Reduction of Lie-Jordan algebras

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In collaboration with: L. Ferro, A. Ibort and G. Marmo

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- We will study the reduction of classical and quantum systems emphasizing its common algebraic structure.



Lie-Jordan Algebras

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- Poisson algebras
 - Reduction by symmetries
 - Dirac reduction
 - Symmetries+constraints. Marsden-Ratiu reduction
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F.F. M. Zambon. Lett. Math. Phys. 85 (2008). arXiv:0806.0638F.F. L. Ferro, A. Ibort and G. Marmo. arXiv:1202.3969

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(*L*, ∘) is associative and (*L*, ∘, { , }) is a Poisson algebra.

Lie-Jordan and C^* **algebras**

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Conversely, given a complex associative algebra (A, ·) with involution *, the selfadjoint elements

$$\mathcal{A}_{\mathrm{sa}} = \{ x \in \mathcal{A} | x^* = x \}$$

form a Lie-Jordan algebra ($A_{sa}, \circ, [,]$) with $\hbar \neq 0$,

$$a \circ b = \frac{1}{2}(a \cdot b + b \cdot a), \qquad [a, b] = \frac{i}{2\hbar}(a \cdot b - b \cdot a)$$

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e.g. if E is the linear span of Hamiltonian vector fields.

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Reduction by symmetries and constraints can be combined

Marsden-Ratiu reduction (constraints + symmetries)

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- Assume $\mathcal B$ is also a Lie subalgebra, i.e. $\{\mathcal B,\mathcal B\}\subset \mathcal B$
- Then if $\mathcal{B}\cap\mathcal{I}$ is a Poisson ideal of $\mathcal{B},$

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BUT \mathcal{B} a Lie subalgebra is a rather strong condition...

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Proof:

Assume $\{f, g\} \notin \mathcal{I}, f \in \mathcal{I}, g \in \mathcal{B}$

i. e. $\{f, g\}(p) \neq 0, p \in N$

 $f^{2} \in \mathcal{B} \Rightarrow f\{f,g\} \in \mathcal{B} \Rightarrow f \in \mathcal{B} \cap \mathcal{I}$ $hf \in \mathcal{B} \forall h \in C^{\infty}(M) \Rightarrow h\{f,g\} + f\{h,g\} \in \mathcal{B} \Rightarrow h \in \mathcal{B}$

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d) If $B \cap TN = B' \cap TN \Leftrightarrow \mathcal{B} + \mathcal{I} = \mathcal{B}' + \mathcal{I}$ $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ $\mathcal{B}'/(\mathcal{B}' \cap \mathcal{I})$

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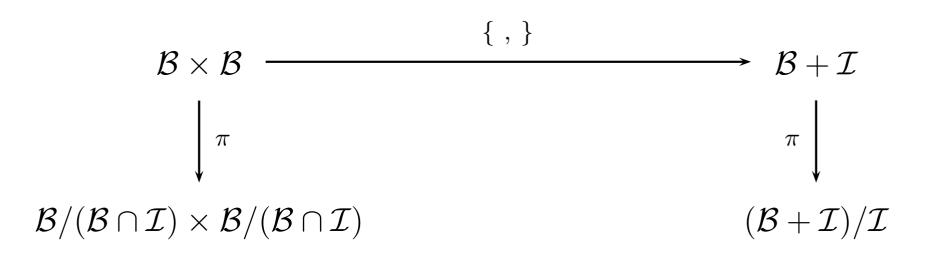
Summarising: MR reduction is nothing but a successive application of constraints and symmetries.!!

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 - $\mathcal{B} \subset \mathcal{L}$ a multiplicative subalgebra
 - $\mathcal{I} \subset \mathcal{L}$ a multiplicative ideal.

