The Quantum Arnold Transformation and its Applications

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Outline

The Arnold Transformation

- Classical Arnold Transformation
- Quantum Arnold Transformation

QAT: Applications

- Harmonic States for the free particle
- Squeezed Coherent States for the free particle
- QAT and density matrices
- The Arnold-Ermakov-Pinney transformation
- Aplications to Inflationary Cosmological models

Generalizations (in progress)

Related Works

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Lie transformations

The problem of Lie symmetries of ordinary differential equations (ODE) is rather old, and S. Lie gave the main results at the end of the nineteenth century. One of these results was that a second order differential equation

$$y'' = F(x, y, y')$$

has the maximal number of Lie symmetries ($SL(3, \mathbb{R})$) if it can be transformed to the free equation by a point transformation:

$$y'' = F(x, y, y') \quad \overset{\substack{\tilde{x} = \tilde{x}(x, y) \\ \tilde{y} = \tilde{y}(x, y)}}{\Longrightarrow} \quad \tilde{y}'' = 0$$

The condition for this linearization is that the ODE must be of the form:

$$y'' = E_3(x, y)(y')^3 + E_2(x, y)(y')^2 + E_1(x, y)y' + E_0(x, y) \qquad (1)$$

with $E_i(x, y)$ satisfying some integrability conditions.

This has a nice geometric interpretation in terms of projective geometry:

- EDO (1) $\Leftrightarrow$ geodesic equations in a 2-dim Riemannian manifold.
- $E_i(x, y) \approx$ Thomas projective parameters Π.
- integrability conditions $\Leftrightarrow$ Riemann tensor $= 0$

V.I. Arnold named this process *rectification* or *straightening* of the trajectories, and studied the case of Linear Second Order Differential Equation (LSODE), giving explicitely the point transformation for this case.

Classical Arnold transformation

General Linear Second Order Differential Equation (LSODE):

$$\boxed{\ddot{x} + \dot{f}\dot{x} + \omega^2 x = \Lambda}$$

$$A: \begin{array}{ccc} \mathbb{R} \times T & \rightarrow & \mathbb{R} \times \mathcal{T} \\ (x, t) & \mapsto & (\kappa, \tau) \end{array} \quad : \begin{cases} \tau = \frac{u_1(t)}{u_2(t)} \\ \kappa = \frac{x - u_p(t)}{u_2(t)} \end{cases} \quad ,$$

$$\ddot{x} + \dot{f}\dot{x} + \omega^2 x = \Lambda \quad \xrightarrow{A} \quad \frac{W}{u_2^3}\,\ddot{\kappa} = 0$$

- T and $\mathcal{T}$ are, in general, open intervals
- u_1 and u_2 are independent solutions of the homogeneous LSODE
- u_p is a particular solution of the inhomogeneous LSODE
- $W(t) = \dot{u}_1 u_2 - u_1 \dot{u}_2 = e^{-f}$ is the Wronskian of the two solutions

The example of the harmonic oscillator

The harmonic oscillator (HO) is the best example to understand how the CAT works. For this case, and considering $\Lambda = 0$

- Classical solutions are: $u_1(t) = \frac{1}{\omega}\sin(\omega t)$ and $u_2(t) = \cos(\omega t)$.
- $T = (-\frac{\pi}{2\omega}, \frac{\pi}{2\omega})$ and $\mathcal{T} = \mathbb{R}$.
- A and its inverse A^{-1} are written as:

$$A:\ \kappa = \frac{x}{u_2(t)} = \frac{x}{\cos(\omega t)}\,, \qquad \tau = \frac{u_1(t)}{u_2(t)} = \frac{1}{\omega}\tan(\omega t) \tag{2}$$

$$A^{-1}:\ x = \cos(\arctan(\omega\tau))\kappa\,, \qquad t = \frac{1}{\omega}\arctan(\omega t)\,. \tag{3}$$

$$= \frac{\kappa}{\sqrt{1+\omega^2\tau^2}} \tag{4}$$

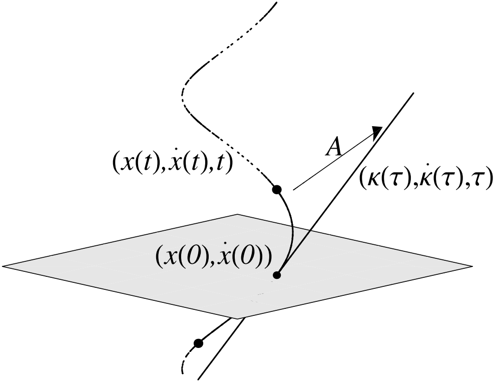

Figure: Depiction of the CAT

Hamiltonian for a LSODE

$$H = \frac{p^2}{2m} e^{-f} + \big(\frac{1}{2} m\omega^2 x^2 - m\Lambda x\big) e^{f}$$

