ICE CSIC IEEC

Chowla-Selberg and other formulas useful for zeta regularization

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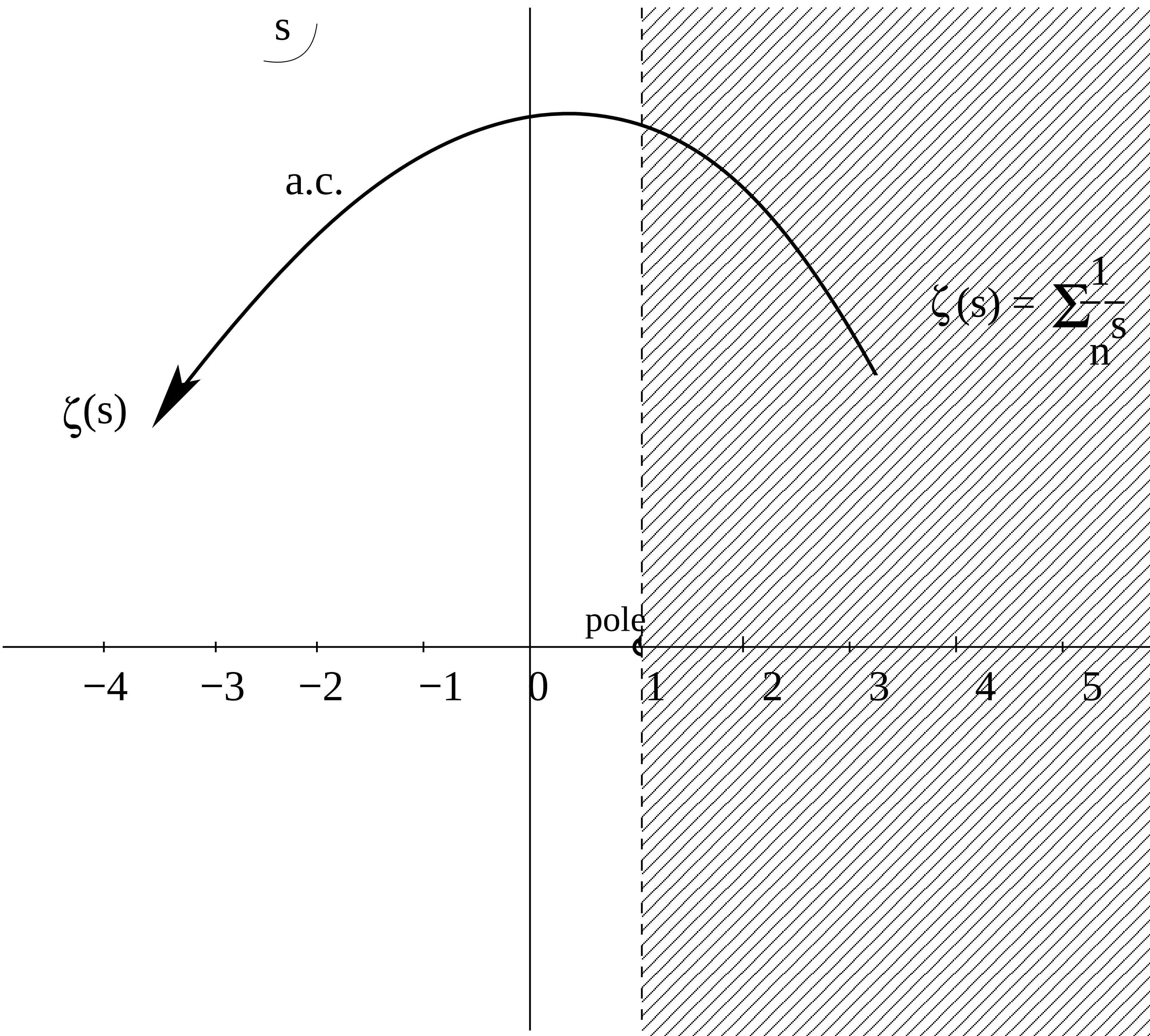
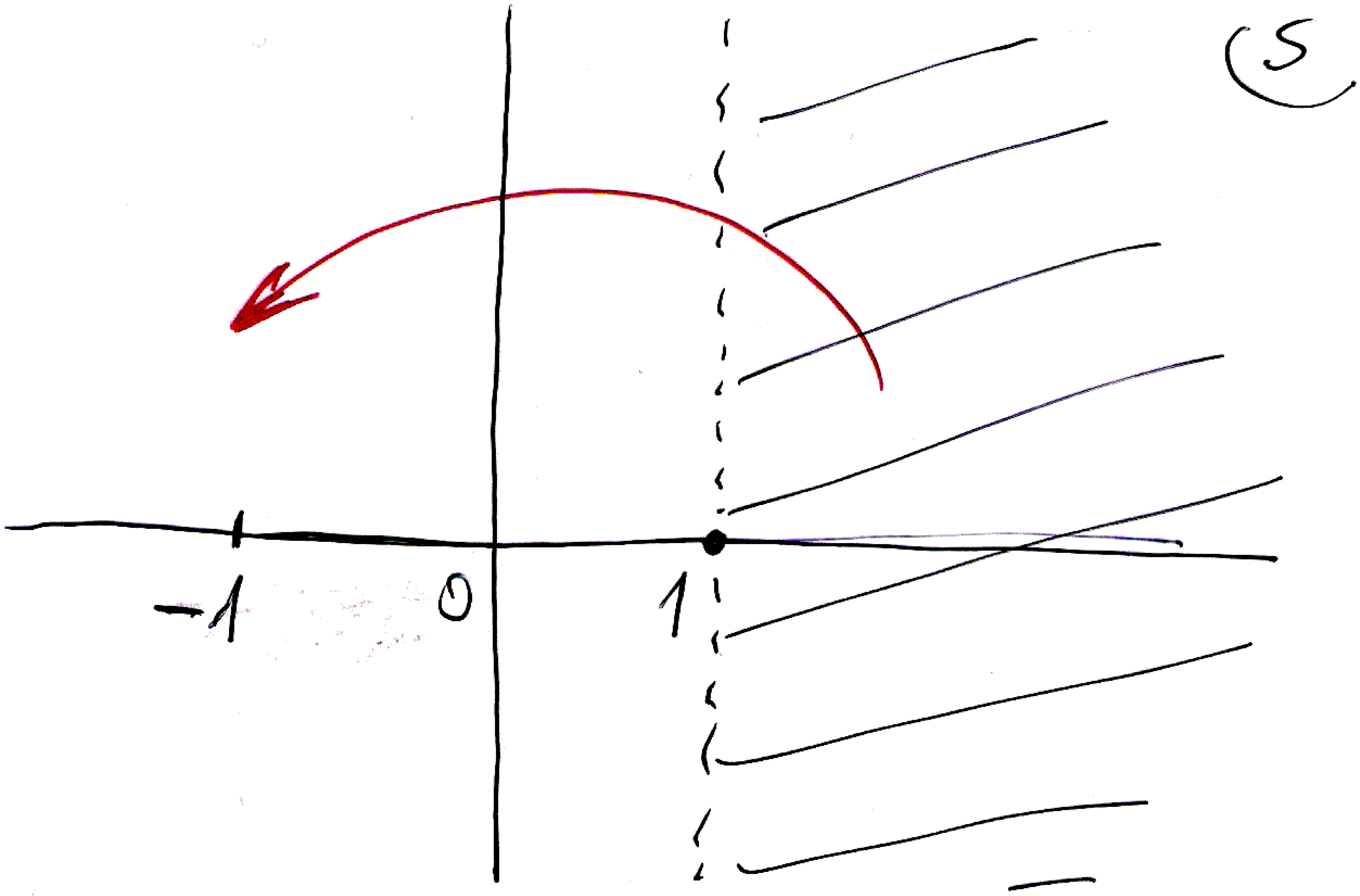
Outline

