S^2K+480S^3+120S_{:a}^{a}\right)
\end{aligned}
$$

Separate bodies and the *TGTG* representation

- Let S be the boundary surface given by

$$\vec{r} = \vec{u}(\vec{\eta}) = \vec{u}(\eta_1, \eta_2)$$

- Reminder

$$E_0 = -\frac{i}{2T} \mathrm{Tr} \ln \tilde{\mathcal{K}}$$

where

$$\tilde{K}(\eta, \eta') = \int d^4x \int d^4x' \; H(\eta, x) G(x, x') H(\eta', x') = G(u(\eta), u(\eta'))$$

with (Dirichlet boundary conditions)

$$H(\eta, x) = \delta^4(x - u(\eta))$$

M. Bordag, G.L. Klimchitskaya, U. Mohideen, V.M. Mostepanenko, Advances in the Casimir Effect, Oxford University Press, Oxford, 2009

- Fourier-transformation for the time variable

$$
\begin{aligned}
E_0 &= -\frac{i}{2T}\mathrm{Tr}\ln\tilde{\mathcal{K}} = -\frac{i}{2T} i \int\limits_{-\infty}^{\infty} dx_4 \ \left\langle x_4 \left| \mathrm{Tr}_3 \ln\tilde{\mathcal{K}} \right| x_4 \right\rangle \\
&= -\frac{i}{2T} i \int\limits_{-\infty}^{\infty} dx_4 \ \int\limits_{-\infty}^{\infty} \frac{d\xi}{2\pi}\mathrm{Tr}_3 \ln\tilde{\mathcal{K}}_\xi = -\frac{i}{2T} iT \ \int\limits_{-\infty}^{\infty} \frac{d\xi}{2\pi}\mathrm{Tr}_3 \ln\tilde{\mathcal{K}}_\xi \\
&= \frac{1}{2\pi}\int\limits_{0}^{\infty} d\xi \mathrm{Tr}_3 \ln\tilde{\mathcal{K}}_\xi
\end{aligned}
$$

where

$$
\tilde{K}_\xi(\vec{\eta},\vec{\eta}') = \int d\vec{r} \int d\vec{r}' H(\vec{\eta},\vec{r}) G_\xi(\vec{r},\vec{r}') H(\vec{\eta}',\vec{r}')
$$

- Let $S = S_A \cup S_B$ with $S_A \cap S_B = \emptyset$.
- Note the block structure of the kernel $\tilde{K}_\xi$

$$\tilde{K}_\xi = \begin{pmatrix} \tilde{K}_{\xi,AA}(\vec{\eta}_A, \vec{\eta}'_A) & \tilde{K}_{\xi,AB}(\vec{\eta}_A, \vec{\eta}'_B) \\ \tilde{K}_{\xi,BA}(\vec{\eta}_B, \vec{\eta}'_A) & \tilde{K}_{\xi,BB}(\vec{\eta}_B, \vec{\eta}'_B) \end{pmatrix}$$

- Isolate 'interaction' between the surfaces

$$\tilde{K}_\xi = \begin{pmatrix} \tilde{K}_{\xi,AA} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{K}_{\xi,BB} \end{pmatrix} \begin{pmatrix} 1 & \tilde{K}^{-1}_{\xi,AA}\tilde{K}_{\xi,AB} \\ \tilde{K}^{-1}_{\xi,BB}\tilde{K}_{\xi,BA} & 1 \end{pmatrix}$$

- Relevant *finite* part of the Casimir energy

$$E = \frac{1}{2\pi} \int_0^\infty d\xi \ \mathrm{Tr}_3 \ln \left(1 - \tilde{K}^{-1}_{\xi,AA}\tilde{K}_{\xi,AB}\tilde{K}^{-1}_{\xi,BB}\tilde{K}_{\xi,BA} \right)$$

- Interaction of quantum field with background field $V(\vec{r})$

$$E_0 = -\frac{1}{2\pi} \int_0^\infty d\xi \;\mathrm{Tr}_3 \ln \mathcal{G}_\xi^{(V)}$$

where

$$\left[\xi^2 - \vec{\nabla}^2 + V(\vec{r})\right] G_\xi^{(V)}(\vec{r}, \vec{r}') = \delta^3(\vec{r} - \vec{r}')$$

- In operator language

$$\left[\xi^2 - \vec{\nabla}^2 + \mathcal{V}\right] \mathcal{G}_\xi^{(V)} = \mathbf{1}$$

$$\left[\xi^2 - \vec{\nabla}^2\right] \mathcal{G}_\xi^{(0)} = \mathbf{1}$$

- Rewriting $\mathcal{G}_\xi^{(V)}$ in terms of $\mathcal{G}_\xi^{(0)}$

$$\left[\xi^2 - \vec{\nabla}^2\right] \mathcal{G}_\xi^{(V)} = \mathcal{G}_\xi^{(0)-1}\mathcal{G}_\xi^{(V)} = \mathbf{1} - \mathcal{V}\mathcal{G}_\xi^{(\mathbf{V})} \Longrightarrow$$
$$\mathcal{G}_\xi^{(V)} = \mathcal{G}_\xi^{(0)} - \mathcal{G}_\xi^{(0)}\mathcal{V}\mathcal{G}_\xi^{(V)}$$

