Kink fluctuation asymptotics and zero modes

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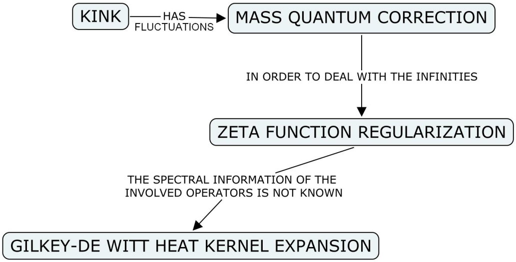
Benasque 2012

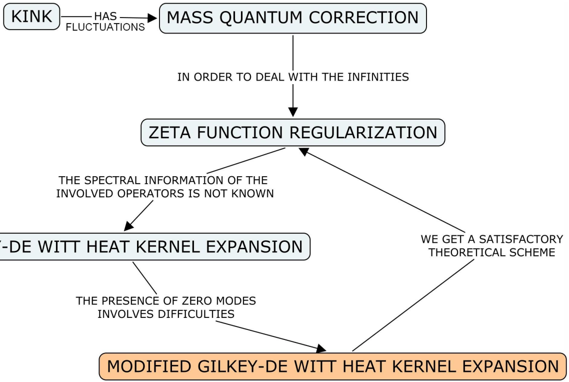
THE AIM OF THE TALK

1. KINK
2. MASS QUANTUM CORRECTION
3. ZETA FUNCTION REGULARIZATION
4. GILKEY-DE WITT HEAT KERNEL EXPANSION
5. MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

The aim of the talk



THE AIM OF THE TALK



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KINK

CLASSICAL PHYSICS ($\hbar = 0$)

- (1+1)-D Scalar field Model:

$$S = \int dx^2 \left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)\right]$$

- Euler-Lagrange Equations

$$\partial_\mu\partial^\mu\phi = -\frac{\partial U}{\partial\phi}$$

SOLUTION: ϕ_S

- Classical Mass:

$$E_{\rm cl} = \int_{-\infty}^{\infty} dx \left[\tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + U(\phi)\right]$$

- Solution stability:

$$K\psi_n = \omega_n^2\psi_n$$

$$K = -\frac{d^2}{dx^2} + \frac{\partial^2 U}{\partial\phi^2}[\phi_S]$$

$\omega_n > 0 \Rightarrow$ Stable solution

A REFERENCE EXAMPLE: MODEL ϕ^4.

KINK

- Action Functional:

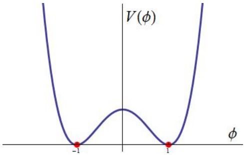
$$S = \int dxdt \left[\frac{1}{2} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial t} - \frac{1}{2} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} - U(\phi) \right]$$

- Potential:

$$U = \frac{1}{2}(\phi^2 - 1)^2$$

- Partial Diferential Equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + 2\phi(\phi^2 - 1) = 0$$



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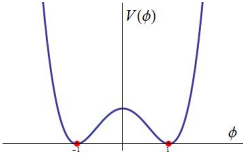
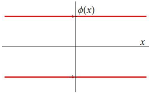
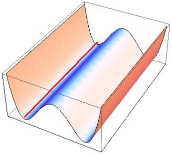


SOLUTIONS:

VACUUM SOLUTION:

$$\phi_V = \pm 1$$





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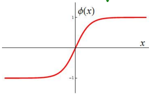

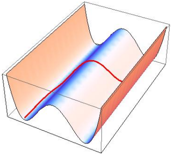

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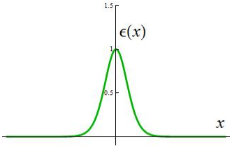
- KINK ENERGY DENSITY:

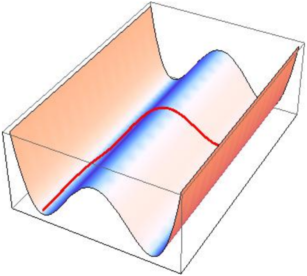
$$\varepsilon(x) = \operatorname{sech}^4 \frac{x + x_0 - vt}{\sqrt{1 - v^2}}$$

- KINK CLASSICAL MASS:

$$E_{\rm cl} = \int_{-\infty}^{\infty} dx \, \operatorname{sech}^4(x + x_0) = \frac{4}{3}$$



CLASSICAL KINK



Cartoon

"What do you mean, 'a quantum fluctuation?' Didn't we discuss cause and effect?"

