Invitation to open quantum systems

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Classical Markovian semigroup

Classical n-state system

$$\mathbf{p} = (p_1, \ldots, p_n)^{\mathrm{T}}$$ — probability vector

$$p_i \geq 0 \; ; \quad \sum_i p_i = 1$$

How to map $\mathbf{p}$ into $\mathbf{q}$?

$$\mathbf{q} = T\,\mathbf{p}$$

T – stochastic matrix

$$T_{ij} \geq 0 \; ; \quad \sum_i T_{ij} = 1$$

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Classical dynamics of n-state system

$$\mathbf{p}(t) = (p_1(t), \ldots, p_n(t))^{\mathrm{T}} \; ; \quad \mathbf{p}(0) = \mathbf{p}$$

$$\mathbf{p}(t) = T(t)\mathbf{p}$$

$$p_i(t) = \sum_j T_{ij}(t) p_j$$

$T(t)$ – classical dynamical map

$T(t)$ – stochastic matrix ; $T(0) = \mathbb{I}_n$

Pauli rate equation

$$\boxed{\frac{d}{dt}\,p_i(t) = \sum_j \Big(\pi_{ij} p_j(t) - \pi_{ji} p_i(t)\Big) \;\; ; \quad \mathbf{p}(0) = \mathbf{p}}$$

$\pi_{ij} \geq 0$; – transition probability from state "j" to state "i" per unit time

$$L_{ij} = \pi_{ij} - \delta_{ij} \sum_k \pi_{kj}$$

$$\frac{d}{dt} p_i(t) = \sum_j L_{ij}\, p_i(t)$$

$$\frac{d}{dt}\,\mathbf{p}(t) = L\,\mathbf{p}(t)$$

L – classical stochastic generator

Pauli rate equation – properties of L

$$\frac{d}{dt}\,\mathbf{p}(t) = L\,\mathbf{p}(t)$$

$$L_{ij} = \pi_{ij} - \delta_{ij}\sum_k \pi_{kj}$$

$$L_{ij} = \pi_{ij}\;;\quad i \neq j$$

$$\sum_i L_{ij} = \sum_i \pi_{ij} - \sum_k \pi_{kj} = 0$$

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Kolmogorov conditions

Suppose you are given L but do not know that

$$L_{ij} = \pi_{ij} - \delta_{ij} \sum_k \pi_{kj} \; ; \quad \pi_{ij} \geq 0$$

How to check that L is legitimate?

Kolmogorov conditions

$$\boxed{L_{ij} \geq 0 \; (i \neq j) \; ; \quad \sum_i L_{ij} = 0}$$

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Classical stochastic dynamics

$$\mathbf{p}(t) = T(t)\,\mathbf{p}$$

$$\frac{d}{dt}T(t) = L\,T(t)\ ; \quad T(0) = \mathbb{I}_n$$

$$T(t) = e^{tL}$$

classical stochastic semigroup

$$T(t+s) = T(t)T(s)$$

Irreversibility

Remark: note, that $T(t)$ is invertible

$$T^{-1}(t) = e^{-tL}$$

however, its is not stochastic matrix !!!

Stochastic dynamics is irreversible

Algebraic approach

Classical algebra of observables

$(\mathbb{R}^n, \circ)$ – commutative algebra

$$\mathbf{a} = (a_1, \ldots, a_n)^{\mathrm{T}}$$

$$(\mathbf{a} \circ \mathbf{b})_k := a_k b_k$$

$$\mathbf{a} \geq 0 \iff \mathbf{a} = \mathbf{x} \circ \mathbf{x} \iff a_k = x_k^2 \geq 0$$

$$\mathbf{e} = (1, \ldots, 1)^{\mathrm{T}} \; ; \quad \mathbf{a} \circ \mathbf{e} = \mathbf{a}$$

Algebraic approach — states

States = normalized positive linear functionals

$$\mathbf{p}(\mathbf{a}) = \mathbf{a}^{\mathrm{T}} \cdot \mathbf{p} = \sum_k p_k a_k$$

$$\mathbf{p}(\mathbf{a}) \geq 0 \; ; \quad \mathbf{a} \geq 0$$

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Algebraic approach — positive maps

A linear map $M : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is positive iff

$$\mathbf{a} \geq 0 \implies M\mathbf{a} \geq 0$$

M is positive iff $M_{ij} \geq 0$

A positive map M is stochastic iff $M^{\mathrm{T}}\mathbf{e} = \mathbf{e}$

A positive map M is doubly-stochastic iff M and M^{T} are stochastic.