- The Riemann zeta function as a regularization tool.
- General scheme for Linear and Quadratic cases. Truncations.
- Spectrum only known Implicitly.
- The Chowla-Selberg formula in Number Theory.
- The Chowla-Selberg series formula (CS). Nontrivial Extensions (ECS).
- Operator Zeta Functions: ζ_A for A a ΨDO, Det's.
- Dixmier trace, Wodzicki Residue.
- Multiplicative Anomaly or Defect.


s
a.c.
ζ(s)
ζ(s) = Σ 1/n^s
pole
−4 −3 −2 −1 0 1 2 3 4 5



$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$$

$\zeta(0) = -\frac{1}{2}$ or $1 + 1 + 1 + \dots \overset{?}{\underset{\zeta}{=}} -\frac{1}{2}$

$\zeta(-1) = -\frac{1}{12}$ or $1 + 2 + 3 + \dots \underset{\zeta}{=} -\frac{1}{12}$

⋮

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^s}}$$

The prime number theorem {Hadamard, de la Vallée Poussin

$$\pi(x) = \#\{\text{primes } p \le x\} \sim \frac{x}{\log x}$$

$$\xi(s) \equiv \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(1-s)$$

(Gelbart + Miller, BAMS '03) (completed ζ f.)

(E) Entirety: $\xi(s)$ meromorphic c., $s=0,1$ poles

(FE) $\xi(s) = \xi(1-s)$

(BV) Bounded in vertical strips:

$\xi(s) + \frac{1}{s} + \frac{1}{1-s}$ bounded $-\infty < a < \operatorname{Re} s < b < +\infty$

Riemann (1859)

Poisson s. for. $\boxed{\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)}$ Fourier t.

$$\hat{f}(r) = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i r x}\, dx, \quad f \text{ Schwartz}$$

1) FE ζ

2) $\hat{f}(r) = \frac{1}{\sqrt{t}} e^{-\pi r^2/t}$ → Jacobi id.

$$\theta(it) = \frac{1}{\sqrt{t}} \theta\left(\frac{i}{t}\right), \quad \theta(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$$

Dirichlet $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}$ Hamburger

Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau \in \mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\, \theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi n z), \qquad z,t \in \mathbb{C},\ \text{Re}\, t > 0$$

- Higher dimensions: Poisson summ formula (Riemann)

$$\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p} \widetilde{f}(\vec{m})$$

$\widetilde{f}$ Fourier transform

[Gelbart + Miller, BAMS '03, Iwaniec, Morgan, ICM '06]

- Truncated sums $\longrightarrow$ asymptotic series

ζ: EXPLICIT CALCULATIONS

Epstein zeta functions (quadratic)

$$\zeta_E = \sum_{\vec{n} \in \mathbb{Z}^d} Q(\vec{n})^{-s}$$ Q quadratic form

Barnes zeta functions (linear)

$$\zeta_B = \sum_{\vec{n} \in \mathbb{N}^d} L(\vec{n})^{-s}$$ L affine form (coeff's $\in \mathbb{Q}^+$)

Extensions:

$\zeta_E \rightarrow Q + L$ affine

$\rightarrow \sum_{\vec{n} \in \mathbb{N}^d}$ (truncation)

$\zeta_B \rightarrow \zeta_B'(0)$ (new formulas)

$\rightarrow \sum'_{\vec{n} \in \mathbb{Z}^d}$ (by analyt. cont.)

Ex

ζ-REGULARIZ: SPECTRUM KNOWN IMPLICITLY

- Example of the ball:
  - Operator
  $$(-\Delta + m^2)$$
  on the D-dim ball $B^D = \{x \in R^D; |x| \le R\}$
  with Dirichlet, Neumann or Robin BC .
  - The zeta function
  $$\zeta(s) = \sum_k \lambda_k^{-s}$$
  - Eigenvalues implicitly obtained from
  $$\boxed{(-\Delta + m^2)\phi_k(x) = \lambda_k \phi_k(x) \quad + \quad BC}$$
  - In spherical coordinates:
  $$\phi_{l,m,n}(r,\Omega) = r^{1-\frac{D}{2}} J_{l+\frac{D-2}{2}}(w_{l,n} r) Y_{l+\frac{D}{2}}(\Omega)$$
  $J_{l+(D-2)/2}$ Bessel functions
  $Y_{l+D/2}$ hyperspherical harmonics
  - Eigenvalues $w_{l,n}$ (>0) determined through BC
  $$\boxed{J_{l+\frac{D-2}{2}}(w_{l,n} R) = 0,} \qquad \text{for Dirichlet BC}$$

$$\frac{u}{R} J_{l+\frac{D-2}{2}}(w_{l,n} R) + w_{l,n} J'_{l+\frac{D-2}{2}}(w_{l,n} r)\,|_{r=R} = 0,$$ for Robin BC

– Now, $\lambda_{l,n} = w_{l,n}^2 + m^2$

$$\zeta(s) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} d_l(D) (w_{l,n}^2 + m^2)^{-s}$$