Canonical quantization leads to:

Generalized Caldirola-Kanai equation

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} e^{-f} \frac{\partial^2 \phi}{\partial x^2} + \big(\frac{1}{2} m\omega^2 x^2 - m\Lambda x\big) e^{f} \phi$$

- No eigenvalue equation makes sense: $\hat{H}$ does not preserve solutions!
- Auxiliary operators, representing integrals of motion, are employed to solve the equation. Where do they come from?

→Quantum Arnold Transformation is very relevant to understand all this.

Quantum free particle and its symmetries

Free Schrödinger equation:

$$i\hbar\frac{\partial\varphi}{\partial\tau} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial\kappa^2}$$

Symmetries: the (centrally extended) Galilei group $+$ scale and "conformal" transformations $=$ the Schrödinger group.

- Basic conserved position and momentum operators:

$$\hat{\kappa} = \kappa + \frac{i\hbar}{m}\tau\frac{\partial}{\partial\kappa}\,, \qquad \hat{\pi} = -i\hbar\frac{\partial}{\partial\kappa}$$

  - Represent constants of motion.
  - Generate symmetries.
  - Preserve the Hilbert space of solutions: $\mathcal{H}^G_\tau$.

(So do the quadratic operators, including $\hat{H}_G = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\kappa^2}$).

Quantum Arnold Transformation $\hat{A}$

$$\hat{A}: \quad \mathcal{H}_t \longrightarrow \mathcal{H}^G_\tau$$

$$\phi(x,t) \longmapsto \varphi(\kappa,\tau) = \hat{A}\big(\phi(x,t)\big) = A^*\Big(\sqrt{u_2(t)}\, e^{-\frac{i}{2}\frac{m}{\hbar}\frac{1}{W(t)}\frac{\dot{u}_2(t)}{u_2(t)}x^2}\phi(x,t)\Big)$$

- $\varphi(\kappa,\tau) \in \mathcal{H}^G_\tau$: solution of the free Schrödinger equation.
- $\phi(x,t) \in \mathcal{H}_t$: solution of the Generalized Caldirola-Kanai (GCK) Schrödinger equation.

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\hat{U}_G(\tau)\uparrow & & \uparrow \hat{U}(t) \\
\mathcal{H}^G_0 \equiv \mathcal{H} & \xrightarrow[\hat{1}]{} & \mathcal{H} \equiv \mathcal{H}_0
\end{array}
$$

- We can relate time-dependent Schrödinger equations.
- Also basic operators in each space (as well as quadratic ones).
- Crucial consequence: realization of the free symmetry on the non-free system.

The Hamiltonians are not connected by this transformation!

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Basic operators

Imported from the free particle system through the QAT:

$$\hat{P} = -i\hbar u_2 \frac{\partial}{\partial x} - mx\frac{\dot{u}_2}{W}$$
$$\hat{X} = \frac{\dot{u}_1}{W}x + \frac{i\hbar}{m}u_1\frac{\partial}{\partial x}$$

Together with the quadratic ones $\hat{P}^2, \hat{X}^2, \hat{X}\hat{P}$:

- represent conserved quantities!
- are symmetry generators!
- their eigenvalue equations make sense!

These imported operators can be used to solve the Generalized Caldirola-Kanai Schrödinger equation.

Simplify computations: Wave functions

Combination of the quadratic operators in the Schrödinger algebra:

$$\hat{H}^* = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\tilde{\omega}^2\hat{X}^2 + \frac{\tilde{\gamma}}{2}\hat{X}\hat{P} \qquad (\tilde{\omega},\ \tilde{\gamma} \text{ arbitrary real constants})$$

