Quantum Symmetries and Non commutative Quantum Field Theories

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Plan

- Some Motivation
- Hopf Symmetries
- Twisted Hopf Symmetries in Moyal Plane
- Twisting of Statistics
- QFT
- Gauge and Diffeos
- Some calculational results

Noncommutative Spacetime

A central feature of Quantum Mechanics is Noncommutativity of quantities

$$[x, p] = i\hbar$$

$$\triangle x \triangle p \sim \hbar$$

A natural extension came soon after

$$[x_i, x_j] \sim i\theta_{ij} \ , \ \triangle x_i \triangle x_j \sim \theta_{ij}$$

Provides small length cuttoff
May regularize Quantum Field Theories

Quant. Degrees of Spacetime

- Quantum Gravity still problematic, canonical quantization not renormalizable
- Fundamental degrees may not be smooth manifolds and metrics
- Casual Sets, Dynamical triangulation

Matrix Models Geometry is emergent

Typically interpreted as a Noncommutative geometry at intermediate steps

Quantum Foam

- The Compton wavelength of the Probe, localizing a length L

$$\frac{\hbar}{Mc} \leq L \quad or \quad M \geq \frac{\hbar}{Lc}$$

for L, Planck length, M is Planck mass

Event horizons will form, limiting the localization

Uncertainty relations among coordinates

$$\Delta x_i \Delta x_j \gtrsim L_p^2$$

The Moyal Plane

The Moyal plane algebra $\mathcal{A}_\theta$

$$[\hat{x}_\mu, \hat{x}_\nu] \;=\; i\theta_{\mu\nu}$$

This algebra is isomorphic to the algebra of functions with star product

$$f * g \;=\; f\, e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial_\nu}}\, g, \quad f, g \in \mathcal{C}^\infty \tag{1}$$

Noncommutative field theory actions are obtained by replacing ordinary products between the fields by star-products, e.g.,

$$S[\phi] \;=\; \int d^d x\, \partial_\mu\phi * \partial^\mu\phi \;+ m\phi(x) * \phi(x) \;+ g\phi(x) * \phi(x) \cdots \phi(x)$$

Breaking of Lorentz Invariance

Because of $\theta_{\mu\nu}$s, Poincaré group is not an automorphism of $\mathcal{A}_\theta$

$$g \triangleright f(x) \longrightarrow f(g^{-1}x)$$

But

$$g \triangleright f_1(x) * g \triangleright f_2(x) \neq f_1 * f_2(g^{-1}x)$$

- Leibnitz rule does not work
- Without the symmetries, there is little justification for choosing actions, and representations of Poincaré groups as fields.

Quantum Symmetries

- One can try to deform (or rather the action of) the symmetries on the non-commutative algebra
- The actions then are invariant under this new symmetry and fields appear as representations
- Hopf algebras are one way to look at the automorphisms of an algebra
- They also naturally describe the symmetries in Hilbert space of a quantum theory
- One can deform some structures in the Hopf algebra related to Poincaré group so that the new Hopf structure is an automorphism of the algebra

Hopf Algebras

Hopf Algebras $(\mathcal{A}, \Delta, \eta, \epsilon, S)$ are generalizations of groups, where Δ is a linear algebra mosphism:

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$$

- $$\Delta(ab) = \Delta(a)\Delta(b).$$
- One easy way to think about them is to think of $\mathcal{A}$ as a group algebra of a group and Δ defining the tensor product of representations
- Consider representation of a Lie group G in a Hilbert space of a multiparticle theory. Then we automatically have the representation of the full group algebra and the enveloping algebra of the Lie group $\mathfrak{U}(g)$

$$g : v \longrightarrow \rho(g)v, \quad g \in \mathfrak{U}(g)$$

Hopf Algebras

- For multiparticle states (tensor products of reps), we want
- $$v_1 \otimes v_2 \longrightarrow \rho \otimes \rho(\Delta(g))v_1 \otimes v_2$$
- Rep. because of homomorphism of coproduct
- We are used to of taking
- $$\Delta_o(g) = g \otimes g$$ for group elements
- $$\Delta_0(L_i) = 1 \otimes L_i + L_i \otimes 1$$ for Lie algebra elemts
- Therefore, the moment we talk about representation of a group on a multiparticle Hilbert space, we have representation of a full Hopf algebra
- The coproduct defines how the symmetry acts in a tensor product of spaces