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Algebraic approach — generator

$$L_{ij} = \pi_{ij} - \delta_{ij} \sum_k \pi_{kj} \; ; \quad \pi_{ij} \geq 0$$

$$(L\mathbf{p})_j = \sum_i \Big(\pi_{ij} p_j - \pi_{ji} p_i \Big)$$

Π – positive map ; $\Pi_{ij} = \pi_{ij} \geq 0$

$$\boxed{L\mathbf{p} = \Pi\,\mathbf{p} - (\Pi^{\mathrm{T}}\mathbf{e}) \circ \mathbf{p}}$$

If Π is stochastic, then

$$L\mathbf{p} = \Pi\,\mathbf{p} - \mathbf{p} \quad \Longrightarrow \quad L = \Pi - \mathbb{I}_n$$

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Quantum Markovian semigroup

Operator algebras — intro

$\mathcal{A}$ – $\mathbb{C}^*$-algebra with identity I

$$a \geq 0 \iff a = xx^* \; ; \quad \mathcal{A}_+$$

A linear map

$$\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$$

is positive iff

$$\Phi(\mathcal{A}_+) \subset \mathcal{A}_+$$

States

A positive map

$$\Phi : \mathcal{A} \longrightarrow \mathbb{C}$$

such that

$$\Phi(I) = 1$$

is called a state.

$$M_k(\mathcal{A}) = M_k(\mathbb{C}) \otimes \mathcal{A}$$

A linear map

$$\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$$

is k-positive iff

$$\Phi_k := \mathbb{1}_k \otimes \Phi : M_k(\mathcal{A}) \longrightarrow M_k(\mathcal{B}(\mathcal{H}))$$

is positive.

A linear map

$$\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$$

is completely positive (CP) iff it is k-positive for $k = 1, 2, \ldots$.

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Example: how Φ_2 operates

$$X \in M_2(\mathcal{A})$$

$$X = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \; ; \quad a_1, a_2, a_3, a_4 \in \mathcal{A}$$

$$\Phi_2(X) = \begin{pmatrix} \Phi(a_1) & \Phi(a_2) \\ \Phi(a_3) & \Phi(a_4) \end{pmatrix}$$

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Stinespring theorem

A map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is CP iff there exists

- Hilbert space $\mathcal{K}$
- representation

$$\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{K})$$

- linear operator

$$V : \mathcal{K} \longrightarrow \mathcal{H}$$

such that

$$\Phi(a) = V\pi(a)V^{\dagger}$$

Stinespring theorem vs. GNS

$$\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$$

If $\mathcal{H} = \mathbb{C}$ or $\mathcal{A}$ is commutative, then Φ is CP iff it is positive.

If $\mathcal{H} = \mathbb{C}$ and Φ is positive (and hence CP), then Φ defines a state. Then

$$\Phi(a) = V\pi(a)V^{\dagger} = \langle\Omega|\pi(a)|\Omega\rangle$$

recovers GNS !

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recovers GNS !

$\mathcal{H} = \mathbb{C}^n$; $\mathcal{A} = \mathcal{B}(\mathcal{H}) = M_n(\mathbb{C})$

$$\Phi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$$

Φ is CP iff Φ is n-positive

Choi, Kraus, Sudarshan

$$\Phi(a) = \sum_{\alpha} K_\alpha \, a \, K_\alpha^\dagger$$

Φ is trace-preserving iff $\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{I}_n$

Φ is unital ($\Phi(\mathbb{I}_n) = \mathbb{I}_n$) iff $\sum_\alpha K_\alpha K_\alpha^\dagger = \mathbb{I}_n$

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States

$$\Phi : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$$

$$\Phi(a) = \mathrm{Tr}(a\rho)$$

quantum states $\longrightarrow$ density operators in $M_n(\mathbb{C})$

$\mathcal{S}(\mathcal{H})$ – space of quantum states

$$\rho \geq 0 \,, \quad \mathsf{Tr}\,\rho = 1$$

Why CP ?

Φ_1 & Φ_2 - positive maps

$\Phi_1 \otimes \Phi_2$ needs NOT be positive !!!

Φ_1 & Φ_2 - CP maps

$\Phi_1 \otimes \Phi_2$ is CP as well !!!

