Local Hamiltonians and Multipartite entangled states

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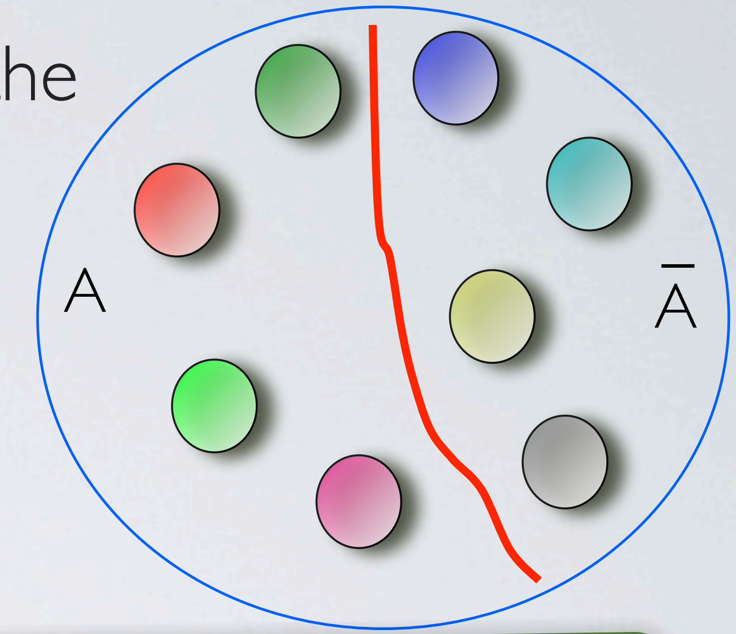
Outline

- Bipartite entanglement and multipartite entanglement
- Construction of local Hamiltonians with highly multipartite entangled states in the spectrum
- Explicit examples for 3, 4 and 5 qubits
- The special case of GHZ states

Entanglement... warm-up

We consider an ensemble of n two-level systems in the pure state

$$\rho = |\psi\rangle\langle\psi|$$



and a partition in two subsystems A (with n_A elements) and Ā (with $n_{\bar{A}}$ elements)

The reduced density matrix of subsystem A is obtained by tracing on the degrees of freedom of Ā and used to evaluate the purity (bipartite entanglement between A and Ā)

$$\pi_A = Tr_A(\rho_A^2)$$

and

$$\rho_A = Tr_{\bar{A}}(\rho)$$

with bounds

$$\frac{1}{2^{n_A}} \leq \pi_A \leq 1$$

reduced density matrix completely mixed

$$|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_{\bar{A}}$$

separable states with respect to the bipartition

maximally entangled states with respect to the bipartition

Entanglement

Bipartite systems VS Multipartite systems

Two subsystems A and B: evaluate entanglement between them

Many subsystems: ?!?!

From Bipartite to Multipartite Entanglement

The quantity π_A completely defines the BIPARTITE ENTANGLEMENT (one number is sufficient). *It depends on the bipartition*.

What about MULTIPARTITE ENTANGLEMENT?
The numbers needed to characterize the system scale exponentially with its size.

Statistical methods

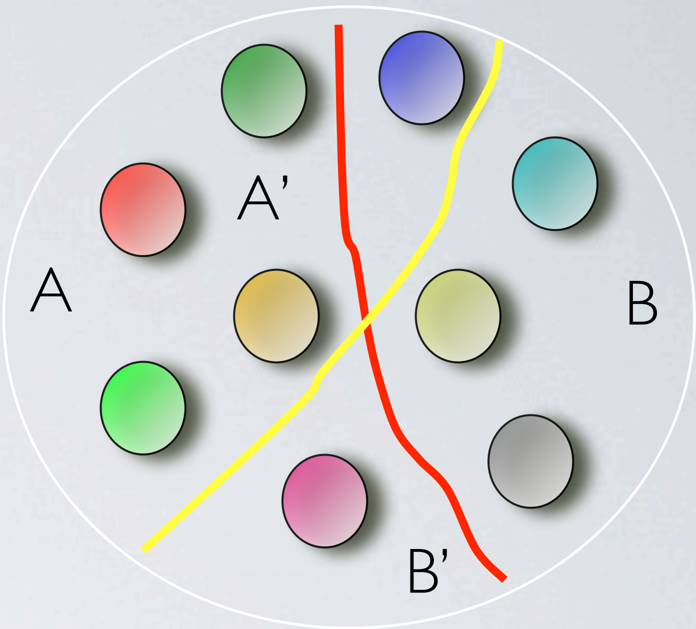
Seminal ideas from
Man'ko, Marmo, Sudarshan, Zaccaria: (J. Phys. A 02-03)
Parisi: complex systems

- The distribution of π_A characterizes the entanglement of the system.