Hopf Algebras

In field theory one has representations of symmetry on fields which themselves form an algebra (easy to construct covariant interactions) Then one has automorphism

$$g \triangleright (\phi \cdot \psi) = g \triangleright \phi \cdot g \triangleright \psi \tag{5}$$

which can be written as (for $\Delta = g \otimes g$)

$$g \rangle m(\phi \otimes \psi) = m\Delta(g)\phi \otimes \psi \tag{6}$$

One can then generalize the idea of symmetry by taking the last equation as the definition for the action of a symmetry on a carrier algebra for any coproduct.

Breaking of Poincaré Invariance

It is clear that Poincaré group is not an automorphism of Moyal algebra. We can define the action of Poincaré group on a functions f as usual

$$\Lambda : f(x) \rightarrow f(\Lambda^{-1}x) \tag{7}$$

but the product does not transform correctly,

$$f(\Lambda^{-1}x) * g(\Lambda^{-1}x) \neq f * g\,(\Lambda^{-1}x)$$

But we can note that the problem occurs when we multiply two fields (Leibnitz rule does not work).

Coproduct and the Leibnitz Rule

We can formalize the Leibnitz rule in the following way

Whenever a symmetry transformation (an algebra G) acts on another algebra $\mathcal{A}$

$$g\, m(f \otimes g) = m(\Delta(g)\, f \otimes g)$$

where Δ is a linear group homomorphism map

$$\Delta : \mathcal{U}G \longrightarrow \mathcal{U}G \otimes \mathcal{U}G \ , \ \ \Delta(g) = g_\alpha^{(1)} \otimes g_\alpha^{(2)}$$

$$\Delta(g_1)\Delta(g_2) = \Delta(g_1 g_2)$$

The homomorphism property ensures that we have correct action.

There can be many coproducts on the group, one example is

Coproduct

$$\begin{array}{ccc} \alpha \otimes \beta & \longrightarrow & (\rho \otimes \rho)\Delta(g)\alpha \otimes \beta \\ m \downarrow & & \downarrow m \\ m(\alpha \otimes \beta) & \longrightarrow & \rho(g)m(\alpha \otimes \beta) \end{array} \tag{8}$$

If such a Δ can be found, G has a well-defined action in $\mathcal{A}$. In the absence of such a Δ, G does not act on $\mathcal{A}$.

The coproduct *defines* how the symmetry acts in the tensor product of representations

$$v_1 \otimes v_2 \longrightarrow \rho \otimes \rho(\Delta(g))v_1 \otimes v_2$$

Hopf Algebra Summary

Hence we have

- The symmetries are characterized by the pair (G, Δ)
- The group action on individual fields in a product is defined using a Leibnitz rule, which in turn comes from coproduct on the group. For this action to be well defined, the coproduct has to be special.
- while (G, Δ_o) does not have action on noncommutative plane another pair (G, Δ_θ) may have , same goes for the invariance of $S[\phi]$
- If we find such a Δ_θ we have recognized a hidden symmetry
- Note that the symmetry "group" structure will not change but its action on multiparticle states (tensor product of representations) will change

The Drinfel'd Twist

Give the action of G with a coproduct Δ_0 on an algebra $\mathcal{A}_0$, we have action of G with a coproduct Δ_θ on another algebra $\mathcal{A}_\theta$ where

$$\Delta_\theta = F\Delta_0 F^{-1}, \quad F \in \mathcal{U}(G) \otimes \mathcal{U}(G)$$

and the product in $\mathcal{A}_\theta$ is given by

$$m_\theta f \otimes g = m_0 F^{-1} f \otimes g$$

The Drinfel'd Twist

Note that the star-product on Moyal plane is of this form (Chaichian et. al., Aschieri et. al.)

$$f * g = m_o \mathcal{F}_\theta^{-1} \; f \otimes g \quad \mathcal{F}_\theta \; = \; e^{\frac{i}{2}\theta_{\mu\nu} P^\mu \otimes P^\nu}$$