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Why CP ?

$$\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

$$\rho = \sum_k p_k \rho_k^A \otimes \rho_k^B$$

$$(\mathbb{1}_A \otimes \Phi)\rho \geq 0$$

If ρ is entangled and Φ positive $(\mathbb{1}_A \otimes \Phi)\rho$ needs NOT be positive !!!

If Φ is CP, then $(\mathbb{1}_A \otimes \Phi)\rho \geq 0$ always !!!

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Dynamical map

$$\Lambda_t \ : \ \mathcal{S}(\mathcal{H}) \ \longrightarrow \ \mathcal{S}(\mathcal{H}) \ \ ; \quad t \geq 0$$

$$\Lambda_0 = \mathbb{1}$$

Λ_t completely positive and trace preserving (CPTP)

$$\rho_0 \quad \longrightarrow \quad \rho_t := \Lambda_t \rho_0$$

Classical $\longrightarrow$ Quantum

Stochastic map $\longrightarrow$ CPTP map

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Markovian semigroup

The quantum analog of the Pauli rate equation

$$\frac{d}{dt}\,\mathbf{p}_t = L\,\mathbf{p}_t$$

is the following Master Equation

$$\frac{d}{dt}\,\rho_t = L\,\rho_t$$

$$L : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$$

generator of the dynamical semigroup ("superoperator")

Markovian semigroup

$$\rho_t = \Lambda_t \rho$$

$$\frac{d}{dt}\Lambda_t = L\,\Lambda_t \ ; \quad \Lambda_0 = \mathbb{1}$$

$$\Lambda_t = e^{tL} = \mathbb{1} + tL + \frac{1}{2}t^2L^2 + \ldots$$

$$\Lambda_t\Lambda_s = \Lambda_{t+s} \ ; \quad t,s \geq 0$$

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What are the properties of L such that $\Lambda_t = e^{tL}$ is CPTP ?

$$\boxed{L\mathbf{p} = \Pi\mathbf{p} - (\Pi^{\mathrm{T}}\mathbf{e}) \circ \mathbf{p}} \;;\quad \Pi_{ij} \geq 0$$

$$\mathbf{a} \circ \mathbf{b} = \frac{1}{2}(\mathbf{a} \circ \mathbf{b} + \mathbf{b} \circ \mathbf{a}) = \frac{1}{2}\{\mathbf{a}, \mathbf{b}\}_{\circ}$$

$$\Pi \longrightarrow \Phi - \text{CP}$$

$$\boxed{L\rho = \Phi\rho - \tfrac{1}{2}\{\Phi^* \mathbb{I}_n, \rho\}}$$

$$\mathrm{Tr}(a[\Phi\rho]) = \mathrm{Tr}([\Phi^* a]\rho)$$

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Gorini-Kossakowski-Sudarshan & Lindblad

$$\boxed{L\rho = \Phi\rho - \tfrac{1}{2}\{\Phi^* \mathbb{I}_n, \rho\}} \quad ; \quad \Phi \text{ - CP}$$

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$$L\rho = \sum_\alpha \left(V_\alpha \rho V_\alpha^\dagger - \frac{1}{2}\{V_\alpha^\dagger V_\alpha, \rho\} \right)$$

$$L\rho = -i[H, \rho] + \sum_\alpha \left(V_\alpha \rho V_\alpha^\dagger - \frac{1}{2}\{V_\alpha^\dagger V_\alpha, \rho\} \right)$$

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Example 1 – pure decoherence of a qubit

$$L\rho \;=\; -i[H,\rho] + \sum_\alpha \left(V_\alpha \rho V_\alpha^\dagger - \tfrac{1}{2}\{V_\alpha^\dagger V_\alpha, \rho\} \right)$$

$$H = 0 \;; \quad V = \sqrt{\gamma/2}\,\sigma_z \;; \quad \gamma > 0$$

$$L\rho = \frac{1}{2}\gamma\left(\sigma_z \rho \sigma_z - \rho\right)$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

e_1 – excited state ; e_0 – ground state

$$\rho_t = \begin{pmatrix} \rho_{11} & \rho_{10}\, e^{-\gamma t} \\ \rho_{01}\, e^{-\gamma t} & \rho_{00} \end{pmatrix} \longrightarrow \begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{00} \end{pmatrix}$$