Multipartite entanglement

Define the Potential of Multipartite Entanglement as the average purity over balanced bipartitions

$$n_A = \left[\frac{n}{2}\right]$$



$$\pi^{(n)}_{\mathrm{ME}}(|\psi\rangle) = \mathbb{E}[\pi_A] = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} \pi_A$$

Minimization of this ***cost function***

States minimizing the potential of multipartite entanglement:

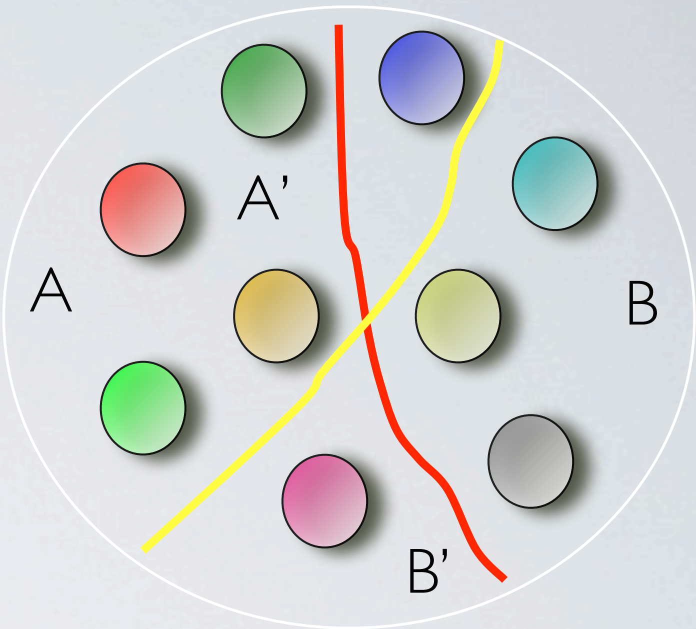
Maximally Multipartite Entangled State (MMES)

(see Facchi, G.F., Parisi, Pascazio PRA 2008; Facchi, G.F., Marzolino, Parisi, Pascazio JPhysA 2009, JPhysA 2010)

Multipartite entanglement

$$\frac{1}{2^{[n/2]}} \leq \pi_{\mathrm{ME}}^{(n)}(|\psi\rangle) \leq 1$$

$$n_A = \left[\frac{n}{2}\right]$$



Maximally Multipartite Entangled State (MMES):

$$\pi_{\mathrm{ME}}(|\varphi\rangle) = \pi_0^{(n)}$$

$$\pi_0^{(n)} = \min\{\pi_{\mathrm{ME}}(|\psi\rangle) \mid |\psi\rangle \in \mathcal{H}_S, \langle\psi|\psi\rangle = 1\}$$

If the lower bound is saturated, the MMES is "perfect"

Perfect MMES exist only for n=2, 3, 5, 6
(Scott 2004)

Entanglement generation

Problem: generation of entangled states

1) Sequence of operations on an initial state

2) Eigenstate of a suitable Hamiltonian

GHZ state (non-degenerate) excited state

Explicit construction of Hamiltonians with highly entangled states in the spectrum

Facchi, GF, Pascazio, Pepe, PRA (2010), PRL (2011)

General strategy for MMES

The problem of finding a Hamiltonian involving local ***(two-body and nearest-neighbor)*** interactions and on site external magnetic fields, one of whose eigenstates is a MMES, is non-trivial

Consider the n-body Hamiltonian

$$H(\epsilon, \mathcal{K}) = \epsilon \mathcal{P} + H^{\perp}(\mathcal{K})$$

$\mathcal{P} = |\varphi\rangle\langle\varphi|$ Projection on MMES $|\varphi\rangle$

$H^{\perp}$ Hermitian operator depending on a set of parameters with $\mathcal{P} H^{\perp} \mathcal{P} = 0$

A simple observation:

$H^{\perp} = 0 \quad \epsilon < 0 \longrightarrow |\varphi\rangle$ nondegenerate ground state of the Hamiltonian

General strategy for MMES

Separate local and non local parts of the Hamiltonian

$$H(\epsilon, \mathcal{K}) = H_{\rm loc}(\epsilon, \mathcal{K}) + H_{\rm nonloc}(\epsilon, \mathcal{K})$$

Objective:

find a set of parameters such that

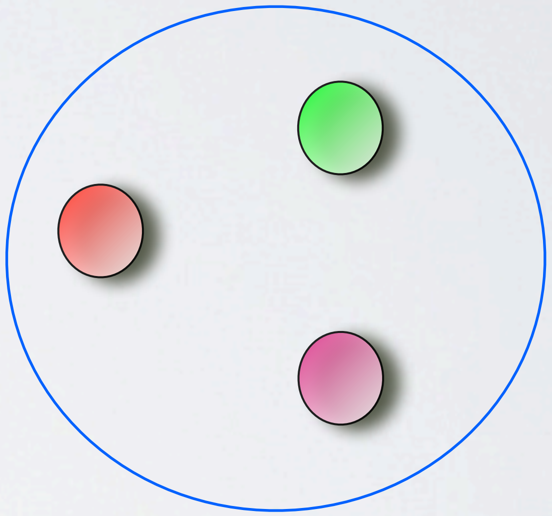
$$H_{\rm nonloc}(\bar{\epsilon}, \bar{\mathcal{K}}) = 0$$

(so that MMES is a non degenerate eigenstate, possibly the ground state)

3 qubits

MMES are equivalent by local unitaries to Greenberger-Horne-Zeilinger (GHZ) states

$$
\begin{aligned}
|G_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|000\rangle \pm |111\rangle\right), \\
|G_2^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|001\rangle \pm |110\rangle\right), \\
|G_3^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|010\rangle \pm |101\rangle\right), \\
|G_4^{\pm}\rangle &= \frac{1}{\sqrt{2}}\left(|011\rangle \pm |110\rangle\right)
\end{aligned}
$$



$$\sigma^z|i\rangle = (-1)^i|i\rangle$$

3 qubits

The most generic Hamiltonian is

$$H = \sum_{i=1}^{4} \left(\epsilon_i^{+} \mathcal{P}_i^{+} + \epsilon_i^{-} \mathcal{P}_i^{-} \right) + H_{\mathrm{M}}$$

$$\mathcal{P}_i^{\pm} = |G_i^{\pm}\rangle\langle G_i^{\pm}| \quad \text{with } i = 1, 2, 3, 4$$

Projections on the GHZ basis

ϵ_i Expectation value of the Hamiltonian on the GHZ basis

H_M Hermitian operator containing terms of the form

$$|G_i^{\pm}\rangle\langle G_j^{\pm}| + \mathrm{H.c.} \quad \text{with } i \neq j$$

3 qubits

Each projection can be decomposed in the form

$$\mathcal{P}_i^{\pm} = \mathcal{Q}_i \pm \mathcal{C}_i$$

product of two Pauli matrices

product of three Pauli matrices

Example:

$$\begin{aligned}
\mathcal{Q}_1 &= \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right), \\
&= \frac{1}{8}\left(1 + \sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z + \sigma_1^z\sigma_3^z\right) \\
\mathcal{C}_1 &= \frac{1}{2}\left(|000\rangle\langle 111| + |111\rangle\langle 000|\right) \\
&= \frac{1}{8}\left(\sigma_1^x\sigma_2^x\sigma_3^x - \sigma_1^x\sigma_2^y\sigma_3^y - \sigma_1^y\sigma_2^x\sigma_3^y - \sigma_1^y\sigma_2^y\sigma_3^x\right)
\end{aligned}$$

3 qubits

The Hamiltonian takes the form:

$$H = \sum_{i=1}^{4} \left(\epsilon_i^+ + \epsilon_i^-\right) \mathcal{Q}_i + \sum_{i=1}^{4} \left(\epsilon_i^+ - \epsilon_i^-\right) \mathcal{C}_i + H_{\mathrm{M}}$$