- Eigenfunctions of $\hat{H}^*$ (normalizability conditions must be imposed!):

$$\phi_\nu(x,t) = \frac{1}{\sqrt{\sqrt{2\pi}\Gamma(\nu+1)\sqrt{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2u_1^2}}} e^{\frac{i}{2\hbar}mx^2\left(\frac{\tilde{\Omega}^2u_1/(u_2-\tilde{\gamma}u_1/2)}{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2u_1^2}+\frac{\dot{u}_2-\tilde{\gamma}\dot{u}_1/2}{(u_2-\tilde{\gamma}u_1/2)W}\right)}$$

$$\left(\frac{u_2-\tilde{\gamma}u_1/2-i\tilde{\Omega}u_1}{\sqrt{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2u_1^2}}\right)^{\nu+\frac{1}{2}}\left(C_1D_\nu\left(\frac{\sqrt{\frac{2m\tilde{\Omega}}{\hbar}}x}{\sqrt{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2u_1^2}}\right)+C_2D_{-1-\nu}\left(\frac{i\sqrt{\frac{2m\tilde{\Omega}}{\hbar}}x}{\sqrt{(u_2-\tilde{\gamma}u_1/2)^2+\tilde{\Omega}^2u_1^2}}\right)\right)$$

- D_ν: parabolic cylinder functions
- C_1 and C_2 arbitrary constants
- $\tilde{\Omega} = \sqrt{\tilde{\omega}^2 - \frac{\tilde{\gamma}^2}{4}}$
- ν in general a complex number

- Spectrum:

$$h^* = \hbar\tilde{\Omega}\left(\nu + \tfrac{1}{2}\right)$$

Simplify computations: Evolution operator

$$
\begin{array}{ccc}
\mathcal{H}^G_\tau & \xleftarrow{\hat{A}} & \mathcal{H}_t \\
\hat{U}_G(\tau)\Big\uparrow & & \Big\uparrow \hat{U}(t) \\
\mathcal{H}^G_0 \equiv \mathcal{H} & \xrightarrow[\hat{1}]{} & \mathcal{H} \equiv \mathcal{H}_0
\end{array}
$$

$$\hat{U}(t)\psi(x) = \hat{A}^{-1}\big(\hat{U}_G(\tau)\psi(\kappa)\big)$$

$$
\begin{aligned}
\hat{U}(t) \equiv \hat{U}(t, t_0) \\
&= e^{\frac{i}{2}\frac{m}{\hbar}\frac{1}{W}\frac{\dot{u}_2}{u_2}x^2} A^{*-1}\big(\hat{U}_G(\tau)\big)\hat{U}_D\big(\tfrac{1}{u_2}\big) \\
&= e^{\frac{i}{2}\frac{m}{\hbar}\frac{1}{W}\frac{\dot{u}_2}{u_2}x^2} e^{\frac{i\hbar}{2m}u_1 u_2 \frac{\partial^2}{\partial x^2}} e^{\log(1/u_2)\left(x\frac{\partial}{\partial x}+\frac{1}{2}\right)} \\
&= e^{i(\alpha(t)\hat{P}^2+\beta(t)\hat{X}^2+\delta(t)\hat{D})}
\end{aligned}
$$

Note that $\hat{U}(t,t_0) \neq e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}$. It is not a one-parameter Lie group of unitary operators. It's just a one-dimensional Lie groupoid!

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Related Works

Humps

Use QAT to connect the free particle to the harmonic oscillator of frequency ω (changed notation!):

$$t = \frac{u_1(t')}{u_2(t')} \quad t' = \frac{1}{\omega}\arctan(\omega t) \qquad (u_1(t') = \frac{1}{\omega}\sin(\omega t'))$$

$$x = \frac{x'}{u_2(t')} \quad x' = \cos(\arctan(\omega t))x = \frac{x}{\sqrt{1+\omega^2 t^2}} \quad (u_2(t') = \cos(\omega t'))$$

We import eigenfunctions of the Hamiltonian of the harmonic oscillator:

Free Hermite-Gauss states in dimension 1

$$\psi_n(x,t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{2^n n! L|\delta|}} e^{-\dfrac{x^2}{4L^2\delta}} \left(\frac{\delta^*}{|\delta|}\right)^{n+\frac{1}{2}} H_n\left(\frac{x}{\sqrt{2}L|\delta|}\right),$$

$\delta \equiv 1 + i\omega t = 1 + i\frac{\hbar t}{2mL^2} = 1 + it/\tau,$ H_n: Hermite polynomials

Humps in dimension 1

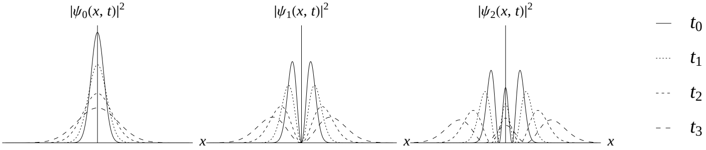

Figure: Spreading under time evolution of wave functions ψ_0, ψ_1 and ψ_2, with $t_k = k\tau$.