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$$L\rho = \frac{1}{2}\gamma\left(\sigma_z \rho \sigma_z - \rho\right)$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

e_1 – excited state ; e_0 – ground state

$$\rho_t \;=\; \begin{pmatrix} \rho_{11} & \rho_{10}\, e^{-\gamma t} \\ \rho_{01}\, e^{-\gamma t} & \rho_{00} \end{pmatrix} \longrightarrow \begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{00} \end{pmatrix}$$

Example 1 – pure decoherence of a qubit

$$\boxed{L\rho \;=\; -i[H,\rho] + \sum_\alpha \left(V_\alpha \rho V_\alpha^\dagger - \tfrac{1}{2}\{V_\alpha^\dagger V_\alpha, \rho\}\right)}$$

$$H = \frac{\omega}{2}\sigma_3 \;;\quad V = \sqrt{\gamma/2}\,\sigma_z \;;\quad \gamma > 0$$

$$L\rho = -\frac{i\omega}{2}[\sigma_3,\rho] \;+\; \frac{1}{2}\gamma\left(\sigma_z\rho\sigma_z - \rho\right)$$

$$\rho_t \;=\; \begin{pmatrix} \rho_{11} & \rho_{10}\, e^{-i\omega t - \gamma t} \\ \rho_{01}\, e^{i\omega t - \gamma t} & \rho_{00} \end{pmatrix} \;\longrightarrow\; \begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{00} \end{pmatrix}$$

Example 2 – dynamics of a qubit

$$\sigma_+ = |1\rangle\langle 0| \; ; \quad \sigma_- = |0\rangle\langle 1|$$

- unitary evolution $L_0\rho = -i[\sigma_3, \rho]$
- dumping $L_-\rho = \sigma_-\rho\sigma_+ \; ; \quad L_-|1\rangle\langle 1| = |0\rangle\langle 0|$
- pumping $L_+\rho = \sigma_+\rho\sigma_- \; ; \quad L_+|0\rangle\langle 0| = |1\rangle\langle 1|$
- pure decoherence $L_z\rho = \sigma_z\rho\sigma_z$

$$L = \frac{\omega}{2} L_0 + \gamma_- L_- + \gamma_+ L_+ + \frac{\gamma}{2} L_z \; ; \quad \gamma_-, \gamma_+, \gamma > 0$$

$$\rho_t \;=\; \begin{pmatrix} p_1(t) & x(t) \\ \overline{x}(t) & p_0(t) \end{pmatrix} \; ; \quad p_0(t) + p_1(t) = 1$$

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$$p_1(t) = 1 - p_0(t)$$

$$x_t = x_0 \, \exp\Big(- i\omega t - [\gamma_- + \gamma_+]t/2 + \gamma t \Big)$$

$$p_0(t) \longrightarrow p_0^* = \frac{\gamma_-}{\gamma_+ + \gamma_-} \; ; \quad p_1(t) \longrightarrow p_1^* = \frac{\gamma_+}{\gamma_+ + \gamma_-}$$

$$\rho_t \longrightarrow \begin{pmatrix} p_1^* & 0 \\ 0 & p_0^* \end{pmatrix}$$ – equilibrium state

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Reduced dynamics

$$\mathcal{H} \otimes \mathcal{H}_R$$

$$\Lambda_t \rho := \mathrm{Tr}_R \left[e^{-iHt} (\rho \otimes \omega_R) e^{iHt} \right]$$

One obtains Markovian semigroup $\Lambda_t = e^{tL}$ only under suitable Born-Markov approximation

Genuine Λ_t is NOT of the form e^{tL} !

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Pure decoherence model

$$H = H_S \otimes \mathbb{I}_R + \mathbb{I}_S \otimes H_R + H_{SR}$$

$|n\rangle$ – orthonormal basis in $\mathcal{H}_S$; $P_n := |n\rangle\langle n|$

$$H_S = \sum_n \epsilon_n P_n$$

$$H_{SR} = \sum_n P_n \otimes B_n$$

$$\mathbb{I}_S = \sum_n P_n$$

$$\boxed{H = \sum_n P_n \otimes Z_n} \quad ; \qquad Z_n = \epsilon_n \mathbb{I}_R + H_R + B_n$$

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$$= \sum_{n,m} c_{mn}(t)\, P_m \rho P_n$$

$$c_{mn}(t) = \mathrm{Tr}\Big(e^{-iZ_m t}\,\omega_R\, e^{iZ_n t}\Big)$$

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$$\boxed{L_t = \dot{\Lambda}_t\, \Lambda_t^{-1}}$$

$$L_t \rho = \sum_{m,n} \frac{\dot{c}_{mn}(t)}{c_{mn}(t)} P_m \rho P_n = \sum_{m \neq n} \alpha_{mn}(t)\, P_m \rho P_n$$

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How to describe general quantum evolution?

$$\rho \quad \longrightarrow \quad \rho_t = \Lambda_t \rho$$

There are several approaches

- local in time master equation (TCL approach)
- non-local master equation (TC – memory kernel)
- stochastic unraveling
- ...