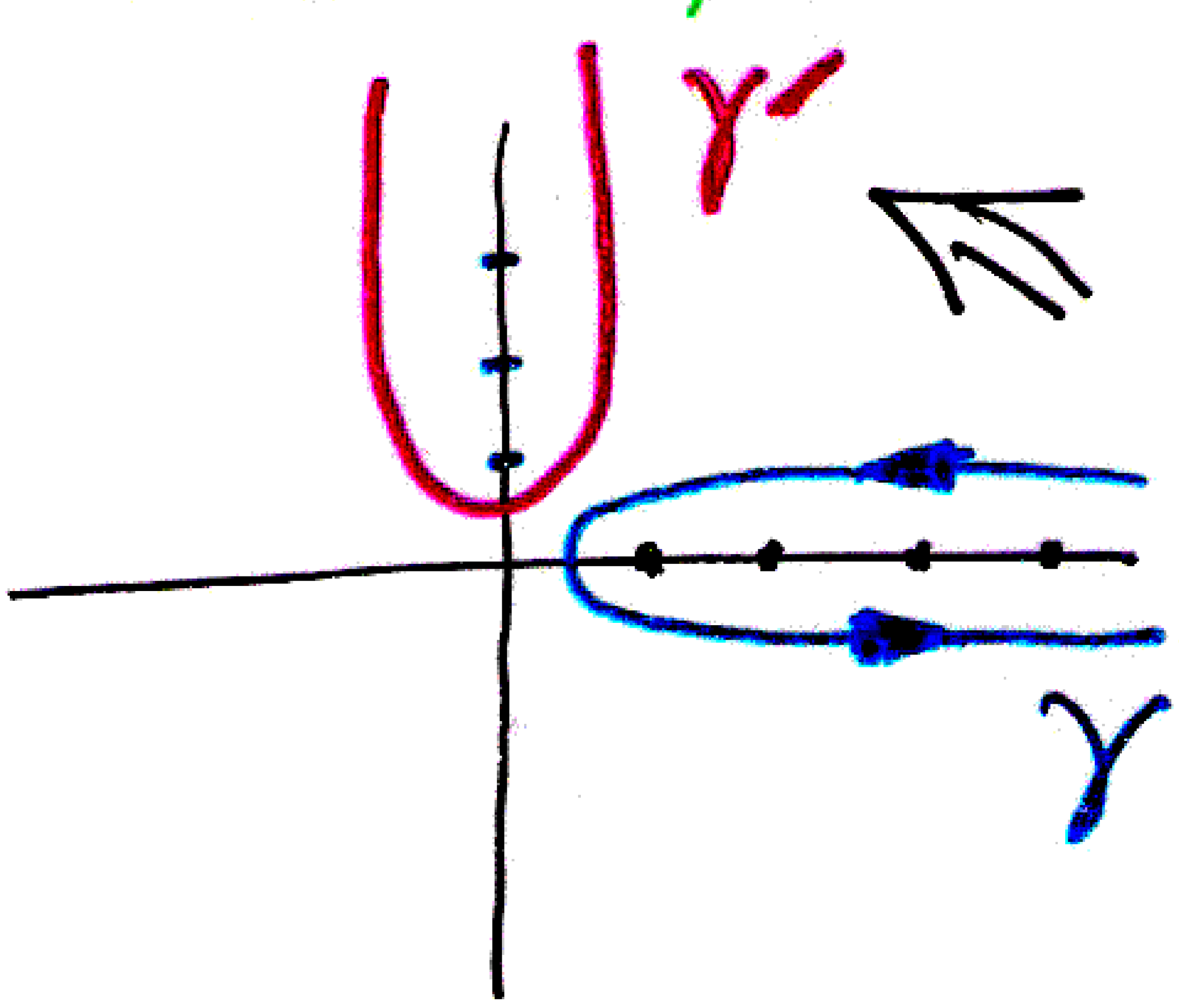
$w_{l,n}$ (> 0) is defined as the n-th root of the l-th equation, $d_l(D) = (2l + D - 2)\frac{(l+D-3)!}{l!\,(D-2)!}$

- **Procedure:**
  - Contour integral on the complex plane

$$\zeta(s) = \sum_{l=0}^{\infty} d_l(D) \int_\gamma \frac{dk}{2\pi i} \, (k^2+m^2)^{-s} \frac{\partial}{\partial k} \ln \Phi_{l+\frac{D-2}{2}}(kR)$$

γ runs counterclockwise and must enclose all the solutions [Ginzburg, Van Kampen, EE + I. Brevik]

- **Obtained:** [with Bordag, Kirsten, Leseduarte, Vassilievich,...]
  - Zeta functions
  - Determinants
  - Seeley [heat-kernel] coefficients



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ON EPSTEIN'S ZETA FUNCTION (I)

BY S. CHOWLA AND A. SELBERG

INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J.

Communicated by H. Weyl, May 18, 1949

1. This paper contains a short account of results whose detailed proofs will be published later.

We define the function $Z(s)$ by

$$Z(s) = \sum{}'(am^2 + bmn + cn^2)^{-s} \tag{1}$$

where $s = \sigma + it$ (σ and t, real), $\sigma > 1$, and the summation is for all integers m, n (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further a and c are positive numbers while b is real and subject to $4ac - b^2 = \Delta > 0$.