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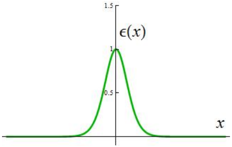
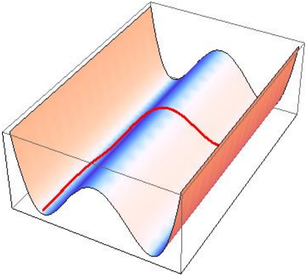


QUANTUM KINK



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MASS QUANTUM CORRECTION

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QUANTUM PHYSICS
$(\hbar \ll 1)$

Quantification of the classical system

SERIES EXPANSIÓN OF THE MASS

$$E_{\rm Q}(\hbar) \approx E_{\rm cl} + \hbar\Delta E$$

SEMICLASSICAL APPROXIMATION:

$$\Delta E = \frac{1}{2}\hbar\sum_r \omega_r$$

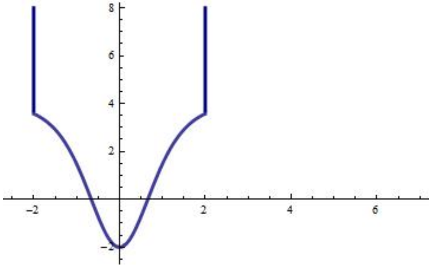
SEMICLASSICAL MASS

$$\Delta E = \frac{1}{2}\hbar\,{\rm tr}(K^{\frac{1}{2}})$$

Quantum correction to the mass:

$$\Delta E = \frac{1}{2}\hbar\,\mathrm{tr}(K^{\frac{1}{2}})$$

- Hessian Operator: $K = -\frac{d^2}{dx^2} + \frac{\partial^2 U}{\partial \phi^2}[\phi_S]$



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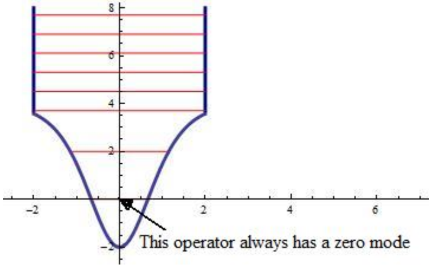


WARNING MESSAGE!!!:

The response will be ∞

SOLUTION:

We need a reference point

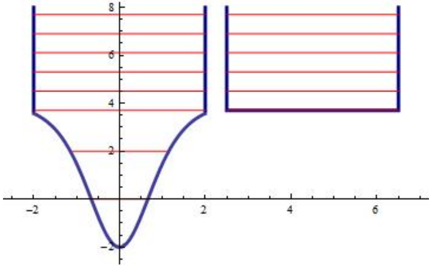
Zero-Point Renormalization

We have to measure the quantum correction with respect to the quantum correction of the vacuum solution ϕ_V.

Quantum correction to the mass:

$$\Delta E = \frac{1}{2}\hbar\, \mathrm{tr}(K^{\frac{1}{2}}) - \frac{1}{2}\hbar\, \mathrm{tr}({K_0}^{\frac{1}{2}})$$

- Hessian Operator: $K = -\frac{d^2}{dx^2} + \frac{\partial^2 U}{\partial \phi^2}[\phi_S]$ $\quad K_0 = -\frac{d^2}{dx^2} + \frac{\partial^2 U}{\partial \phi^2}[\phi_V]$