Hence Poincaré group has a well defined action on the Moyal plane with the coproduct

$$\Delta_\theta(g) \; = \; \mathcal{F}\, \Delta_\circ(g)\, \mathcal{F}_\theta^{-1}$$

$$\Delta_\theta(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}[(p\cdot\theta)_\mu \otimes p_\nu - p_\nu \otimes (p\cdot\theta)_\mu - \mu \leftrightarrow \nu]$$

Statistics and the Twist

For identical particles we must have

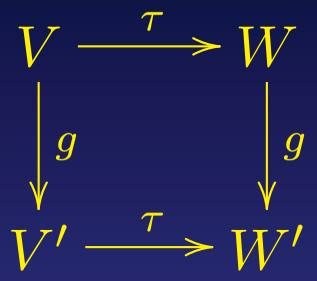
$$\phi \otimes \chi(k_1, k_2) \;=\; \alpha(k_1, k_2)\chi \otimes \phi(k_1, k_2).$$

Compatibility of Lorentz invariance with statistics means that for two states V and W, which are related as above we must have

$$\begin{array}{ccc} V & \sim & W \\ \downarrow g & & \downarrow g \\ V' & \sim & W' \end{array}$$

Statistics and the Twist

Or in other words, the operation of $V \xrightarrow{\tau} W$ should commute with the action of g



i.e., the statistics operator τ is superselected

One can show that the usual statistics is not compatible with the twisted coproduct

$$\tau\Delta_\theta \neq \Delta_\theta\tau$$

The Twisted Statistics

The reason behind this contradiction is the fact that the new coproduct $\Delta_\theta(g)$ is not cocommutative. i.e.,

$$\Delta_\theta(g) = g_\alpha^{(1)} \otimes g_\alpha^{(2)} \neq \sigma\Delta_\theta(g) = \Delta'_\theta(g) = g_\alpha^{(2)} \otimes g_\alpha^{(1)}. \quad (9)$$

But there exists an element belonging to $\mathfrak{G} \otimes \mathfrak{G}$ called an $\mathcal{R}$-matrix, such that

$$\mathcal{R}\,\Delta_\theta = \Delta'_\theta\,\mathcal{R} \quad (10)$$

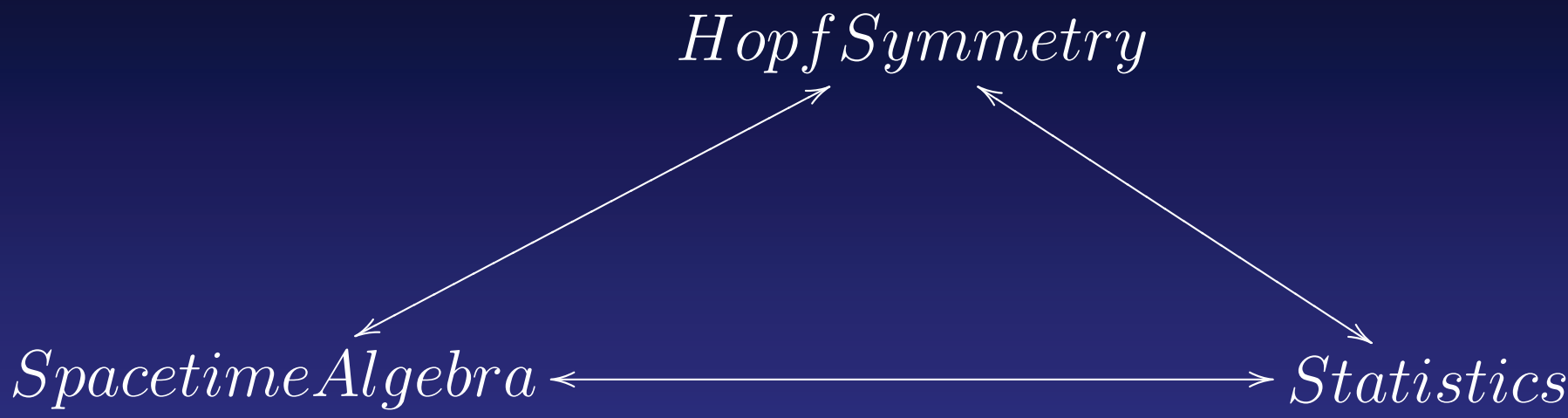
In this case the correct statistics to use is

$$\phi \otimes \chi = \sigma\mathcal{R}\,\phi \otimes \chi \quad (11)$$

for Δ_θ in , the $\mathcal{R}-$ matrix is

$$\mathcal{R} = \mathcal{F}^{-2} \quad (12)$$

Connections in some algebraic structures



Quantum Field Theory and Twisted Poincaré Symmetry

We can construct quantum field theories with the twisted Poincaré as the invariance