It is well known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole s-plane, and satisfies a functional equation similar to the one satisfied by the Riemann Zeta Function. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

Deuring (*Math. Ztschr.*, **37**, 403–413 (1933)) obtained an important formula for $Z(s)$. Deuring's work led Heilbronn (*Quart. J. Maths., Oxford*, **5**, 150 (1934)) to the proof of the following famous conjecture of Gauss on the class-number of binary quadratic forms with a negative fundamental discriminant: let $h(-\Delta)$ denote the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$, then

$$h(-\Delta) \to \infty \quad \text{as} \quad \Delta \to \infty \tag{2}$$

Again using the ideas of Heilbronn and Deuring, Siegel proved that

$$h(-\Delta) > \Delta^{1/2 - \epsilon} \; [\Delta > \Delta_0(\epsilon)] \tag{3}$$

which is a great advance on (2).

Our starting point is the formula:

$$Z(s) = 2\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Gamma(s)\,\Delta^{s-1/2}}\zeta(2s-1)\Gamma(s - 1/2) + Q(s) \tag{4}$$

where

$$Q(s) = \frac{\pi^s \cdot 2^{s+3/2}}{a^{1/2}\Gamma(s)\,\Delta^{s/2-1/4}} \sum_{n=1}^{\infty} n^{s-1/2}\,\sigma_{1-2s}(n)\cos\left(\frac{n\pi b}{a}\right) \int_0^\infty \phi^{s-3/2} \exp\left\{-\frac{\pi n \Delta^{1/2}}{2a}(\phi + \phi^{-1})\right\} d\phi \tag{4}$$

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Berlin 1967

On Epstein's Zeta-function

By *Atle Selberg* at Princeton (N. J.), and *S. Chowla* at State College (Pa.)

Introduction

This paper was written in the Spring of 1949, and a resumé appeared in the note: On Epstein's zeta Function (I), Proceedings of the National Academy of Sciences (U. S. A.), **35** (1949), 371--374.

Meanwhile, the following papers which have reference to the Proceedings paper, came to our attention:

1. *J. B. Rosser*, Real roots of real Dirichlet *L*-series, Jour. Research National Bureau of Standards, **45** (1950), 505—514.

2. *E. A. Anferteva*, On an identity of Chowla and Selberg (Russian), Izvestija Vyšsik Učebnyh Zavadeniĭ Mathematika (Kazan), No. 3 (10) (1959), 13—21.

3. *P. T. Bateman* and *E. Grosswald*, On Epstein's zeta Function, Acta Arithmetica, **9** (1964), 365—373.

4. *K. Ramachandra*, Some applications of Kronecker's limit formulas, Annals of Mathematics **80** (1964), 104—148.

§ 1.

We define the function $Z(s)$ by

$$Z(s) = \Sigma' (am^2 + bmn + cn^2)^{-s} \tag{1}$$

where $s = \sigma + it$ (σ and t, real), $\sigma > 1$, and the summation is for all integers m, n (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further a and c are positive numbers while b is real and subject to $4ac - b^2 = \Delta > 0$.

It is well-known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole s-plane. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

In 1933, Deuring obtained an important formula for $Z(s)$. Deuring's work led Heilbronn to his proof of a famous conjecture of Gauss on the class number of binary quadratic forms with a negative fundamental discriminant. If $h(-\Delta)$ is the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$, Gauss conjectured that

$$h(-\Delta) \to \infty \text{ as } \Delta \to \infty. \tag{2}$$

Transforming this we get

$$\sum_{j=1}^{h} \log \Delta\left(\frac{b_j + i\sqrt{|d|}}{2a_j}\right) = 6\left\{h\gamma + \log \frac{\prod_{j=1}^{h} a_j}{|d|^h}\right\} - \frac{3w}{\pi}\sqrt{|d|}\, L_d'(1).$$

Inserting here the value (obtained like (58))

$$L_d'(1) = -\frac{\pi}{\sqrt{|d|}} \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) \log \Gamma\left(\frac{m}{|d|}\right) + \frac{2h\pi(\gamma + \log 2\pi)}{w\sqrt{|d|}}$$

one gets, writing $\tau_j = \dfrac{b_j + i\sqrt{|d|}}{2a_j}$,

$$\prod_{j=1}^{h} \Delta(\tau_j) = \frac{\prod_{j=1}^{h} a_j^6}{(2\pi\,|d|)^{6h}} \left\{\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{3w} \tag{2}$$

Now let $\tau = i\dfrac{K'}{K}$ be a number from the field $k(\sqrt{d})$, then from Lemma 3 we get

$$\frac{\Delta(\tau_j)}{\Delta(\tau)} = \lambda_j,$$

where λ_j are algebraic numbers. Thus (2) gives

$$\Delta(\tau) = \frac{\lambda'}{\pi^6} \left\{\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{\frac{3w}{h}}, \tag{3}$$

where λ' is an algebraic number. Finally we have from (48)

$$\Delta(\tau) = \left(\frac{2K}{\pi}\right)^{12} \cdot 2^{-8} (kk')^4 = \lambda'' \left(\frac{K}{\pi}\right)^{12},$$

where λ'' is an algebraic number. This gives, when inserted in (3)

$$K = \lambda''' \sqrt{\pi} \left\{\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{\frac{w}{4h}}, \tag{4}$$

which is the desired expression for K in finite terms.