Local in time master equation – TCL

$$\frac{d}{dt}\Lambda_t = L_t\Lambda_t \ , \quad \Lambda_0 = \mathbb{1}$$

$$\Lambda_t = \mathrm{T}\ \exp\left(\int_0^t L_u du\right) =$$

$$= \mathbb{1} + \int_0^t dt_1\, L_{t_1} + \int_0^t dt_1 \int_0^{t_1} dt_2\, L_{t_1} L_{t_2} + \ldots$$

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Basic question

What are condition for L_t such that the solution to

$$\frac{d}{dt}\Lambda_t = L_t \Lambda_t \ , \quad \Lambda_0 = \mathbb{1}$$

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is legitimate — CPTP?

Conditions for L_t are NOT known

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is legitimate — CPTP?

Conditions for L_t are NOT known

Special classes

1. C1 – Markovian semigroup (K-L generator)
2. C2 – Divisible maps
3. C3 – Commutative dynamics

$$C1 \subset C2 \cap C3$$

Divisible maps

Λ_t – dynamical map

Λ_t is divisible iff

$$\boxed{\Lambda_t = V_{t,s}\Lambda_s}$$

$$t \geq s \geq 0$$

$V_{t,s}$ completely positive maps for all $t \geq s$

Divisible maps

$$\frac{d}{dt}\Lambda_t = L_t \Lambda_t \ , \quad \Lambda_0 = \mathbb{1}$$

$$\frac{d}{dt}V_{t,s} = L_t V_{t,s} \ , \quad V_{s,s} = \mathbb{1}$$

$$V_{t,s} = \mathrm{T}\,\exp\left(\int_s^t L_u du\right)$$

$$\Lambda_t = V_{t,0}$$

$$\boxed{V_{t,s}V_{s,u} = V_{t,u}}$$

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Divisible maps

Λ_t is divisible iff

L_t – legitimate Kossakowski-Lindblad generator for all t

$$L_t\rho \;=\; -i[H(t),\rho] + \sum_\alpha \left(V_\alpha(t)\rho V_\alpha^\dagger(t) - \frac{1}{2}\{V_\alpha^\dagger(t)V_\alpha(t),\rho\} \right)$$

Divisible vs. Markovian

$$\Lambda_t = V_{t,s}\Lambda_s$$

$$\boxed{V_{t,s}V_{s,u} = V_{t,u}}$$

Some authors call the quantum evolution Markovian iff Λ_t defines divisible dynamical map.

$$\boxed{\text{Markovianity} = \text{Divisibility}}$$

Commutative dynamics

$$[L_t, L_u] = 0$$

$$\Lambda_t = \mathrm{T}\ \exp\left(\int_0^t L_u du\right) = \exp\left(\int_0^t L_u du\right)$$

L_t defines a legitimate generator iff

$\int_0^t L_u du$ has K-L form for all $t \geq 0$

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Example: qubit dynamics

commutative pure decoherence

$$L_t\rho = \frac{1}{2}\gamma(t)\Big(\sigma_z\rho\sigma_z - \rho\Big)$$

$$\rho_t \; = \left(\begin{array}{cc} \rho_{00} & \rho_{01}\, e^{-\Gamma(t)} \\ \rho_{10}\, e^{-\Gamma(t)} & \rho_{11} \end{array} \right) \; ; \quad \Gamma(t) := \int_0^t \gamma(u)\, du$$

- Λ_t is CPTP iff $\Gamma(t) \geq 0$
- Λ_t is divisible (Markovian) iff $\gamma(t) \geq 0$
- Λ_t is a Markovian semigroup iff $\gamma(t) = \gamma > 0$