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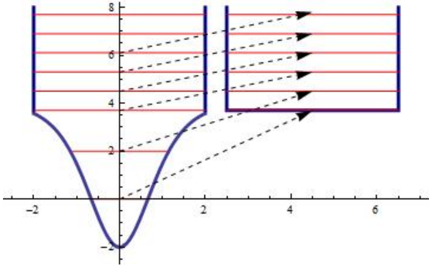


WARNING MESSAGE!!!:

The response is $\infty - \infty$

SOLUTION:

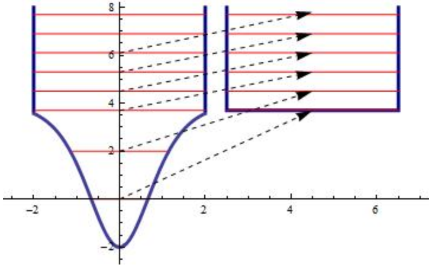
We choose a prescription

Mode Number
Cut-off Regularization

Quantum correction to the mass:

$$\Delta E = \frac{1}{2}\hbar\, \mathrm{tr}\left(K^{\frac{1}{2}} - {K_0}^{\frac{1}{2}}\right)$$

- Hessian Operator: $K = -\frac{d^2}{dx^2} + \frac{\partial^2 U}{\partial \phi^2}[\phi_S]$ $\quad K_0 = -\frac{d^2}{dx^2} + \frac{\partial^2 U}{\partial \phi^2}[\phi_V]$



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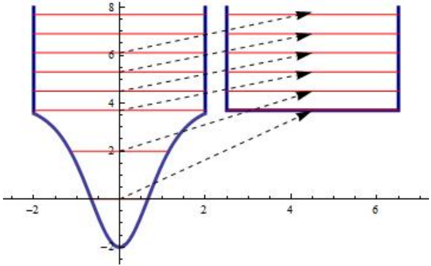

WARNING MESSAGE!!!:

The response is still ∞

SOLUTION:

The mass coupling constant

is infinite

Mass Renormalization

We have to introduce the counterterms (well stablished procedure in Physics)

Quantum correction to the mass:

$$\Delta E = \Delta E_1 + \Delta E_2$$

- Kink Casimir Energy:

$$\Delta E_1 = \frac{1}{2}\hbar\, \mathrm{tr}(K^{\frac{1}{2}} - K_0{}^{\frac{1}{2}})|_{M.C.}$$

- Counterterms:

$$\Delta E_2 = \hbar\, \langle V(x)\rangle \int \frac{dk}{4\pi}\frac{1}{\sqrt{k^2+v^2}}, \qquad \langle V(x)\rangle = \int_{-\infty}^{\infty} dx V(x)$$

- Kink fluctuation operator:

$$K_0 = -\frac{d^2}{dx^2} + v^2 \qquad \text{where} \qquad v^2 = \frac{\partial^2 U}{\partial \phi^2}[\phi_V]$$

- Vacuum fluctuation operator:

$$K = -\frac{d^2}{dx^2} + v^2 + V(x) \qquad \text{where} \qquad V(x) = \frac{\partial^2 U}{\partial \phi^2}[\phi_S] - \frac{\partial^2 U}{\partial \phi^2}[\phi_V]$$

MASS QUANTUM CORRECTION

MODEL ϕ^4.

- Action Functional:

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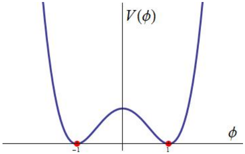

SOLUTIONS:

VACUUM SOLUTION:

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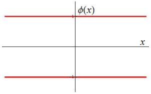

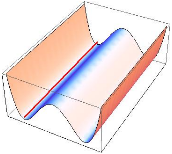

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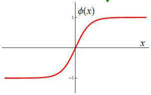

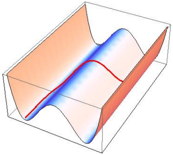

Model ϕ^4.

