Spontaneous Symmetry Breaking in curved space: renormalization and vacuum stress-tensor

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Contents:

- General motivations
- SSB in curved space at classical level
- Induced action of gravity and nonlocal terms
- Renormalization in SSB theories.
- Decoupling theorem with SSB.
- Derivation of $\langle T_{\mu\nu} \rangle$.

General motivations

The Spontaneous Symmetry Breaking is very important aspect of our understanding of vacuum.

Therefore, it is important to understand how it works in curved space-time.

The two most important applications are the induced gravity and cosmological constant problem from one side

and the decoupling (Appelquist & Carazzone) theorem from another one.

Scalar field

The minimal action for a real scalar field is

$$S_0 = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu\varphi \, \partial_\nu\varphi - V_{min}(\varphi) \right\} ,$$

where $\quad V_{min}(\varphi) = -\frac{1}{2} m^2\varphi^2 - \frac{\lambda}{4!} \varphi^4$

is a minimal potential term.

The possible nonminimal structure is

$$S_{non-min} = \frac{1}{2} \int d^4x \sqrt{-g} \, \xi \, \varphi^2 R \,.$$

The new quantity ξ is called nonminimal parameter.

Since the non-minimal term does not have derivatives of the scalar field, it should be included into the potential term, and thus we arrive at the new definition of the classical potential.

$$V(\varphi) = -\frac{1}{2} \left(m^2 + \xi R \right) \varphi^2 + \frac{f}{4!} \varphi^4 \,.$$

In case of the multi-scalar theory the nonminimal term is

$$\int d^4x\sqrt{-g}\,\xi_{ij}\;\varphi^i\varphi^j\,R\,.$$

Further non-minimal structures involving scalar are indeed possible, for example

$$\int R^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\,.$$

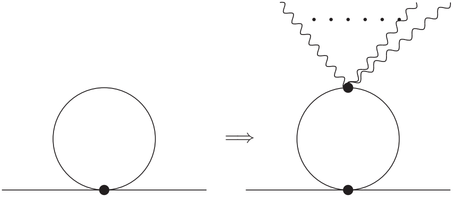
However, these structures include constants of inverse mass dimension, therefore do not fit the principles declared above.

In fact, these terms are not necessary for the construction of consistent quantum theory.

Types of the counterterms:

- **Minimal, e.g.,** $m^2\varphi^2$, $(\nabla\varphi)^2$, $i\bar{\psi}\gamma^\mu\nabla_\mu\psi$.
- **Non-minimal in the scalar sector,** $R\varphi^2$.

E.g., the quadratically divergent diagram



in the $\lambda\varphi^4$ **theory produces log. divergences corresponding to** $\int d^4\sqrt{-g}R\varphi^2$ **counterterm.**

- **Vacuum terms** Λ, R, R^2, C^2, **etc.**

Renormalization doesn't depend on the choice of the metric!

Along with the non-minimal term, covariance and locality admit some terms which involve only metric. These terms are conventionally called "vacuum action" and their general form is the following

$$S_{vac} = S_{EH} + S_{HD}$$

where $$S_{EH} = \frac{1}{16\pi G} \int d^4x\sqrt{-g}\,\{R + 2\Lambda\}\,.$$

is the Einstein-Hilbert action with the CC

S_{HD} **includes higher derivative terms. The most useful form is**

$$S_{HD} = \int d^4x\sqrt{-g}\,\big\{a_1C^2 + a_2E + a_3\Box R + a_4R^2\big\}\,,$$

where $$C^2(4) = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + 1/3\,R^2$$

is the square of the Weyl tensor in $n = 4$,

$$E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2$$

is the integrand of the n=4 **Gauss-Bonnet topological invariant.**

SSB in curved space at classical level

Classical action of scalar φ coupled to the Abelian gauge vector

$$S = \int d^4x\sqrt{-g}\,\Big\{ -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + g^{\mu\nu}\,(\partial_\mu - ieA_\mu)\varphi^*\,(\partial_\nu + ieA_\mu)\varphi +$$

$$+\,\mu_0^2\,\varphi^*\varphi - \lambda(\varphi^*\varphi)^2 + \xi\,R\,\varphi^*\varphi\Big\}\,.$$

No much reason to consider non-Abelian theory, since for the one-loop vacuum effects and the results are indeed the same.

The VEV for the scalar field is defined as

$$-\Box v + \mu_0^2\,v + \xi R\,v - 2\lambda v^3 = 0\,. \tag{1}$$

If the interaction is minimal $\xi = 0$, the SSB is simple, because the vacuum solution of the last equation is constant

$$v_0^2 = \frac{\mu_0^2}{2\,\lambda}\,. \tag{2}$$

However, in the general case one can not neglect $\Box$ in Eq. (1).

Let us consider

$$v_0^2 \,=\, \frac{\mu_0^2}{2\,\lambda}\,. \tag{3}$$

as zero-order approximation and find the solution of the Eq. (2) in the form of the power series in ξ

$$v(x) \,=\, v_0 \,+\, v_1(x) \,+\, v_2(x) \,+\, \dots .$$

For the first order term $v_1(x)$ we have

$$-\Box v_1 \,+\, \mu^2 v_1 \,+\, \xi R\, v_0 \,-\, 6\lambda v_0^2\, v_1 \,=\, 0\,,$$

and the solution has the form

$$v_1 \,=\, \frac{\xi\, v_0}{\Box - \mu^2 + 6\lambda v_0^2}\, R \,=\, \frac{\xi\, v_0}{\Box \,+\, 4\lambda v_0^2}\, R\,,$$

In a similar way, we find

$$v_2 \,=\, \frac{\xi^2\, v_0}{\Box + 4\lambda v_0^2}\, R\, \frac{1}{\Box + 4\lambda v_0^2}\, R \,-\, \frac{6\,\lambda\,\xi^2\, v_0^3}{\Box + 4\lambda v_0^2}\, \Big(\frac{1}{\Box + 4\,\lambda v_0^2}\, R \Big)^2,$$

where the operator in each parenthesis acts only on the curvature inside this parenthesis.

