Reduction of Lie-Jordan algebras

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- We will study the reduction of classical and quantum systems emphasizing its common algebraic structure.

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  - Dirac reduction
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F.F, M. Zambon. Lett. Math. Phys. 85 (2008). arXiv:0806.0638
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- $\hbar=0$
  ($\mathcal{L},\circ$) is associative and ($\mathcal{L},\circ,\{\ ,\ \}$) is a Poisson algebra.

Lie-Jordan and C^* algebras

- Define a product in $\mathcal{L}^{\mathbb{C}}$ by

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- Conversely, given a complex associative algebra $(\mathcal{A}, \cdot)$ with involution $*$, the selfadjoint elements

$$\mathcal{A}_{\mathrm{sa}} = \{x \in \mathcal{A} | x^* = x\}$$

form a Lie-Jordan algebra $(\mathcal{A}_{\mathrm{sa}}, \circ, [\ ,\ ])$ with $\hbar \neq 0$,

$$a \circ b = \frac{1}{2}(a \cdot b + b \cdot a), \qquad [a, b] = \frac{i}{2\hbar}(a \cdot b - b \cdot a)$$

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e.g. if E is the linear span of Hamiltonian vector fields.

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Dirac bracket on (first class) functions restricted to N.

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Reduction by symmetries and constraints can be combined

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Marsden-Ratiu reduction (constraints + symmetries)

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BUT $\mathcal{B}$ a Lie subalgebra is a rather strong condition...

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Proof:

Assume $\{f, g\} \notin \mathcal{I}, f \in \mathcal{I}, g \in \mathcal{B}$

i. e. $\{f, g\}(p) \neq 0, p \in N$

$f^2 \in \mathcal{B} \Rightarrow f\{f, g\} \in \mathcal{B} \Rightarrow f \in \mathcal{B} \cap \mathcal{I}$

$hf \in \mathcal{B}\, \forall h \in C^\infty(M) \Rightarrow h\{f, g\} + f\{h, g\} \in \mathcal{B} \Rightarrow h \in \mathcal{B}$

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Summarising: MR reduction is nothing but a successive application of constraints and symmetries.!!

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- The flaw: Jacobi identity is not guaranteed.
- A solution: introduce Jordan subalgebras $\mathcal{B}_-$ and $\mathcal{B}_+$

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Reduction of Poisson algebras

- The data: - $\mathcal{L}$ a Poisson algebra
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  Then, we can show:

  $\{\mathcal{B}_-, \mathcal{B}_-\} \subset \mathcal{B}_+$, $\quad \{\mathcal{B}_-, \mathcal{B}_+ \cap \mathcal{I}\} \subset \mathcal{I} \quad \Rightarrow$ Jacobi identity.

Reduction of Poisson algebras

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- $\| a^2 \| = \| a \|^2$
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A state is a positive, normalised functional in $\mathcal{L}$, i.e.

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Some properties, $\sigma \in \mathfrak{S}(\mathcal{L})$:

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Reduction of LJB algebras

Reduction by symmetries

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Consider the space of Jordan derivations

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Reduction by constraints (T-procedure): $\mathcal{C} \subset \mathcal{L}$

H. Grundling, F. Lledo. Rev. Math. Phys. 12 (2000) 1159. [math-ph/9812022]

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- Reduced space: $\mathcal{B} = \{a \in \mathcal{L} \text{ s.t. } \delta a \in \mathcal{S}, \forall \delta \in D\}$ is a closed subspace and $\mathcal{B} \cap \mathcal{S}$ is a closed LJ subalgebra.

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- Reduced LJB algebra:

If $\mathcal{B}$ is a LJ subalgebra and $\mathcal{B} \cap \mathcal{S}$ is its LJ ideal

$$\mathcal{B}/(\mathcal{B} \cap \mathcal{S})$$

inherits the structure of a LJB algebra.

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- Reduction by symmetries: $\mathcal{C} = \{0\}$

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MORE GENERAL REDUCTIONS?

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Proof: $[[a + \mathcal{S}, b + \mathcal{S}], c + \mathcal{S}] = [([a_-, b_-])_-, c_-] + \mathcal{S}$

but $[a_-, b_-] - ([a_-, b_-])_- \in \mathcal{B}_+ \cap \mathcal{S}$.

Therefore $[[a + \mathcal{S}, b + \mathcal{S}], c + \mathcal{S}] = [[a_-, b_-], c_-] + \mathcal{S}$.

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