Orthogonality holds:

$$\mathcal{C}_i \mathcal{C}_j = 0 \qquad \forall i \neq j$$

does not contain product of three Pauli matrices

Cubic terms can be eliminated imposing

$$\epsilon_i^+ = \epsilon_i^- \qquad \forall i$$

3 qubits

We obtain:

$$\langle G_i^+ | H_2 | G_i^+ \rangle = \langle G_i^- | H_2 | G_i^- \rangle$$

with

$$H_2 = \sum_{i,j,\alpha,\beta} C_{ij}^{\alpha\beta} \sigma_i^\alpha \sigma_j^\beta$$

Suppose $|G_1^+\rangle$ ground state of this Hamiltonian

$|G_1^-\rangle$ eigenstate

degeneracy

$|G_1^-\rangle$ not an eigenstate

$$E_0 < \langle G_1^- | H_2 | G_1^- \rangle < E_{\max}$$

$|G_1^+\rangle$ not the ground state

GHZ states can never be nondegenerate ground states of this Hamiltonian

3 qubits

$|G_1^+\rangle$ excited state?

$$H_2 = \sum_{i<j}(J_{ij}^x\sigma_i^x\sigma_j^x + J_{ij}^y\sigma_i^y\sigma_j^y + J_{ij}^z\sigma_i^z\sigma_j^z) + \sum_{i\neq j}(K_{ij}\sigma_i^x\sigma_j^y + X_{ij}\sigma_i^x\sigma_j^z + Y_{ij}\sigma_i^y\sigma_j^z) + \sum_i(h_i^x\sigma_i^x + h_i^y\sigma_i^y + h_i^z\sigma_i^z)$$

The most general two-body Hamiltonian

YES if

$$\sum_{i=1}^{3} h_i^z = 0$$

$$\sum_{j\neq i} X_{ij} = 0 \quad \text{for} \quad i = 1,2,3,$$

$$\sum_{j\neq i} Y_{ij} = 0 \quad \text{for} \quad i = 1,2,3,$$

$$h_1^x = J_{23}^y - J_{23}^x\,, \quad h_2^x = J_{13}^y - J_{13}^x,$$
$$h_3^x = J_{12}^y - J_{12}^x\,, \quad h_1^y = K_{23} + K_{32},$$
$$h_2^y = K_{13} + K_{13}\,, \quad h_3^y = K_{12} + K_{21}$$

3 qubits

In order to check if there is no degeneracy we use a simplified version

$$H_{Jk} = J\sum_{i=1}^{3}\sigma_i^z\sigma_{i+1}^z + k\sum_{i=1}^{3}\left(\sigma_i^x\sigma_{i+1}^x - \sigma_i^x\right)$$