MASS QUANTUM CORRECTION

Homogeneous Solution

$$\phi_V = \pm 1$$

- Hessian Operator:

$$K_0 = -\frac{d^2}{dx^2} + 4$$

$$\mathrm{Spec}^d K_0 = \{4\}_{\frac{1}{2}}$$

$$\mathrm{Spec}^c K_0 = \{k^2 + 4\}_{k\in\mathbb{R}}$$

- Spectral Density:

$$\rho_0(k) = \frac{L}{2\pi}$$

Kink Solution:

$$\phi_K(x) = \pm \tanh x$$

- Hessian Operator:

$$K = -\frac{d^2}{dx^2} + 4 - 6\,\mathrm{sech}^2 x$$

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$$\Delta E = \frac{\hbar m}{2}\left(\sqrt{3} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\,\sqrt{q^2+4}\,\frac{d\delta(q)}{dq}\right) + \Delta E_2 =$$

$$= \frac{\sqrt{3}\hbar m}{2} - \frac{\hbar m}{2\pi}\int_{-\infty}^{\infty} dq\,\frac{3\sqrt{q^2+4}\,(q^2+2)}{q^4+5q^2+4} + \frac{3\hbar m}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{k^2+4}} + \frac{\hbar m}{4\pi}\langle \mathcal{V}(x) - 4\rangle =$$

$$= \frac{\sqrt{3}\,\hbar m}{2} - \frac{\hbar m}{\sqrt{3}} + \frac{\hbar m}{4\pi}\int_{-\infty}^{\infty} dx(-6\,\mathrm{sech}^2 x) = \hbar m\left(\frac{1}{2\sqrt{3}} - \frac{3}{\pi}\right) \approx \boxed{-0.666255\,\hbar m}$$

MODEL ϕ^4.

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$$\Delta E = \frac{\hbar m}{2}\left(\sqrt{3} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, \sqrt{q^2+4}\,\frac{d\delta(q)}{dq}\right) + \Delta E_2 =$$

$$= \frac{\sqrt{3}\hbar m}{2} - \frac{\hbar m}{2\pi}\int_{-\infty}^{\infty} dq \frac{3\sqrt{q^2+4}(q^2+2)}{q^4+5q^2+4} + \frac{3\hbar m}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{k^2+4}} + \frac{\hbar m}{4\pi}\langle \mathcal{V}(x) - 4\rangle =$$

$$= \frac{\sqrt{3}\,\hbar m}{2} - \frac{\hbar m}{\sqrt{3}} + \frac{\hbar m}{4\pi}\int_{-\infty}^{\infty} dx(-6\mathrm{sech}^2 x) = \hbar m\left(\frac{1}{2\sqrt{3}} - \frac{3}{\pi}\right) \approx \boxed{-0.666255\hbar m}$$

• R. Dashen, B. Hasslacher, A. Neveu, Phys. Rev. D 10 (1974) 4130

Goal:

Invent a procedure to approximately compute the semiclassical mass of any kink in any $(1+1)$-dimensional scalar field theory.

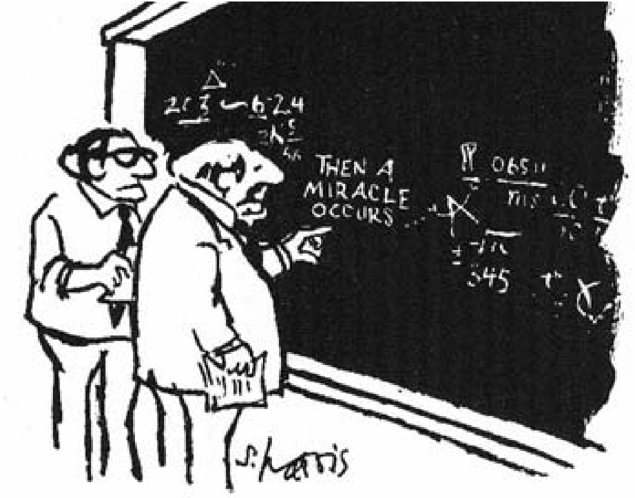
Clue:

We will need to estimate the trace of differential operators without knowing the spectra, even asymptotically.