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Stochastic unraveling

$$\boxed{\dot{\psi}_t = \tfrac{i}{2}\,\omega_t\,\sigma_z\,\psi_t}$$

$$\langle\langle\,\omega_t\,\omega_s\,\rangle\rangle =: \alpha(t,s)\ , \quad \langle\langle\,\omega_t\,\rangle\rangle = 0$$

$$\rho_t := \langle\langle\,|\psi_t\rangle\langle\psi_t|\,\rangle\rangle \quad \longrightarrow \quad \boxed{\dot{\rho}_t = \gamma(t)\Big(\sigma_z\rho_t\sigma_z - \rho_t\Big)}$$

$$\gamma(t) := \int_0^t \alpha(t,s)\,ds$$ needs not be positive!

$$\Gamma(t) := \int_0^t \gamma(u)\,du = \int_0^t\int_0^t \alpha(u,s)\,du\,ds \geq 0$$

$$\boxed{\text{Markovian} \quad \longleftrightarrow \quad \alpha(t,s) = \gamma\delta(t-s) \quad \longleftrightarrow \quad \gamma(t) = \gamma}$$

Markovianity – various concepts

- Markovian semigroup (white noise) $\longrightarrow \mathcal{M}_1$

$$\boxed{\Lambda_t \Lambda_u = \Lambda_{t+u}}$$

- divisibility $\longrightarrow \mathcal{M}_2$

$$\boxed{V_{t,s} V_{s,u} = V_{t,u}}$$

- distinguishability of states (Breuer et al) $\longrightarrow \mathcal{M}_3$

NOT related to any composition law

$$\boxed{\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3}$$

Distinguishability

$$D[\rho_1, \rho_2] := \frac{1}{2}||\rho_1 - \rho_2||_1$$

$$||A||_1 := \mathrm{Tr}|A| = \mathrm{Tr}\sqrt{AA^\dagger}$$

$$D[\rho_1, \rho_2] = \text{distinguishability of } \rho_1 \text{ and } \rho_2$$

$$\Lambda \quad \text{— CPTP map (quantum channel)}$$

$$\boxed{D[\Lambda\,\rho_1, \Lambda\,\rho_2] \leq D[\rho_1, \rho_2]}$$

Divisibility vs. distinguishability

Λ_t – dynamical map

$$\rho_1(t) = \Lambda_t \rho_1 \ , \qquad \rho_2(t) = \Lambda_t \rho_2$$

$$\boxed{D[\rho_1(t), \rho_2(t)] \ \leq \ D[\rho_1, \rho_2] \quad ; t \geq 0}$$

$$\rho_t = U_t \rho U_t^\dagger \quad \Longrightarrow \quad \frac{d}{dt} D[\rho_1(t), \rho_2(t)] \ = \ 0$$

$$\rho_t = e^{Lt} \rho \quad \Longrightarrow \quad \frac{d}{dt} D[\rho_1(t), \rho_2(t)] \ < \ 0$$

Divisibility vs. distinguishability

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Markovianity — new definition

Breuer, Lane and Piilo, PRL 2008

$$\sigma[\rho_1, \rho_2; t] := \frac{d}{dt} D[\rho_1(t), \rho_2(t)] = \text{flux of information}$$

Λ_t is Markovian iff $\sigma[\rho_1, \rho_2; t] \leq 0$

Divisibility vs. information flow

- Markonianity I $\longleftrightarrow$ semigroup $\Lambda_t = e^{tL}$
- Markonianity II $\longleftrightarrow$ divisibility
- Markovianity III $\longleftrightarrow$ $\sigma[\rho_1, \rho_2; t] \leq 0$

$$\text{Divisibility} \implies \sigma[\rho_1, \rho_2; t] \leq 0$$

$\sigma[\rho_1, \rho_2; t] \leq 0$ does not imply Divisibility

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Example: decoherence of qubit (Sabrina talk)

$$L_t\rho = \frac{1}{2}\gamma(t)\Big[\sigma_z\rho\sigma_z - \rho\Big]$$

$$\boxed{\mathcal{M}_1 \subset \mathcal{M}_2 = \mathcal{M}_3}$$

$$\gamma(t) \geq 0 \quad \Longleftrightarrow \quad \sigma[\rho_1, \rho_2; t] \leq 0$$

Special class of generators

$\mathcal{P}$ — CPTP projector

$$\mathcal{P}^2 = \mathcal{P}$$

$L := \gamma(\mathcal{P} - \mathbb{1})$; legitimate K-L generator

$$\boxed{\Lambda_t = e^{tL} = e^{-\gamma t}\mathbb{1} + (1 - e^{-\gamma t})\mathcal{P}}$$

$$t \longrightarrow \infty \quad \Longrightarrow \quad \rho_t \longrightarrow \mathcal{P}\rho$$