- They are not eigenstates of the free Hamiltonian.
- Import creation and annihilation operators:

$$\hat{a} = L\delta \frac{\partial}{\partial x} + \frac{x}{2L} \qquad \hat{a}^\dagger = -L\delta^* \frac{\partial}{\partial x} + \frac{x}{2L}$$

Humps in dimension 2: Laguerre-Gauss states

Import from the HO eigenstates of $\hat{H}_{HO}$ and the angular momentum $\hat{L}$:

$$\hat{L}\psi_{n,l}(r,\phi,t) = l\psi_{n,l}(r,\phi,t) \qquad (L_n^{|l|} : \text{Laguerre polynomials})$$

Free Laguerre-Gauss states

$$\psi_{n,l}(r,\phi,t) = \sqrt{\frac{n!}{2\pi\Gamma(n+|l|+1)L^2|\delta|}}\left(\frac{\delta^*}{|\delta|}\right)^{2n+|l|+1} e^{il\phi}e^{-\frac{r^2}{4L^2\delta}}\left(\frac{r}{\sqrt{2}L|\delta|}\right)^{|l|} L_n^{|l|}\left(\frac{r^2}{2L^2|\delta|^2}\right)$$

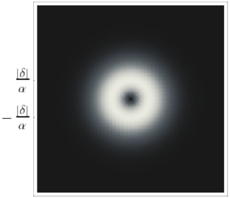

Figure: $|\psi_{0,1}|^2$

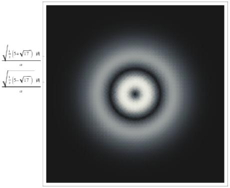

Figure: $|\psi_{1,1}|^2$

Wave function for Squeezed Coherent States (HO)

Displacement operator: $\hat{D}(a) = e^{a\hat{a}^\dagger - a^*\hat{a}}$

(Radial) squeezing operador: $\hat{S}(\xi) = e^{\frac{1}{2}(\xi^*\hat{a}^2 - \xi(\hat{a}^\dagger)^2)}$

$$a = \sqrt{\frac{m\omega}{2\hbar}}x_0 + i\frac{1}{\sqrt{2m\hbar\omega}}p_0, \qquad \xi = r \in \mathbb{R}.$$

$$\boxed{|n,\xi,a\rangle = \hat{D}(a)\hat{S}(\xi)|n\rangle}$$

Time-evolving Squeezed Coherent State wave function for the HO

$$\varphi'^{n}_{(a,r)}(x',t') = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \left(\frac{|\delta'|}{|\delta'_r|}\right)^{\frac{1}{2}} \left(\frac{\delta'^*_r}{|\delta'_r|}\right)^{n+\frac{1}{2}} e^{i\theta(x',t')} e^{-q'^2/2} H_n(q')$$

$\delta' = 1 + i\tan(\omega t'),$ $\delta'_r = 1 + ie^{2r}\tan(\omega t')$

$q' = \frac{\sqrt{\frac{m\omega}{\hbar}}(x' - x_0\cos(\omega t') - \frac{p_0}{m\omega}\sin(\omega t'))}{(e^{2r}\sin^2(\omega t') + e^{-2r}\cos^2(\omega t'))^{1/2}}$ $\theta(x',t')$: huge...

Wave function for Squeezed Coherent States (free particle)

To perform the QAT, we use the "dictionary":

$$\begin{aligned} \omega t' &\to \tan^{-1}(\omega t) & x' &\to \frac{x}{|\delta|} \\ \cos(\omega t') &\to \frac{1}{|\delta|} & \sin(\omega t') &\to \frac{\omega t}{|\delta|} \\ \delta' &\to \delta & \delta_r' &\to \delta_r = 1 + ie^{2r}\omega t \\ \varphi &\to \psi = \frac{1}{\sqrt{|\delta|}} e^{i\omega t \frac{x^2}{4L^2|\delta|^2}} \varphi \end{aligned}$$

$$q' \to q = \frac{x - x_0 + \frac{p_0}{m}\tau}{\sqrt{2}Le^{-r}|\delta_r|}$$

Free time-evolving Squeezed State wave function

$$\psi^n_{(a,r)}(x,t) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \frac{1}{\sqrt{|\delta_r|}} \left(\frac{{\delta_r}^*}{|\delta_r|}\right)^{n+\frac{1}{2}} e^{i\omega t \frac{x^2}{4L^2|\delta|^2}} e^{i\theta(x,t)} e^{-q^2/2} H_n(q)$$

- Time-evolution can be transferred from one quadratic system to another by QAT.