with periodic boundary conditions

$|G_1^+\rangle$ corresponds to the eigenvalue (nondegenerate) $3J$

for the set of parameters

$$J \neq 0, \qquad k \neq 0, \qquad J \neq -\frac{k}{2}$$

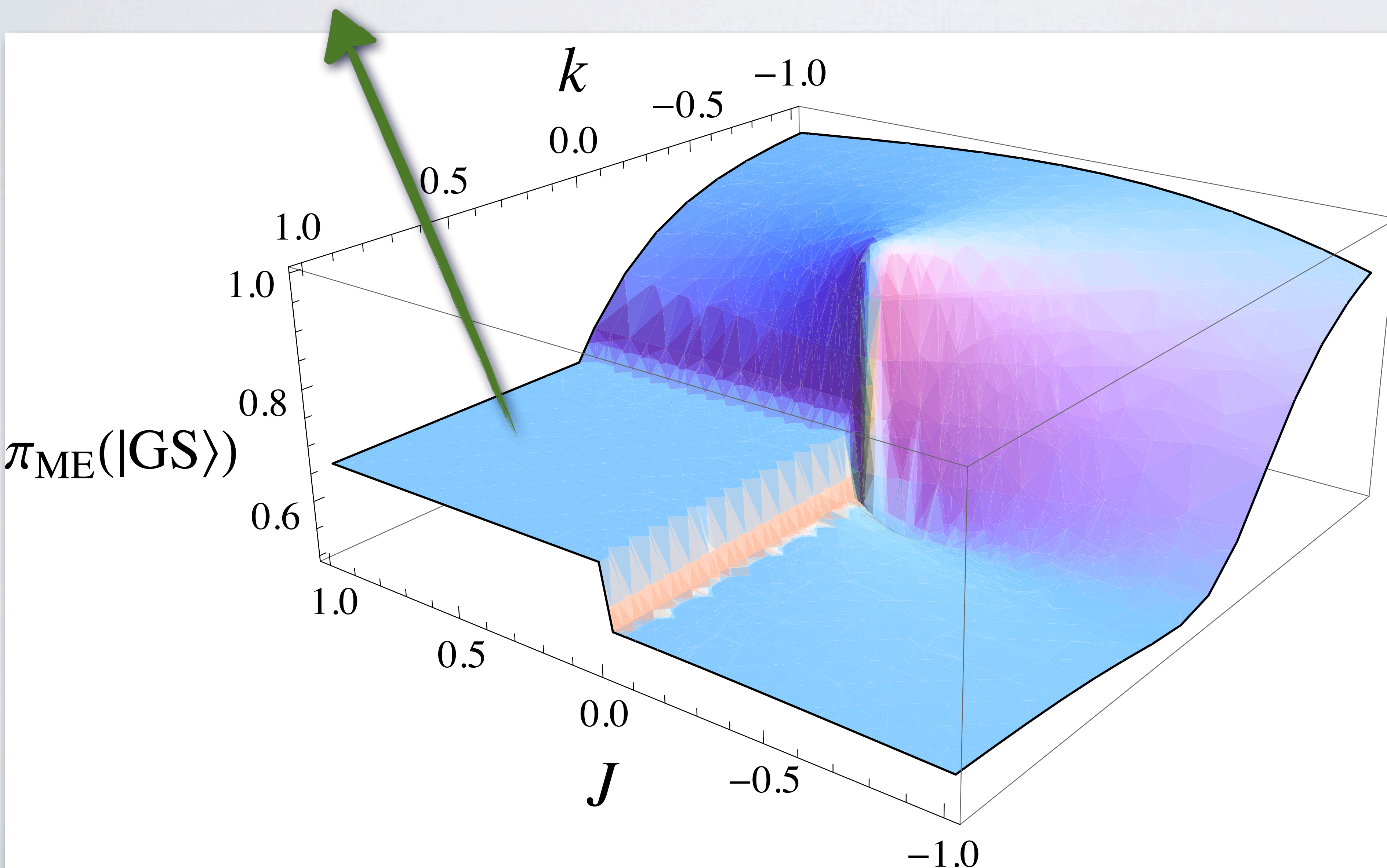
3 qubits

$|G_1^+\rangle$ is the first nondegenerate excited state if

$J < 0, k < 0$

$J < -k/2, k > 0$

$E_0 = -(J + 2k), \pi_{ME} = 2/3$



the GS has

$\pi_{ME} \leq 0.556$

4 qubits

Is it possible in general to have for all bipartitions $\pi_A = 1/2^{n_A}?$ (and saturate the lower bound)

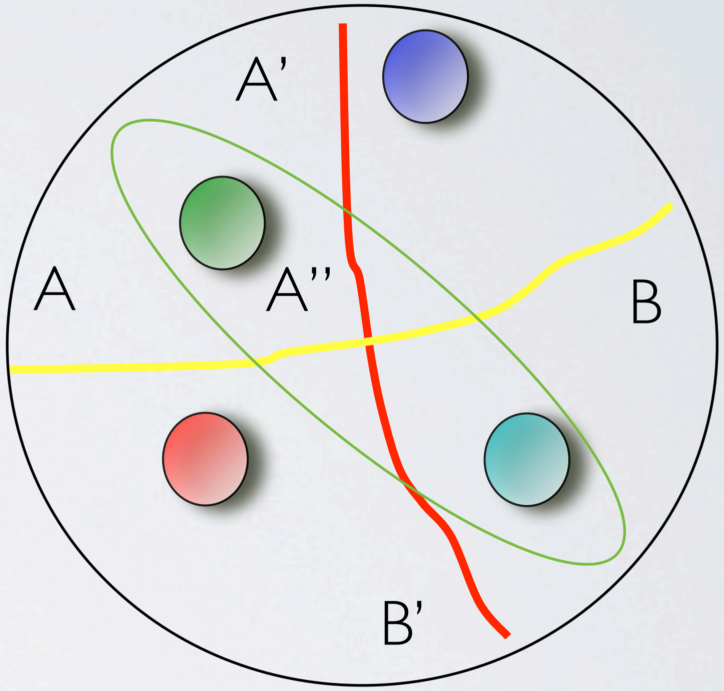
if NO → FRUSTRATION

$$n = 4$$

$$min\{\pi_A\} = \frac{1}{4}$$

$$min\{\frac{1}{2}(\pi_A + \pi_{A'})\} = \frac{1}{4}$$

π_A → $\pi_A + \pi_{A'}$ → $\pi_A + \pi_{A'} + \pi_{A''}$



$$min\{\pi^{(4)}_{ME}\} = \frac{1}{3}(\pi_A + \pi_{A'} + \pi_{A''}) = \frac{1}{3} \neq \frac{1}{4}$$