No country for numerical analysis:

The difference between the traces of two operators demands to compute the sum of infinite infinitesimal values.

Cartoon



"I think you should be more explicit here in step two."

ZETA FUNCTION REGULARIZATION

Quantum Correction

$$\Delta E = \frac{1}{2}\hbar\left[\mathrm{tr}\left(K^{\frac{1}{2}} - {K_0}^{\frac{1}{2}}\right)\right] - \frac{\hbar}{8\pi}\langle V(x)\rangle \int_{-\infty}^{\infty}\frac{dk}{\sqrt{k^2+v^2}}$$

Generalized Zeta Function:

$$\zeta_K(s) = \mathrm{Tr}\,K^{-s} = \sum_{n=0}^{\infty}(\omega_n^2)^{-s}$$

Quantum Correction

$$\Delta E = \lim_{s\to -\frac{1}{2}} \frac{\hbar}{2}(\mu^2)^{s+\frac{1}{2}}\left[\ (\zeta_K(s) - \zeta_{K_0}(s)) \ + \lim_{l\to\infty}\frac{1}{l}\frac{\Gamma[s+1]}{\Gamma[s]}\zeta_{K_0}(s+1)\right]$$

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Heat Function

$$h_K(\beta) = \mathrm{Tr}\, e^{-\beta K} = \sum_{n=0}^{\infty} e^{-\beta\omega_n^2}$$

↘ ↙

Mellin Transform

$$\zeta_K(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} d\beta\, \beta^{s-1}\, h_K(\beta)$$

Quantum Correction

$$\Delta E = \lim_{s\to -\frac{1}{2}} \frac{\hbar}{2}(\mu^2)^{s+\frac{1}{2}} \left[\left(\zeta_K(s) - \zeta_{K_0}(s)\right) + \lim_{l\to\infty} \frac{1}{l} \frac{\Gamma[s+1]}{\Gamma[s]} \zeta_{K_0}(s+1) \right]$$

$$\frac{1}{\Gamma(s)} \int_0^{\infty} d\beta\, \beta^{s-1} \left(h_K(\beta) - h_{K_0}(\beta)\right)$$

GILKEY-DE WITT HEAT KERNEL EXPANSION

- Heat function: $h_K(\beta) = \mathrm{Tr}_{\mathbb{L}^2}(e^{-\beta K}) = \int_\Omega dx\, \underbrace{K_K(x,x,\beta)}$

$$K_K(x,y;\beta) = \psi_0^*(y)\psi_0(x) + \sum_n \psi_n(y)^*\psi_n(x)e^{-\beta\omega_n^2} + \int dk\psi_k^*(y)\psi_k(x)e^{-\beta\omega^2(k)}$$

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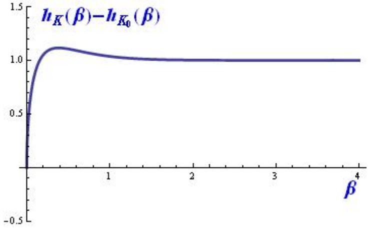
OUR REFERENCE EXAMPLE: MODEL ϕ^4.

$$h_K(\beta) - h_{K_0}(\beta) = e^{-3\beta}\mathrm{erf}\sqrt{\beta} - \mathrm{erf}\,2\sqrt{\beta}$$



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– We can calculate exactly the heat function.

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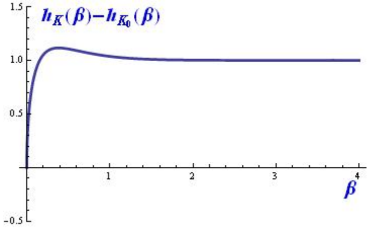

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Plugging the Series Expansion into the PDE:

$$K_K(x,y;\beta) = K_{K_0}(x,y;\beta)A(x,y;\beta)$$

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starting with $a_0(x,y) = 1$