Start with a multiparticle Hilbert space, which carries representation of the twisted symmetry group and operators $U(g)$ representing the twisted Poincaré group

$$U(g)|k_1, k_2> = \rho_{k_1' k_1}(f^{\alpha} g f_{\beta})\, \rho_{k_2' k_2}(f_{\alpha} g f^{\beta})\, |k_1', k_2'>$$

where

$$\rho_{k'k}(g) =< k'|U(g)|k >$$

The Statistics

A compatible statistics

$$|k_1, k_2 >= e^{i\theta_{\mu\nu} k_1^\mu k_2^\nu} |k_2, k_1 >$$

and scalar product

$$(|k_1, k_2 > \, , \; |p_1, p_2 >) =$$
$$k_1^0 k_2^0 \, [\, \delta^3(k_1 - p_1)\delta^3(k_2 - p_2) + \; e^{-i\theta_{\mu\nu} p_2^\mu p_1^\nu} \delta^3(k_1 - p_2)\delta^3(k_2 - p_1) \,]$$

Creation/Annihilation Operators

Creation/annihilation operators. Defining $a^\dagger$ to be the operator which adds a particle to the right of the particle list.

$$a_k^\dagger |k_1, k_2, \cdots, k_n> = |k_1, k_2, \cdots, k_n, k>$$

One can derive

$$a_{k_2}^\dagger a_{k_1}^\dagger = e^{i\theta_{\mu\nu} k_2^\mu k_1^\nu} a_{k_1}^\dagger a_{k_2}^\dagger$$

$$a_{k_2} a_{k_1} = e^{i\theta_{\mu\nu} k_2^\mu k_1^\nu} a_{k_1} a_{k_2}$$

$$a_{k_2} a_{k_1}^\dagger = e^{i\theta_{\mu\nu} k_2^\mu k_1^\nu} a_{k_1}^\dagger a_{k_2} + k_1^0 \delta^3(k_1 - k_2)$$

Interactions

Construction of interactions leading to an invariant S-Matrix under very general conditions demands that

$$[U(g), H_I(t)] = 0$$

If one then wants to construct H_I by fields

$$\Phi(x) = \int d^3k [e^{ik\cdot x} a_k + c.c.]$$

one is lead to the form

$$H_I(x) = \Phi(x) * \Phi(x) * \cdots * \Phi(x)$$

Noncommutativity emerges as a consequence of requiring invariance under twisted Poincaré group

Invariance of N-Point Functions

One can show that the Wightman functions of the twisted theory are twisted Poincaré invariant, while that of usual noncommutative theory are not.

$$< 0|\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)|0 >= \int \frac{d^3p d^3k}{(2\pi)^6 4p_o k_o} e^{-i\theta_{\mu\nu}p^\mu k^\nu} e^{-ip\cdot(x_1-x_3)} e^{-ik\cdot(x_2-x_3)} + D(x_1-x_2)\ D(x_3-x_4) + D(x_1-x_4)\ D(x_2-x_4).$$

Even the free theory is non-local and non-gaussian.

Some Results

1-One can show that for a general interacting theory all the dependence on θ comes from the statistics of in and out states. i.e., for

$$S_\theta(in|out) =_\theta < in|out >_\theta =_\theta < free\ particles|S|free\ particles >_\theta$$

one can express the states $|free\ particles >_\theta$ in terms of $|free\ particles >_{\theta=0}$ states and then S is the same operator as in

$$S_{\theta=0}(in|out) =_{\theta=0} < freeparticles|S|freeparticles >_\theta = 0$$

No UV-IR mixing.