4 qubits

Consider

$$|M_4^1\rangle = \frac{1}{4}\sum_{k=0}^{15} \zeta_k^{(4)} |k\rangle \qquad \pi_{ME} = 1/4$$

$$\zeta^{(4)} = \{1,1,1,1,1,1,-1,-1,1,-1,1,-1,-1,1,1,-1\}$$

The generic Hamiltonian is

$$H = \sum_{i=1}^{16} \epsilon_i \mathcal{M}_4^i + H_{\mathrm{M}}$$

Hermitian operator containing terms of the form

$$|M_4^i\rangle\langle M_4^j| + \mathrm{H.c.} \quad \text{with } i \neq j$$

does not contain product of three and four Pauli matrices

$\mathcal{M}_4^i$ projection on $|M_4^i\rangle$

ϵ_i Expectation value of the Hamiltonian on the basis states

4 qubits

Example:

$$\begin{aligned}
\mathcal{M}_4^1 &= \frac{1}{16}\big(\mathbb{I} + \sigma_1^z\sigma_4^x + \sigma_2^z\sigma_3^x + \sigma_1^x\sigma_2^z\sigma_4^z + \sigma_1^x\sigma_3^x\sigma_4^z \\
&+ \sigma_1^y\sigma_2^z\sigma_4^y + \sigma_1^y\sigma_3^x\sigma_4^y + \sigma_2^x\sigma_1^z\sigma_3^z + \sigma_2^x\sigma_3^z\sigma_4^x \\
&+ \sigma_2^y\sigma_3^y\sigma_1^z + \sigma_2^y\sigma_3^y\sigma_4^x + \sigma_1^x\sigma_2^x\sigma_3^y\sigma_4^y - \sigma_1^x\sigma_2^y\sigma_3^z\sigma_4^y \\
&+ \sigma_1^y\sigma_2^y\sigma_3^z\sigma_4^z - \sigma_1^y\sigma_2^x\sigma_3^y\sigma_4^z + \sigma_1^z\sigma_2^z\sigma_3^x\sigma_4^z\big)
\end{aligned}$$

A difficult analysis...

4 qubits

From an analysis similar to the three qubits cas one can see that MMES can never be nondegenerate ground states for local Hamiltonian!

General conditions for non degeneracy are difficult to obtain. We consider

$$
\begin{aligned}
H_{Jk} &= J(\sigma_4^x\sigma_1^z + \sigma_3^x\sigma_2^z) \\
&+ k\left(\sigma_1^x\sigma_4^z + \sigma_2^x\sigma_3^z + \sigma_2^x\sigma_1^z + \sigma_1^x\sigma_2^z - \sum_{i=1}^{4}\sigma_i^z\right)
\end{aligned}
$$

4 qubits

$|M_4^1\rangle$ corresponds to the eigenvalue (nondegenerate) $2J$

for the set of parameters

$$J \neq 0, \quad k \neq 0, \quad J \neq \pm\sqrt{\frac{3}{2}}k, \quad J \neq \pm\sqrt{3}k$$

We have numerically generated a large sample of sets of parameters and analyzed the position of the MMES in the spectrum.