GILKEY-DE WITT HEAT KERNEL EXPANSION

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$$\widetilde{h}_K(\beta) - \widetilde{h}_{K_0}(\beta) = \frac{e^{-\beta v^2}}{2\sqrt{\pi\beta}} \sum_{n==1}^{N} a_n \, \beta^n$$

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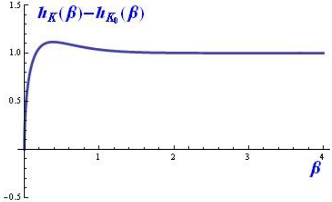
OUR REFERENCE EXAMPLE: MODEL ϕ^4:

EXACT HEAT FUNCTION

$$h_K(\beta) - h_{K_0}(\beta) =$$
$$+e^{-3\beta}\text{erf}\sqrt{\beta} - \text{erf}\,2\sqrt{\beta}$$



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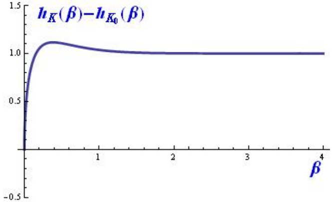

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Gilkey-De Witt Expansion

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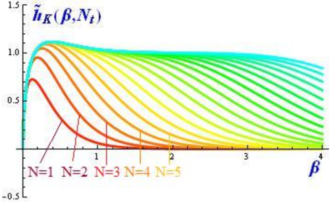

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Conclusion: The Gilkey-De Witt keat kernel expansion does not work properly with zero modes

GILKEY-DE WITT HEAT KERNEL EXPANSION

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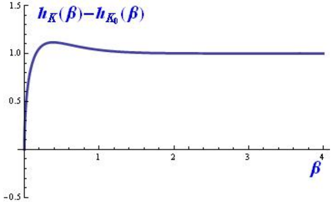

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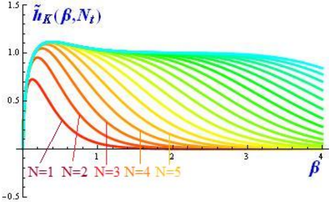

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with $c_0(x,y) = 1$

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$$^{(k)}C_n(x) = \frac{1}{n+k}\left[\, ^{(k+2)}C_{n-1}(x) - \sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j V}{\partial x^j}\, ^{(k-j)}C_{n-1}(x)\right] - -2v f_0(x)\frac{df_0^k}{dx^k}\delta_{0,n-1} - f_0(x)\frac{df_0^k}{dx^k}\frac{2^n v^{2n-1}}{(2n-1)!!}(1+2k)$$

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

- Recurrence Relations:

$$(n+1)\, c_{n+1}(x,y) + (x-y)\frac{\partial c_{n+1}(x,y)}{\partial x} + V(x) c_n(x,y) - \frac{\partial^2 c_n(x,y)}{\partial x^2} + 2v f_0^*(y) f(x)\delta_{0n} + f_0^*(y) f(x)\frac{2^{n+1} v^{2n+1}}{(2n+1)!!} + (x-y) f_0^*(y)\frac{df_0(x)}{dx}\frac{2^{n+2} v^{2n+1}}{(2n+1)!!} = 0$$

$y \longrightarrow x$ (delicate limit)

$${}^{(k)}C_n(x) = \lim_{y\to x}\frac{\partial^k c_n(x,y)}{\partial x^k}$$

$${}^{(0)}C_n(x) = c_n(x,x) \qquad {}^{(k)}C_0(x) = \delta^{k0}$$

$${}^{(k)}C_n(x) = \frac{1}{n+k}\left[{}^{(k+2)}C_{n-1}(x) - \sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j V}{\partial x^j}\, {}^{(k-j)}C_{n-1}(x)\right] - 2v f_0(x)\frac{df_0^k}{dx^k}\delta_{0,n-1} - f_0(x)\frac{df_0^k}{dx^k}\frac{2^n v^{2n-1}}{(2n-1)!!}(1+2k)$$

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MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