QAT and density matrices

- The density matrix $\hat{\rho}'$ of a mixed GCK oscillator state can be mapped into the density matrix $\hat{\rho}$ of a mixed free particle state:

$$\hat{\rho} = \hat{A}\hat{\rho}'\hat{A}^{\dagger}$$

- The unitarity of $\hat{A}$ guaranties that $\hat{\rho}$ is a proper density matrix, provided that $\hat{\rho}'$ is.
- If $\hat{\rho}'$ satisfies the quantum Liouville equation for the GKC oscillator then $\hat{\rho}$ satisfies the free particle counterpart (use the evolution operator):

$$\frac{\partial \hat{\rho}'}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}'] \quad \Rightarrow \quad \frac{\partial \hat{\rho}}{\partial \tau} = -\frac{i}{\hbar}[\hat{H}_G, \hat{\rho}]$$

- All the properties of $\hat{\rho}'$ are transferred to $\hat{\rho}$, such as characteristic functions, quasi probability distributions, etc. In particular, if $\hat{\rho}'$ describes a Gaussian state, also $\hat{\rho}$ does.
- However a thermal equilibrium state the GCK oscillator is not mapped to a free particle thermal equilibrium state.
- It would be interesting to apply the QAT to Kossakovsky-Lindblad type equations, and study open systems under the QAT point of view.

The Arnold-Ermakov-Pinney transformation

- Two arbitrary LSODE systems can be related composing a CAT and an inverse CAT:
  $E = A_1^{-1}A_2$ relates LSODE-system 2 to LSODE-system 1. E can be written as:

$$
\begin{aligned}
E : & \mathbb{R} \times T_2 \to \mathbb{R} \times T_1 \\
& (x_2, t_2) \mapsto (x_1, t_1) = E(x_2, t_2)
\end{aligned}
$$

- The explicit form of the transformation can be easily computed by composing the two CATs, resulting in:

$$
x_1 = \frac{x_2}{b(t_2)} \qquad W_1(t_1)dt_1 = \frac{W_2(t_2)}{b(t_2)^2}dt_2
$$

- where $b(t_2) = \frac{u_2^{(2)}(t_2)}{u_2^{(1)}(t_1)}$ satisfies the non-linear SODE:

$$
\ddot{b} + \dot{f}_2\dot{b} + \omega_2 b = \frac{W_2^2}{W_1^2}\frac{1}{b^3}\left[\omega_1^2 + \dot{f}_1\frac{\dot{u}_2^{(1)}}{u_2^{(1)}}(1 - b^2\frac{W_1}{W_2})\right]
$$

The Arnold-Ermakov-Pinney transformation and BEC

- For the particular case where LSODE-system 1 is a harmonic oscillator ($\omega_1(t_1) \equiv \omega_0$ and $\dot{f}_1 = 0$), these expression symplify:

$$\ddot{b} + \dot{f}_2 \dot{b} + \omega_2 b = \frac{W_2^2}{b^3} \omega_0^2$$

  and this is the Generalized Ermakov-Pinney equation. For $\dot{f}_2 = 0$ the Ermakov-Pinney equation is recovered.
- For $\omega_0 = 0$, $E = A$.
- The quantum version of the Arnold-Ermakov-Pinney transformation, $\hat{E}$, is given by:

$$\begin{aligned} \hat{E} : \quad \mathcal{H}_{t_2}^{(2)} & \longrightarrow \quad \mathcal{H}_{t_1}^{(1)} \\ \phi(x_2, t_2) & \longmapsto \varphi(x_1, t_1) = \hat{E}\left(\phi(x_2, t_2)\right) \\ & = E^*\left(\sqrt{b(t_2)}\, e^{-\frac{i}{2}\frac{m}{\hbar}\frac{1}{W_2(t_2)}\frac{\dot{b}(t_2)}{b(t_2)}x_2^2} \phi(x_2, t_2)\right) \end{aligned}$$

- This transformation has been extensively used in BEC, known as scaling transformation to transform the time-dependent potential (oscillator traps with time-dependent frequencies) into a time-independent harmonic oscillator potential.