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Example 1

$\mathcal{P}$ – CPTP projector

$|1\rangle, |2\rangle \ \ldots$ – orthonormal basis in $\mathcal{H}$

$$P_k := |k\rangle\langle k|$$

$$\boxed{\mathcal{P}\rho = \sum_k P_k \, \rho P_k} \qquad \mathcal{P}^2 = \mathcal{P}$$

$$\rho_{ii}(t) = \rho_{ii} \ ; \qquad \rho_{ij}(t) = e^{-\gamma t}\rho_{ij}$$

pure decoherence of a qudit

Example 2

ω – density matrix ; $\gamma > 0$

$$\boxed{\mathcal{P}\rho = \omega \operatorname{Tr} \rho} \qquad \mathcal{P}^2 = \mathcal{P}$$

$L\rho = \gamma\Big(\omega \operatorname{Tr} \rho - \rho\Big)$; – K-L generator

$$\boxed{\rho \longrightarrow \rho_t = e^{-\gamma t} \rho + \Big(1 - e^{-\gamma t}\Big)\, \omega}$$

$t \longrightarrow \infty \quad \Longrightarrow \quad \rho_t \longrightarrow \omega$ (asymptotic state)

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ω – density matrix ; $\gamma > 0$

$$\gamma \quad \longrightarrow \quad \gamma(t) \geq 0$$

$\omega \quad \longrightarrow \quad \omega_t$ – time-dependent density matrix for $t \geq 0$

$$L_t\rho = \gamma(t)\Big(\omega_t\,\mathrm{Tr}\,\rho - \rho\Big)\ ; \quad \text{– K-L generator for } t \geq 0$$

Λ_t – divisible map

Example 2

$$L\rho = \gamma\Big(\omega\,\mathrm{Tr}\,\rho - \rho\Big)\ ; \quad \text{– K-L generator}$$

$$\omega \quad \text{– density matrix} \quad ; \quad \gamma > 0$$

$$\gamma \quad \longrightarrow \quad \gamma(t) \geq 0$$

$$\omega \quad \longrightarrow \quad \omega_t \quad \text{– time-dependent density matrix for} \quad t \geq 0$$

$$L_t\rho = \gamma(t)\Big(\omega_t\,\mathrm{Tr}\,\rho - \rho\Big)\ ; \quad \text{– K-L generator for} \quad t \geq 0$$

$$\Lambda_t \quad \text{– divisible map}$$

Example 2

$$L\rho = \gamma\Big(\omega \operatorname{Tr}\rho - \rho\Big) \;; \quad \text{– K-L generator}$$

ω – density matrix ; $\gamma > 0$

$\gamma \longrightarrow \gamma(t)$ (needs not be positive!!!)

$\omega \longrightarrow \omega_t$ – hermitian matrix ; $\operatorname{Tr}\omega_t = 1$

$$L_t\rho = \gamma(t)\Big(\omega_t \operatorname{Tr}\rho - \rho\Big)$$

Example 2

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Example 2

$$L\rho = \gamma\Big(\omega \,\mathrm{Tr}\, \rho - \rho\Big) \;; \quad \text{– K-L generator}$$

$$\boxed{\rho \;\longrightarrow\; \Lambda_t \rho = e^{-\gamma t}\, \rho + \Big(1 - e^{-\gamma t}\Big)\, \omega\, \mathrm{Tr} \rho}$$

$$L_t \rho := \gamma(t)\Big(\omega_t \,\mathrm{Tr}\, \rho - \rho\Big)$$

$$\Lambda_t \rho = e^{-\Gamma(t)} \rho + \Big(1 - e^{-\Gamma(t)}\Big)\, \Omega_t\, \mathrm{Tr} \rho$$

$$\Gamma(t) = \int_0^t \gamma(u) du \;; \quad \Omega_t = \frac{1}{e^{\Gamma(t)} - 1} \int_0^t \gamma(u) e^{\Gamma(u)}\, \omega_u \, du$$

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Example: Λ_t legitimate

$$\Lambda_t \rho = e^{-\Gamma(t)} \rho + \left(1 - e^{-\Gamma(t)}\right) \Omega_t \, \mathrm{Tr} \rho$$

$$\Gamma(t) = \int_0^t \gamma(u) du \; ; \quad \Omega_t = \frac{1}{e^{\Gamma(t)} - 1} \int_0^t \gamma(u) e^{\Gamma(u)} \, \omega_u \, du$$