2-Statistical potential for two twisted fermions is not infinite at same position.

Gauge Symmetries

One can think of realizing gauge symmetries in a similar fashion in order to:

- Preserve the original structure of the gauge symmetry
- Treat gauge groups like $SU(N)$ and not just $U(N)$
- Preserve the semi-direct product structure $G \ltimes P$ of the gauge and Poincaré group as in the commutative case

$$(g', d') \cdot (g, d) = (g'd' \triangleright g, d'd)$$

Then the gauge transformation just depend on commuting coordinates but act with twisted coproduct. The matter fields are $\mathcal{A}_\theta$ modules. Compatibility with this module structure or preserving the semi-direct product structure of gauge and Poincaré group enforces us to conclude the twist of the coproduct

Commutative Action

For $\hat{x}_\mu$ we have two operators, $\hat{x}^L_\mu$ and $\hat{x}^R_\mu$, acting on $\mathcal{A}_\theta$

$$\begin{aligned} \hat{x}^L_\mu \hat{f} &= \hat{x}_\mu \hat{f} \\ \hat{x}^R_\mu \hat{f} &= \hat{f} \hat{x}_\mu \end{aligned}$$

From these operators we can form the operator

$$\hat{x}^C_\mu = \frac{1}{2}(\hat{x}^L_\mu + \hat{x}^R_\mu)$$

which forms representation of commutative algebra

$$[\hat{x}^C_\mu, \hat{x}^C_\mu] = 0$$

also we can have the operator

$$\hat{p}_\nu = -\frac{1}{\theta_{\mu\nu}} ad\hat{x}_\mu = -\frac{1}{\theta_{\mu\nu}}(\hat{x}^L_\mu - \hat{x}^R_\mu)$$

Commutative Action

We then have

$$[\hat{x}^C_\mu, \hat{p}_\nu] = i\delta_{\mu\nu}$$

Since the algebra of $\hat{x}^C$ and $\hat{p}$ is exactly as in commutative case, we have the representatives of the Poincaré generators,

$$M_{\mu\nu} = \hat{x}^C_\mu \hat{p}_\nu - \hat{x}^C_\nu \hat{p}_\mu$$

In the star-product langauge the operators $\hat{x}^C$ ad $\hat{p}$ take the form of point-wise product and the usual derivative.

$$\begin{aligned} \hat{x}^C_\mu \hat{f} &\rightarrow x_\mu f(x) \\ \hat{p}_\mu \hat{f} &\rightarrow -i\partial_\mu f(x) \end{aligned}$$

This also applies to general vector fields

$$v(x) : \phi(x) \longrightarrow v^\mu(x^C)\partial_\mu \phi(x)$$

Commutative Gauge Transformation

Hence in this approach, gauge transformations depend on x^C.
The matter fields are elements of $\mathcal{A}_\theta$. As we saw that $\mathcal{A}_\theta$ is module for $\mathcal{A}_0$, matter fields are gauge modules,

$$\phi(x) \xrightarrow{g(x)} g(x^C)\phi(x)$$

The covariant derivative transforms as

$$D = d + A \longrightarrow g(x^c)Dg(x^c)^{-1}$$

A possible choice for D is to be constructed also from x^C in that case A will transform in a simple way

$$A \longrightarrow [g, A] - dg$$

Quantum Gauge Theories

If we make that choice then the pure gauge sector is completely independent of θ. Only matter fields see the noncommutativity of the spacetime, while interaction terms couple the two sectors in interesting way.

We can write terms like

$$\overline{D_\mu \phi} * D^\mu \phi$$

which are gauge and Poincaré invariant.

It is possible to write the full standard model using this approach.

Consistent quantization requires that fermion-fermion and fermion-gluon statistics is twisted while gluon-gluon statistics in untwisted.

The full structure of the Poincaré group acting on the full Hilbert space forms a quasi-Hopf algebra.

Conclusions

- Although the classical Actions of noncommutative theories do not have Poincaré symmetry, they can be thought of as having a twisted Hopf Poincaré symmetry
- Implementation of twisted symmetry requires twisting of statistics
- Conventional NCQFT are not invariant even under twisted Poincaré group
- Gauge transformations can also be treated using Drinfel'd twist.