Also

$$\left(\mathcal{G}_\xi^{(0)-1} + \mathcal{V}\right) \mathcal{G}_\xi^{(V)} = \mathcal{G}_\xi^{(0)-1}\left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}\right) \mathcal{G}_\xi^{(V)} = \mathbf{1} \Longrightarrow$$
$$\mathcal{G}_\xi^{(V)} = \left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}\right)^{-1} \mathcal{G}_\xi^{(0)}$$

- Introduce T-matrix

$$\begin{aligned} \mathcal{T} &= \mathcal{V}\left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}\right)^{-1} \Longrightarrow \\ \mathcal{G}_\xi^{(V)} &= \mathcal{G}_\xi^{(0)} - \mathcal{G}_\xi^{(0)}\mathcal{T}\mathcal{G}_\xi^{(0)} \end{aligned}$$

- Exploit structure when

$$V(\vec{r}) = V_A(\vec{r}) + V_B(\vec{r})$$

Separation of A-only and B-only contribution

$$\begin{aligned} \mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}(\mathcal{V}_\mathbf{A} + \mathcal{V}_\mathbf{B}) &= \left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}_\mathbf{A}\right)\left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}_\mathbf{B}\right) - \mathcal{G}_\xi^{(0)}\mathcal{V}_A\mathcal{G}_\xi^{(0)}\mathcal{V}_B \\ &= \left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}_\mathbf{A}\right)(\mathbf{1} - \mathcal{M}_\xi)\left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}_\mathbf{B}\right) \end{aligned}$$

where

$$\mathcal{M}_\xi = \left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}_\mathbf{A}\right)^{-1}\mathcal{G}_\xi^{(0)}\mathcal{V}_A\mathcal{G}_\xi^{(0)}\mathcal{V}_B\left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}_\mathbf{B}\right)^{-1}$$

- Express this in terms of the T-matrix

$$\mathcal{T}_i = \mathcal{V}_i\left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}\mathcal{V}_\mathbf{i}\right)^{-1} \Longrightarrow \mathcal{M}_\xi = \mathcal{G}_\xi^{(0)}\ \mathcal{T}_A\ \mathcal{G}_\xi^{(0)}\ \mathcal{T}_B$$

- Isolate interaction terms in the Casimir energy

$$
\begin{aligned}
\mathrm{Tr}_3 \ln \mathcal{G}_\xi^{V_A+V_B} &= \mathrm{Tr}_3 \ln \left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})}(\mathcal{V}_{\mathbf{A}} + \mathcal{V}_{\mathbf{B}})\right)^{-1} \mathcal{G}_\xi^{(0)} \\
= \mathrm{Tr} \ln \mathcal{G}_\xi^{(0)} &- \mathrm{Tr} \ln \left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})} \mathcal{V}_{\mathbf{A}}\right) - \mathrm{Tr} \ln \left(\mathbf{1} + \mathcal{G}_\xi^{(\mathbf{0})} \mathcal{V}_{\mathbf{B}}\right) - \mathrm{Tr} \ln (\mathbf{1} - \mathcal{M}_\xi) \\
= -\mathrm{Tr} \ln \mathcal{G}_\xi^{(0)} &+ \mathrm{Tr} \ln \mathcal{G}_\xi^{(V_A)} + \mathrm{Tr} \ln \mathcal{G}_\xi^{(V_B)} - \mathrm{Tr} \ln (\mathbf{1} - \mathcal{M}_\xi)
\end{aligned}
$$

- Finite Casimir interaction energy

$$
\begin{aligned}
E &= \frac{1}{2\pi} \int_0^\infty d\xi \; \mathrm{Tr}_3 \ln (\mathbf{1} - \mathcal{M}_\xi) \\
&= \frac{1}{2\pi} \int_0^\infty d\xi \mathrm{Tr} \ln \left(1 - \mathcal{G}_\xi^{(0)} \mathcal{T}_A \mathcal{G}_\xi^{(0)} \mathcal{T}_B\right)
\end{aligned}
$$

- More explicit

$$
\begin{aligned}
E &= \frac{1}{2\pi} \int_0^\infty d\xi \ \mathrm{Tr}_3 \ln \left(\mathbf{1} - \mathcal{M}_\xi \right) \\
&= \frac{1}{2\pi} \int_0^\infty d\xi \mathrm{Tr} \ln \left(1 - \mathcal{G}_\xi^{(0)} \mathcal{T}_A \mathcal{G}_\xi^{(0)} \mathcal{T}_B \right)
\end{aligned}
$$

with

$$
\mathcal{M}_\xi(\vec{r}, \vec{r}') = \int_A dr'' \int_B d\tilde{r} \int_B d\tilde{r}' \, T^A(\vec{r}, \vec{r}'') G_\xi^{(0)}(\vec{r}'', \tilde{\vec{r}}) T^B(\tilde{\vec{r}}, \tilde{\vec{r}}') G_\xi^{(0)}(\tilde{\vec{r}}', \vec{'})
$$

- Compare with

$$
E = \frac{1}{2\pi} \int_0^\infty d\xi \ \mathrm{Tr}_3 \ln \left(1 - \tilde{K}_{\xi,AA}^{-1} \tilde{K}_{\xi,AB} \tilde{K}_{\xi,BB}^{-1} \tilde{K}_{\xi,BA} \right)
$$

A. Bulgac, P. Magierski and A. Wirzba, Phys. Rev. D**73** (2006) 025007 (14pp)
O. Kenneth and I. Klich, Phys. Rev. Lett. **97** (2006) 160401 (4pp)