Equation of motion of the Inflaton

The simplest inflationary models consists of an empty universe in which there exists a self-interacting scalar field which finds itself in a very flat region of its potential ("slow roll" approximation). In this configuration the expansion of the universe is dominated by the potential energy density of the scalar field and the expansion proceeds exponentially, like in a de Sitter universe. The action that describes this system is

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{2}\int d^4x\sqrt{-g}\left[\partial_\mu\phi\partial^\mu\phi + 2V(\phi)\right] \qquad (5)$$

The equation of motion for this field is:

$$\Box\phi - V'(\phi) = 0 \qquad (6)$$

Assuming a sufficiently large homogeneous and isotropic patch in the universe, the metric in that patch can be written in the usual FRLW form.

The scalar field equation then becomes

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{7}$$

This is the equation of a non-linear oscillator with a (approx. constant) damping term. Exactly solvable models are obtained when $V(\phi)$ is exponential, or when $V(\phi)$ is quadratic (corresponding to a damped harmonic oscillator), that is used to model the last epoch of inflation (known as reheating).

Quatum fluctuations of the metric

The CMB anisotropies are explained by the stretching of the quantum fluctuations of the metric. Assuming a flat FRLW metric, the fluctuation h_{ij} they are $ds^2 = -dt^2 + a(t)(\delta_{ij} + h_{ij})dx^i dx^j$ satisfy the linearized Einstein equations:

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \nabla h_{ij} = 0 \tag{8}$$

Expanding h_{ij} in Fourier modes as usual $h_{ij} = \int \mathrm{d}^3 k h_{\vec{k}}(t) e_{ij} e^{i\vec{k}\cdot\vec{x}}$, we obtain the equation for each mode:

$$\ddot{h}_{\vec{k}} + 3H\dot{h}_{\vec{k}} + \frac{k^2}{a^2} h_{\vec{k}} = 0 \tag{9}$$

Since at the beginning of inflation the size of the universe was so small, and the quantum fluctuations were dominant, the fluctuations h_{ij} must be treated as a quantum field and its normal modes as quantum damped harmonic oscillators.

Thus, all the discussion concerning the Caldirola-Kanai and Bateman systems apply here.

In particular, and due to the necessarily autonomous character of the Universe evolution, a quantum treatment "à la Bateman" is specially in order. This would mean that an extra "mirror" fluctuations field is required.

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Quantum Arnold Transformation

QAT: Applications
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Aplications to Inflationary Cosmological models

Generalizations (in progress)

Related Works

Generalizations

- Extend the QAT to the relativistic case. Two possible directions:
  1. As the quantum version of a CAT for geodesic equations in a fixed background and with external forces.
  2. As the quantum version of Geodesic Mappings, that transforms geodesic of a metric into geodesic of a different metric (Beltrami Theorem).
- Extend the QAT to non-linear potentials $\rightarrow$ Quantum Lie Transformation. Second order Ricatti equation.
- Extend the QAT to non-local potentials, like Gross-Pitaevskii equation.

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Related Works

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- Lewis & Riesenfeld (1969) introduced a technique to obtain solutions of the time dependent Schrödinger equation (TDSE) for a time dependent quadratic Hamiltonian (TDQH) as eigenfunctions of quadratic invariants. For that purpose they wrote the solutions in terms of auxiliary variables that satisfy the classical equations of motion (something that resembles the CAT)
- Dodonov & Man'ko (1979) constructed invariant operators for the damped harmonic oscillator and introduced coherent states, using a method similar to that of Lewis and Riesenfeld.
- Jackiw (1980) gave the quantum transformation from the harmonic oscillator (even with a $1/x^2$ term) to the free particle when studying the symmetries of the magnetic monopole.
- Duru & Kleinert (1982) gave the transformation of the propagator, in a path integral appoach, for the hidrogen atom into the harmonic oscillator one (this could be seen as the Quantum KS transformation).

- Junker & Inomata (1985) gave the transformation of the propagator, in a path integral approach, for an arbitrary quadratic potential, into the free propagator (the equivalent of the QAT).
- Takagi (1990) gave the quantum transformation from the harmonic oscillator to the free particle, interpreted as the change to comoving coordinates.
- Bluman & Shtelen (1996) gave the (non-local) transformation of the TDSE for a TDQH plus a non-linear term into the free particle one, in the context of transformations of PDEs.
- Kagan et al. and independently Castin & Dum (1996) introduced a scaling transformation in the Gross-Pitaevskii equation describing Bose-Einstein Condensates (BEC) which is related to the QAT.
- Suslov et al. (2010) computed the propagator for a time-dependent quadratic Hamiltonian using the classical equations. Later (2011) they use the transformation point of view which can be considered equivalent to our QAT.

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