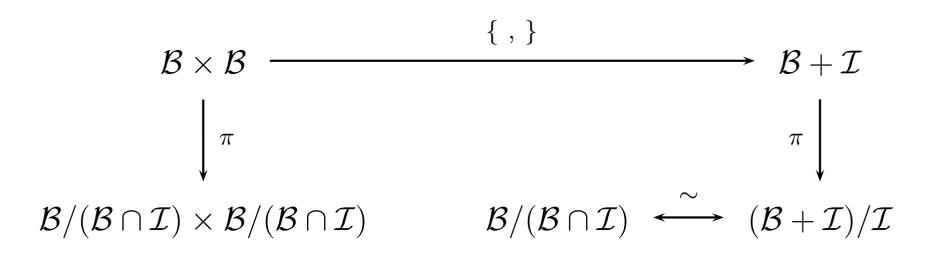
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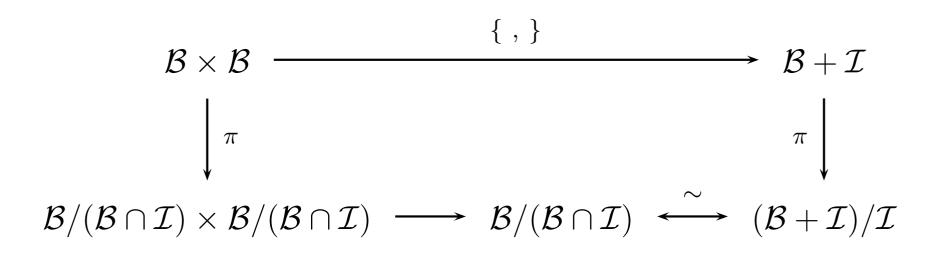
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 - s. t. $\mathcal{B}_{-} \subset \mathcal{B} \subset \mathcal{B}_{+}, \qquad \mathcal{B}_{\pm} + \mathcal{I} = \mathcal{B} + \mathcal{I}.$

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- The flaw: Jacobi identity is not guaranteed.
- A solution: introduce Jordan subalgebras \mathcal{B}_{-} and \mathcal{B}_{+} s. t. $\mathcal{B}_{-} \subset \mathcal{B} \subset \mathcal{B}_{+}, \quad \mathcal{B}_{\pm} + \mathcal{I} = \mathcal{B} + \mathcal{I}.$ Then, we can show: $\{\mathcal{B}_{-}, \mathcal{B}_{-}\} \subset \mathcal{B}_{+}, \quad \{\mathcal{B}_{-}, \mathcal{B}_{+} \cap \mathcal{I}\} \subset \mathcal{I} \Rightarrow Jacobi identity.$

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Then: $\mathcal{B} + \mathcal{I} = C^{\infty}(M)$ and $\{C^{\infty}(M), \mathcal{B} \cap \mathcal{I}\} \subset \mathcal{I}$

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Some properties, $\sigma \in \mathfrak{S}(\mathcal{L})$:

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Consider the space of Jordan derivations

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For $D \subset Der_J(\mathcal{L})$ let $\mathcal{F}_D = \{a \in \mathcal{L} \text{ s.t. } \delta a = 0 \ \forall \ \delta \in D\}$

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Take $D_{\mathcal{X}} = \{ [x, .] | x \in \mathcal{X} \subset \mathcal{L} \}$ then $\mathcal{F}_{D_{\mathcal{X}}}$ is a LJB subalgebra.

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MORE GENERAL REDUCTIONS?

Assume now \mathcal{B} is not a LJ subalgebra but there is \mathcal{S} s. t. $\mathcal{B} \circ \mathcal{B} \subset \mathcal{B} + \mathcal{S}$ and $[\mathcal{B}, \mathcal{B}] \subset \mathcal{B} + \mathcal{S}$

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 - $[\mathcal{B}_{-},\mathcal{B}_{-}] \subset \mathcal{B}_{+} \quad [\mathcal{B}_{-},(\mathcal{B}_{+} \cap \mathcal{S})] \subset \mathcal{S}$

Jacobi, Leibniz and associator identities are fulfilled

Assume now \mathcal{B} is not a LJ subalgebra but there is \mathcal{S} s. t.

- $\mathcal{B} \circ \mathcal{B} \subset \mathcal{B} + \mathcal{S}$ and $[\mathcal{B}, \mathcal{B}] \subset \mathcal{B} + \mathcal{S}$
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Then $\mathcal{B}/(\mathcal{B}+\mathcal{S})\sim (\mathcal{B}+\mathcal{S})/\mathcal{S}$ inherits \circ and [,] operations

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Jacobi, Leibniz and associator identities are fulfilled Proof: $[[a + S, b + S], c + S] = [([a_-, b_-])_-, c_-] + S$ but $[a_-, b_-] - ([a_-, b_-])_- \in \mathcal{B}_+ \cap S$. Therefore $[[a + S, b + S], c + S] = [[a_-, b_-], c_-] + S$.

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MORE WORK IS NEEDED