On the average, the MMES is inte center of the energetic band (in the best case is the second excited level)

5 qubits

Consider

$$|M_5^1\rangle = \frac{1}{4\sqrt{2}} \sum_{k=0}^{31} \zeta_k^{(5)} |k\rangle \qquad \pi_{ME} = 1/4$$

$$\zeta^{(5)} = \{1,1,1,1,1,-1,-1,1,1,-1,-1,1,1,1,1,1,1,1,-1,-1,1,-1,1,-1,\\ -1,1,-1,1,-1,-1,1,1\}$$

The generic Hamiltonian is

$$H = \sum_{i=1}^{32} \epsilon_i \mathcal{M}_5^i + H_{\mathrm{M}}$$

Hermitian operator containing terms of the form

$$|M_5^i\rangle\langle M_5^j| + \mathrm{H.c.} \quad \text{with } i \neq j$$

does not contain product of three and four Pauli matrices

$\mathcal{M}_5^i$ projection on $|M_5^i\rangle$

ϵ_i Expectation value of the Hamiltonian on the basis states

5 qubits

Also in this case is it non possible to have a MMES as nondegenerate ground state for a local Hamiltonian (same expectation value of the Hamiltonian if only two-body interaction terms are present)

It is possible to find local Hamiltonians witha MMES as a nondegenerate excited state (but the expressions are very complicated).

Numerical analysis based on analytical results show that MMES are placed in the center of the spectrum. Apparently it is not possible to reach low-lying excited states (not better than the 14th level).

GHZ states

$$|G^n_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes n} \pm |1\rangle^{\otimes n}\right) \qquad \sigma^z|i\rangle = (-1)^i|i\rangle$$

Hamiltonian with n-body interactions with GHZ as a nondegenerate ground state

$$H = E_0|G^n_+\rangle\langle G^n_+|, \quad E_0 < 0$$

In the case of m-body interactions (m<n) it is not possible to have GHZ as non degenerate ground state

$|G^n_+\rangle$ $|G^n_-\rangle$ → They share the same m-body reduced density matrices and, thus, the same expectation values of m-body interaction terms

GHZ states

Is it possible to have a GHZ as nondegenerate excited state for an Hamiltonian with m-body interaction terms (m<n)?

Hamiltonian with m-body interactions (m < n)

$$H^{(m)} = \sum_{j_1<j_2<\cdots<j_m} \sum_{\alpha_1,\ldots,\alpha_m} J^{\alpha_1\ldots\alpha_m}_{j_1\ldots j_m} \sigma^{\alpha_1}_{j_1} \ldots \sigma^{\alpha_m}_{j_m}$$

$$\alpha_i = 0, x, y, z \quad \sigma_i^0 \equiv 1_i$$

It is possible to show that if the Hamiltonian contains terms that couple less than

$$\boxed{m_n^* = [(n+1)/2]}$$

the GHZ state $|G_+^n\rangle$ and any equivalent state by local unitaries cannot be a nondegenerate eigenstate

GHZ states

If interaction terms involve a sufficient number of bodies, degeneration can be avoided!

Consider

$$H_0 = -\sum_{j=1}^{n} \sigma_j^z \sigma_{j+1}^z \qquad (\text{with } \sigma_{n+1} \equiv \sigma_1)$$

GHZ states $|G_+^n\rangle$ $|G_-^n\rangle$ are degenerate eigenstates with energy

$$E_0 = -n$$

Excited states are grouped in degenerate multiplets with energy

$$E^{(k)} = -n + 4k, \qquad \text{with } k = 1, 2, \ldots, 2[n/2]$$

$$\Delta E = -4$$

GHZ states

In order to lift the degeneracy we consider the Hamiltonian

$$H(\lambda) = H_0 + \lambda H_1$$

$$H_1 = \sigma_1^x \sigma_2^x \ldots \sigma_{[n/2]}^x - \sigma_{[n/2]+1}^x \ldots \sigma_n^x$$

two string of of spin flipping matrices acting on one half of the system

It is not possible to reduce the number of addenda nor to reduce the range of the couplings.

We are interested in determining the position of the eigenstate

$$|G_+^n\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes n} + |1\rangle^{\otimes n} \right)$$