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[2] *S. Chowla*, Acta Arithmetica **1** (1935), 113—114.
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History

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$$\sum_{\lambda=1}^{|D|} \left(\frac{D}{\lambda}\right) \log \Gamma \left(\frac{\lambda}{D}\right) = h \log |D| - \frac{h}{3} \log(2\pi) - \sum_{(a,b,c)} \log a + \frac{2}{3} \sum_{(a,b,c)} \log \left[\theta_1'(0|\alpha)\theta_1'(0|\beta)\right]$$

D discriminant, $\theta_1' \sim \eta^3$

h class number of binary quadratic forms (a, b, c)

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h class number of binary quadratic forms (a,b,c)

- Eta evaluations Dedekind eta function for Im $(\tau) > 0$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q := e^{2\pi i\tau}$$

It is a 24-th root of the discriminant func $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ from a lattice $L = \{a\tau + b \,|\, a,b \in \mathbb{Z}\}$

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}$$

Properties & Recent Results

$\Longrightarrow$ The C-S formula gives the value of a product of eta functions

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Extended CS Series Formulas (ECS)

- Consider the zeta function (Re$s > p/2, A > 0,$ Re$q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A\left(\vec{n}+\vec{c}\right)+q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q\left(\vec{n}+\vec{c}\right)+q\right]^{-s}$$

prime: point $\vec{n}=\vec{0}$ to be excluded from the sum
(inescapable condition when $c_1 = \cdots = c_p = q = 0$)

$$Q\left(\vec{n}+\vec{c}\right)+q = Q(\vec{n}) + L(\vec{n}) + \bar{q}$$

Extended CS Series Formulas (ECS)

- Consider the zeta function (Re$s > p/2, A > 0,$ Re$q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}(\vec{n}+\vec{c})^T A(\vec{n}+\vec{c}) + q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q(\vec{n}+\vec{c}) + q\right]^{-s}$$

prime: point $\vec{n} = \vec{0}$ to be excluded from the sum

(inescapable condition when $c_1 = \cdots = c_p = q = 0$)

$$Q(\vec{n}+\vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}$$

- Case $q \neq 0$ (Re$q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\,\Gamma(s)}$$

$$\times \sum_{\vec{m}\in\mathbb{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c}) \left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right)$$

EE JPA34 (2001) 3025 [ECS1]

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- Pole: $s = p/2$ Residue:

$$\text{Res}_{s=p/2}\,\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A)^{-1/2}$$

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- K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one may select those vectors in $\mathbb{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive]

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- Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s}\, \Gamma\left(s - \frac{j}{2}\right) \zeta_R(2s-j) + 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right)\right]$$

[ECS3d]

QFT in s-t with non-comm toroidal part

- $D-$dim non-commut manifold: $M = \mathbb{R}^{1,d} \bigotimes \mathbb{T}^p_\theta, \quad D = d+p+1$
  $\mathbb{T}^p_\theta$ a $p-$dim non-commutative torus: $[x_j, x_k] = i\theta\sigma_{jk}$
  σ_{jk} a real, nonsingular, antisymmetric matrix of ± 1 entries
  θ the non-commutative parameter. [Quresh's talk]

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- Interest recently, in connection with $M-$theory & string theory [Connes,Douglas,Seiberg,Cheung,Chu,Chomerus,Ardalan, . . .]

- Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with # of compact, noncompact, and noncommutative dimensions

$$\zeta_\alpha(s) = \frac{V\,\Gamma(s-(d+1)/2)}{(4\pi)^{(d+1)/2}\,\Gamma(s)} \sum_{\vec{n}\in\mathbb{Z}^p}{}' Q(\vec{n})^{(d+1)/2-s}\left[1+\Lambda\theta^{2-2\alpha}Q(\vec{n})^{-\alpha}\right]^{(d+1)/2-s}$$

$\alpha = 2$ bos, $\alpha = 3$ ferm, $V = \text{Vol}\,(\mathbb{R}^{d+1})$ of non-compact part

$Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$ a diag quadratic form, $R_j = a_j^{-1/2}$ compactific radii

$$\zeta_\alpha(s) = \frac{2^{s-d}\, V}{(2\pi)^{(d+1)/2}\Gamma(s)} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)} (-2^{\alpha}\Lambda\theta^{2-2\alpha})^l \sum_{j=0}^{p-1} (\det A_j)^{-\frac{1}{2}}$$

$$\times \Big[\pi^{j/2} a_{p-j}^{-s-\alpha l+(d+j+1)/2} \Gamma(s+\alpha l-(d+j+1)/2) \zeta_R(2s+2\alpha l-d-j-1)$$

$$+ 4\pi^{s+\alpha l-(d+1)/2} a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{(d+j+1)/2-s-\alpha l}$$

$$\times \left(\vec{m}_j^t A_j^{-1} \vec{m}_j \right)^{(s+\alpha l)/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l} \left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j} \right) \Big]$$

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p \ D	even	odd
odd	(1a) pole / finite ($l \geq l_1$)	(2a) pole / pole
even	(1b) double pole / pole ($l \geq l_1, l_2$)	(2b) pole / double pole ($l \geq l_2$)

– General pole structure of $\zeta_\alpha(s)$, for the possible values of D and p being odd or even. Magenta, type of behavior corresponding to lower values of l; behavior in blue corresponds to larger values of l

- After some calculations,

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-\frac{d+1}{2})}{l!\,\Gamma(s)} \left(-\Lambda\theta^{2-2\alpha}\right)^l \zeta_{Q,\vec{0},0}(s+\alpha l - \frac{d+1}{2})$$

for all radii equal to R, with $I(\vec{n}) = \sum_{j=1}^{p} n_j^2$,

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where we use the notation $\zeta_E(s) := \zeta_{I,\vec{0},0}(s)$

e.g., the Epstein zeta function for the standard quadratic form

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- Classify the different possible cases according to the values of d and $D = d+p+1$. We obtain, at $s = 0$:

$$\text{For } d = 2k \quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = 0 \\ \text{if } D = \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \end{cases}$$

$$\text{For } d = 2k-1 \quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} \left\{ \begin{array}{ll} \text{finite}, & \text{for } l \leq k \\ 0, & \text{for } l > k \end{array} \right\} \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \\ \\ \text{if } D = 2\alpha l \left\{ \begin{array}{ll} \text{pole}, & \text{for } l \leq k \\ \text{finite}, & \text{for } l > k \end{array} \right\} \Longrightarrow \quad \zeta_\alpha(0) = \text{pole} \end{cases}$$

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– Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\dot{\overline{2\alpha}}$ means multiple of 2α)

$\Longrightarrow$ Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$, & specific pole structure

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m $\qquad$ M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x,\xi) \right| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function

$$f \in \mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}^n);\ \sup_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n \right\}$$

- $\mathcal{S}'$ space of tempered distributions

- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

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– – ΨDOs are basic tools both in Mathematics & in Physics – –

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

[K Fredenhagen, R Brunetti, . . . R Wald ’06, R Haag EPJH35 ’10]

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr}\, A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re}\, s > \frac{n}{m} := s_0$$

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(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n-j)/m, \qquad j = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

Definition of Determinant

H ΨDO operator $\{\varphi_i, \lambda_i\}$ spectral decomposition

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Riemann zeta func: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \ Re\, s > 1$ (& analytic cont)

Definition: zeta function of H $\zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr } H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1} \text{tr } e^{-tH}, \ \text{Re} s > s_0$

Derivative: $\zeta_H'(0) = -\sum_{i \in I} \ln \lambda_i$

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Determinant: Ray & Singer, '67

$$\det_\zeta H = \exp\left[-\zeta_H'(0)\right]$$

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Derivative: $\zeta_H'(0) = -\sum_{i\in I} \ln \lambda_i$

Determinant: Ray & Singer, '67

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Weierstrass def: subtract leading behavior of λ_i in i, as $i \to \infty$, until series $\sum_{i\in I} \ln \lambda_i$ converges $\Longrightarrow$ non-local counterterms !!

Definition of Determinant

H ΨDO operator $\{\varphi_i, \lambda_i\}$ spectral decomposition

$\prod_{i\in I} \lambda_i$?! $\ln \prod_{i\in I} \lambda_i = \sum_{i\in I} \ln \lambda_i$

Riemann zeta func: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \ Re\, s > 1$ (& analytic cont)

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C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...