Λ_t is CPTP iff

- $\Gamma(t) \geq 0$
- $\Omega_t \geq 0$, that is, Ω_t defines a density matrix

Example: Λ_t divisible

$$\Lambda_t\rho = e^{-\Gamma(t)}\rho + \left(1 - e^{-\Gamma(t)}\right)\Omega_t \,\mathrm{Tr}\rho$$

$$\Gamma(t) = \int_0^t \gamma(u)du \; ; \quad \Omega_t = \frac{1}{e^{\Gamma(t)} - 1}\int_0^t \gamma(u)e^{\Gamma(u)}\,\omega_u\, du$$

Λ_t is divisible iff

- $\gamma(t) \geq 0$
- $\omega_t \geq 0$, that is, ω_t defines a density matrix

Example: information flow

$$\Lambda_t \rho = e^{-\Gamma(t)} \rho + \left(1 - e^{-\Gamma(t)}\right) \Omega_t \, \mathrm{Tr} \rho$$

$$\Gamma(t) = \int_0^t \gamma(u) du \; ; \quad \Omega_t = \frac{1}{e^{\Gamma(t)} - 1} \int_0^t \gamma(u) e^{\Gamma(u)} \, \omega_u \, du$$

$$\Delta = \rho_1 - \rho_2 \; ; \quad \mathrm{Tr} \, \Delta = 0$$

$$\Lambda_t \Delta = e^{-\Gamma(t)} \Delta$$

Negative information flow is controlled ONLY by $\gamma(t) \geq 0$ and NOT by ω_t !!!

Divisible maps $\subset \; \sigma[\rho_1, \rho_2; t] \leq 0$

Divisibility and contractivity

$$\sigma(\rho_1, \rho_2; t) := \frac{d}{dt} || \Lambda_t \Delta ||_1 \; ; \qquad \Delta := \rho_1 - \rho_2$$

$$\boxed{\mathrm{Tr}\, \Delta = 0}$$

Δ traceless hermitian $\longrightarrow$ Δ hermitian

Let Λ_t be a dynamical map. Λ_t is divisible if and only if

$$\frac{d}{dt} || \Lambda_t \Delta ||_1 \leq 0$$

for all $\Delta^\dagger = \Delta \in \mathcal{B}(\mathcal{H})$

Divisibility implies monotonicity of several well known quantities.

- distinguishability,
- fidelity,
- relative entropy,
- entanglement measures, ...

Divisibility vs. fidelity

$$F(\rho,\sigma) = \left(\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right]\right)^2$$

$$F(\rho,\sigma) \le F(\rho(t),\sigma(t))$$

If Λ_t is a divisible map, then

$$\frac{d}{dt} F(\rho(t),\sigma(t)) \ge 0 \ .$$

Divisibility vs. entropy

$$S(\rho \,||\, \sigma) = \mathrm{Tr}\Big[\rho(\log \rho - \log \sigma)\Big]$$

$$S(\rho(t) \,||\, \sigma(t)) \leq S(\rho \,||\, \sigma)$$

If $\Lambda(t)$ is a divisible map, then

$$\frac{d}{dt} S(\rho(t) \,||\, \sigma(t)) \leq 0 \ .$$

Divisibility vs. entropy

The same works for relative Rényi and Tsallis entropies

$$S_\alpha(\rho\,||\,\sigma) = \frac{1}{\alpha - 1}\log\left[\mathrm{Tr}\,\rho^\alpha\sigma^{1-\alpha}\right] \;; \qquad \alpha \in [0,1)\cup(1,\infty)$$

$$T_q(\rho\,||\,\sigma) = \frac{1}{1-q}\left[1 - \mathrm{Tr}\,\rho^q\sigma^{1-q}\right] \;; \qquad q \in [0,1)$$

$$\lim_{\alpha\to 1} S_\alpha(\rho\,||\,\sigma) = \lim_{q\to 1} T_q(\rho\,||\,\sigma) = S(\rho\,||\,\sigma)$$

Divisibility vs. entanglement

W – an arbitrary density matrix in $\mathcal{H} \otimes \mathcal{H}'$

$$W_t = (\Lambda_t \otimes \mathbb{1})W$$

If $\mathcal{E}$ is an entanglement measure then

$$\mathcal{E}[(\Phi \otimes \Phi')W] \leq \mathcal{E}[W] \ ,$$

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