GHZ states

It is easy to check that

$$H(\lambda)|G_+^n\rangle = -n|G_+^n\rangle$$

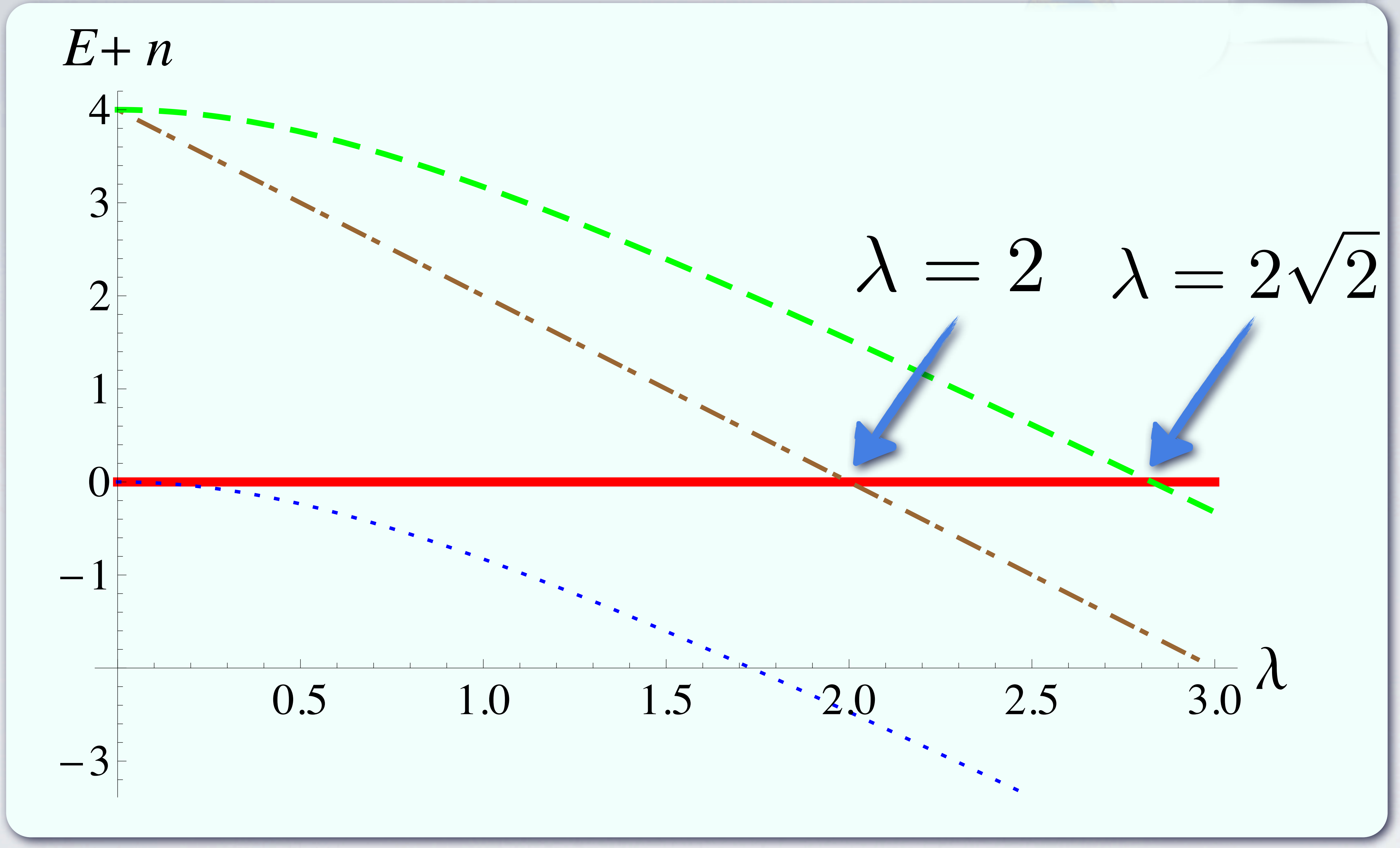
The twofold degenerate the ground state is split in two energy levels by the perturbation!

The state $|G_+^n\rangle$ is left unchanged by the presence of the perturbation (is an excited state); the lower one represents the nondegenerate ground state for the Hamiltonian

$$H(\lambda) = H_0 + \lambda H_1$$

energy of the ground state: $E_0 = -n - 2(\sqrt{1+\lambda^2} - 1)$

GHZ



It is possible to see that $|G_+^n\rangle$ is nondegenerate if

$$\lambda \neq \pm 2k \quad \text{and} \quad \lambda \neq \pm 2\sqrt{k(k+1)} \qquad k = 1, 2, \ldots, 2[n/2]$$

Conclusions

- We constructed local Hamiltonians with highly multipartite entangled states in the spectrum
- We have found necessary and sufficient conditions for encoding GHZ states into the nondegenerate eigenstate of an Hamiltonian
- General conditions for MMES?
- Experimental feasibility?