Properties

- The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the cut

- A zeta function (and corresponding determinant) with the same meromorphic structure in the complex s-plane and extending the ordinary definition to operators of complex order $m \in \mathbb{C} \backslash \mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik]

- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = {\sum}'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \, \text{Res}_{s=s_j} \, \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[\text{PP } \zeta_A(-k) + \psi(k+1) \, \text{Res}_{s=-k} \, \zeta_A(s) \right], \qquad s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \, \zeta_A(s), \quad k \in \mathbb{N} \backslash \{0\}$$

$$\text{PP } \phi := \lim_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p} \, \phi(s)}{s-p} \right]$$

The Dixmier Trace

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- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \qquad \mu_0 \geq \mu_1 \geq \cdots$$

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$$\text{Dtr}\, T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

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- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ [Connes]

$$\text{Dtr}\, T = \lim_{s\to 1^+} (s-1)\zeta_{T^{-1}}(s)$$

The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to ΨDOs which are not in $\mathcal{L}^{(1,\infty)}$
- Only trace one can define in the algebra of ΨDOs (up to multipl const)
- Definition: $\text{res}\, A = 2\, \text{Res}_{s=0}\, \text{tr}\,(A\Delta^{-s})$, Δ Laplacian
- Satisfies the trace condition: $\text{res}\,(AB) = \text{res}\,(BA)$
- Important!: it can be expressed as an integral (local form)

$$\text{res}\, A = \int_{S^*M} \text{tr}\, a_{-n}(x,\xi)\, d\xi$$

  with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler-Manin residue)
- If $\dim M = n = -\,\text{ord}\, A$ (M compact Riemann, A elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and $\text{Res}_{s=1}\zeta_A(s) = \frac{1}{n}\,\text{res}\, A^{-1}$
- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

Singularities of ζ_A

- A complete determination of the meromorphic structure of some zeta functions in the complex plane can be also obtained by means of the Dixmier trace and the Wodzicki residue

- Missing for full descript of the singularities: residua of all poles

- As for the regular part of the analytic continuation: specific methods have to be used (see later)

- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x,\xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1}=0$:

  $\text{Res}_{s=s_k}\zeta_A(s) = \frac{1}{m}\,\text{res}\, A^{-s_k} = \frac{1}{m}\int_{S^*M} \text{tr}\, a_{-n}^{-s_k}(x,\xi)\, d^{n-1}\xi$

- Proof. The homog component of degree $-n$ of the corresp power of the principal symbol of A is obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1}=0$:

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial \xi^{-1}}\right)^k \left[\xi^{n-k} a^{(k-n)/m}(x,\xi)\right]\Big|_{\xi^{-1}=0} \xi^{-n}$$

Multipl or N-Comm Anomaly, or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det_\zeta(AB) \neq \det_\zeta A \, \det_\zeta B$$

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- The multiplicative (or noncommutative) anomaly (defect) is defined as

$$\delta(A,B) = \ln \left[\frac{\det_\zeta(AB)}{\det_\zeta A \ \det_\zeta B} \right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

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- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{ [\ln \sigma(A,B)]^2 \right\}}{2 \, \text{ord}\, A \; \text{ord}\, B \, (\text{ord}\, A + \text{ord}\, B)}$$

where $\quad \sigma(A,B) = A^{\text{ord}\, B} B^{-\text{ord}\, A}$

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi] \; \exp\left\{ - \int d^D x \left[\Phi^\dagger(x) (\quad) \Phi(x) + \cdots \right] \right\}$$

Gaussian integration: $\longrightarrow \quad \det\,(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

$\det(AB)$ or $\det A \;\cdot\; \det B$?

- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$

- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

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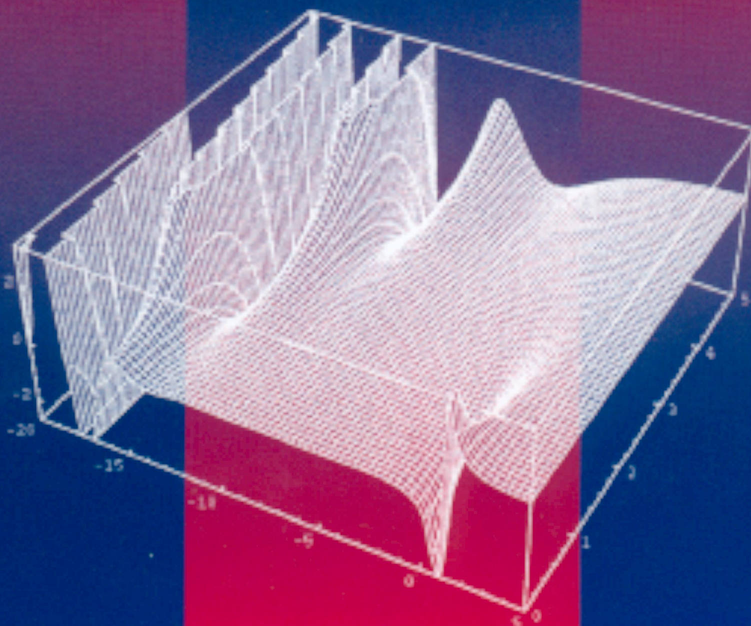
Zeta Regularization Techniques with Applications



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