- **OUTLINE:**

$$\overline{h}_K(\beta) - \overline{h}_{K_0}(\beta) = \frac{e^{-\beta v^2}}{2\sqrt{\pi\beta}} \sum_{n==1}^{N} c_n \ \beta^n$$

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$\downarrow$

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$\downarrow$

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$$ -2v f_0(x) \frac{d f_0^k}{dx^k} \delta_{0,n-1} - f_0(x) \frac{d f_0^k}{dx^k} \frac{2^n v^{2n-1}}{(2n-1)!!}(1+2k) $$

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

STANDARD GILKEY-DE WITT HEAT KERNEL EXPANSION

$$a_0(x,x) = {}^{(0)}A_0(x) = 1$$

$$a_1(x,x) = {}^{(0)}A_1(x) = -V(x)$$

$$a_2(x,x) = {}^{(0)}A_2(x) = -\frac{1}{6}\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\left(V(x)\right)^2$$

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

$$c_0(x,x) = {}^{(0)}C_0(x) = 1$$

$$c_1(x,x) = {}^{(0)}C_1(x) = -V(x) - 4vf_0^2(x)$$

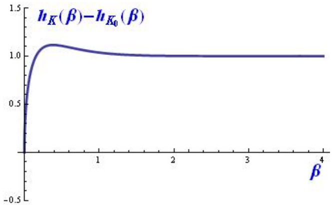
$$c_2(x,x) = {}^{(0)}C_2(x) = -\frac{1}{6}\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\left(V(x)\right)^2 + \frac{4}{3}v^3 f_0^2(x) + 4vf_0^2(x)V(x)$$

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

OUR REFERENCE EXAMPLE: MODEL ϕ^4

EXACT HEAT FUNCTION

$$h_K(\beta) - h_{K_0}(\beta) = \frac{e^{-4\beta}}{8\pi\beta} +$$
$$+e^{-3\beta}\,\mathrm{erf}\sqrt{\beta} - \mathrm{erf}\,2\sqrt{\beta}$$



$$\lim_{\beta\to 0}[h_K(\beta) - h_{K_0}(\beta)] = 0$$
$$\lim_{\beta\to \infty}[h_K(\beta) - h_{K_0}(\beta)] = 1$$

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

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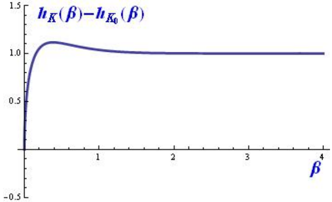

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MODIFIED G-DW EXPANSION

$$\overline{h}_K(\beta) - \overline{h}_{K_0}(\beta) = \frac{e^{-4\beta}}{4\pi\beta} \sum_{n=1}^{N} c_n \, \beta^n$$

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

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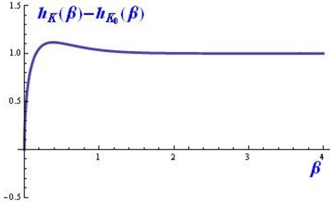

$$\lim_{\beta\to 0} [h_K(\beta) - h_{K_0}(\beta)] = 0$$
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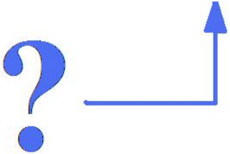

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

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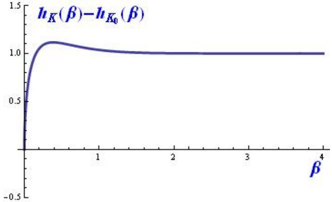

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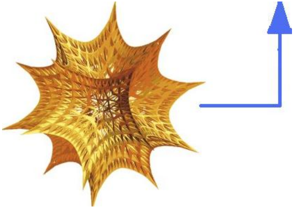

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

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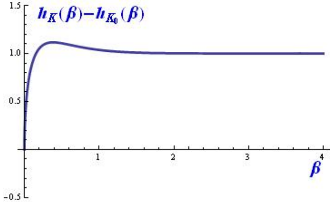

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MODIFIED G-DW EXPANSION

$$\bar{h}_K(\beta)-\bar{h}_{K_0}(\beta)=\frac{e^{-4\beta}}{4\pi\beta}\sum_{n=1}^{N}\boxed{c_n}\,\beta^n$$

Kink Seeley Coefficients

n	$c_n(K)$
1	4.00000
2	2.66667
3	1.06667
4	0.304762
5	0.0677249
6	0.0123136
7	0.0018944
8	0.000252587
9	0.0000297161
10	$3.12801\cdot 10^{-6}$

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

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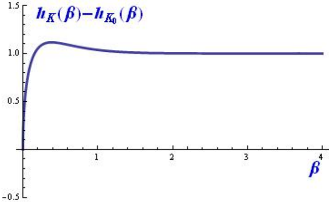

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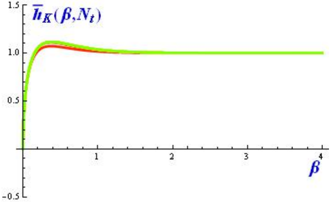

$$\lim_{\beta\to 0}[\widetilde{h}_K(\beta)-\widetilde{h}_{K_0}(\beta)]=0$$
$$\lim_{\beta\to\infty}[\widetilde{h}_K(\beta)-\widetilde{h}_{K_0}(\beta)]=1$$

Conclusion: The modified Gilkey-De Witt keat kernel expansion does work very properly with zero modes

MODIFIED GILKEY-DE WITT HEAT KERNEL EXPANSION

Kink Mass Quantum Correction

$$\frac{\Delta E[\phi_k]}{\hbar} = -\frac{v}{\pi} - \frac{1}{8\pi}\sum_{n=2}^{N} c_n(K)(v^2)^{1-n}\Gamma[n-1]$$

Exact quantum correction to the kink mass

$\Delta E = -0.666255\hbar m$

Estimated quantum correction to the kink mass

`quantumcorrection[1/2(y^2-1)^2,-1,1,10]` $= -0.666255\hbar m$

Virtue of the Approach

Applicable to every model

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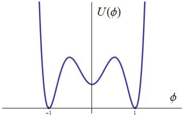
Applicable to every model

Model ϕ^6.

MASS QUANTUM CORRECTION

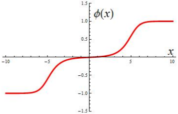
- Potential:

$$U = \frac{1}{2}(\phi^2 + a^2)(\phi^2 - 1)^2, \quad a = \frac{1}{2}$$



- Kink Solution:

$$\phi_K(x) = \frac{a(-1+e^{2\sqrt{1+a^2}x})}{\sqrt{4e^{2\sqrt{1+a^2}x}+a^2(1+e^{2\sqrt{1+a^2}x})^2}}$$



- Second Order Small Fluctuation Operator:

$$K = -\frac{d^2}{dx^2} + 4(1+a^2) + \frac{15(4a+1)^2}{[2a\cosh(2\sqrt{a^2+1}x) + 2a + 1]^2} - \frac{6(a^2+3)(4a+1)}{2a\cosh(2\sqrt{a^2+1}x) + 2a + 1}$$

UNKNOWN SPECTRAL INFORMATION !!!!

Estimated quantum correction to the kink mass

`quantumcorrection[1/2(y^2-1)^2,-1,1,8]` $= -1.0748\hbar m$

MODEL ϕ^6.

MASS QUANTUM CORRECTION

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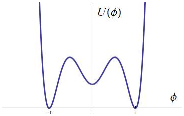
$$U = \frac{1}{2}(\phi^2 + a^2)(\phi^2 - 1)^2, \quad a = \frac{1}{2}$$



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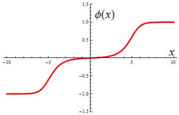

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End of the talk

Conclusions

1. The Gilkey-de Witt heat kernel expansion has been adapted to operators which involve zero modes.

2. This offers a tool to compute the kink mass quantum correction with a high precision.

3. The computation associated with this scheme can be automatized by means of a Mathematica program.