Induced gravity action

Of course, one can continue the expansion of v to any desirable order.

If we replace the SSB solution $v(x)$ back into the scalar action, we arrive at the following induced low-energy action of vacuum:

$$S_{ind} = \int d^4x\sqrt{-g}\,\Big\{ g^{\mu\nu}\,\partial_\mu v\,\partial_\nu v + (\mu_0^2+\xi R)\,v^2 - \lambda\,v^4 \Big\}. \qquad (3)$$

Making an expansion in the powers of the curvature tensor, in the second order we obtain

$$S_{ind} = \int d^4x\sqrt{-g}\,\Big\{ -\, v_1\Box v_1 + \mu^2\,(v_0^2+2v_0v_1+2v_0v_2+v_1^2)$$

$$-\,\lambda\,(v_0^4+4v_0^3v_1+4v_0^3v_2+6v_0^2v_1^2) + \xi R\,(v_0^2+2v_0v_1)\Big\} + \mathcal{O}(R^3)\,.$$

and, finally, to

$$S_{ind} = \int d^4x\sqrt{-g}\,\Big\{ \lambda v_0^4 + \xi R v_0^2 + \xi^2\,v_0^2\,R\,\frac{1}{\Box+4\,\lambda v_0^2}\,R + \,...\Big\}.$$

$$S_{ind} = \int d^4x\sqrt{-g}\,\Big\{\,\lambda v_0^4 + \xi R v_0^2 + \xi^2\, v_0^2\, R\,\frac{1}{\Box + 4\,\lambda v_0^2}\,R + ...\Big\}\,.$$

The first term here is the induced cosmological constant, which is supposed to almost cancel with its vacuum counterpart.

The second term is an induced Einstein-Hilbert action, which also has to be summed up with the corresponding vacuum term.

The observable cosmological constant is extremely small compared to the magnitude of λv_0^4 in the SM of particle physics. We need a precise cancelation between the vacuum and induced cosmological constants (Cosmological Constant Problem).

At the same time, for the Einstein-Hilbert term the vacuum coefficient is the inverse Newton constant $1/16\pi G = M_P^2/16\pi$, where $M_P \approx 10^{19}\,GeV$ is a Planck mass.

Obviously, the induced contribution becomes relevant only at the GUT scale.

$$S_{ind} \,=\, \int d^4x\sqrt{-g}\,\Big\{\,\lambda v_0^4 \,+\, \xi R v_0^2 \,+\, \xi^2\, v_0^2\, R\,\frac{1}{\Box + 4\,\lambda v_0^2}\,R \,+\, ...\Big\}\,.$$

The nonlocal term in the induced action of gravity was first noticed in

Ed. Gorbar and I.Sh. JHEP (2004); hep-ph/0311190.

Of course, this term becomes relevant only for a very small mass field, for otherwise

$$R\,\frac{1}{\Box + 4\,\lambda v_0^2}\,R \,=\, R^2 \,-\, R\left(\frac{\Box}{4\,\lambda v_0^2}\right)R \,+\, \dots\,.$$

But, even if the mass is large, it is important to understand how the renormalization in such a theory is performed because, due to the uncontrolled divergences even for the weak curvature values the higher derivative terms can pose a huge problem.

Let us start from the simplest $\xi = 0$ minimal interaction case.

The effective action $\Gamma[\varphi, g_{\mu\nu}]$ of the scalar field can be presented as the perturbative expansion

$$\Gamma[\varphi, g_{\mu\nu}] = S_{cl}[\varphi, g_{\mu\nu}] + \hbar\,\bar{\Gamma}^{(1)}[\varphi, g_{\mu\nu}] + \mathcal{O}(\hbar^2)\,.$$

At one-loop order we consider only the $\bar{\Gamma}^{(1)}[\varphi, g_{\mu\nu}]$ term. Then the effective equation for the VEV is

$$\frac{\delta S_{cl}}{\delta\varphi} + \hbar\,\frac{\delta\bar{\Gamma}^{(1)}}{\delta\varphi} = 0\,.$$

This equation can be rewritten as

$$-\Box\varphi + \mu^2\varphi - 2\lambda(\varphi^*\varphi)\varphi + \hbar\,\frac{\delta\bar{\Gamma}^{(1)}}{\delta\varphi} = 0\,.$$

Let us present the scalar field as $\varphi = v + \hbar\phi$, where v is the solution of the classical equation $\delta S_{cl}/\delta\varphi = 0$ and $\hbar\phi$ is a quantum correction. Then we find, in the first order in $\hbar$, the following relation:

$$-\Box\phi + \mu^2\phi - 6\lambda v^2\phi + \hbar\,\frac{\delta\bar{\Gamma}^{(1)}[v,g]}{\delta v} = 0\,.$$

The expansion in $\hbar$ yields

$$\Gamma[\varphi, g_{\mu\nu}] = S_{cl}[v + \hbar\phi,\, g_{\mu\nu}] + \hbar\,\bar{\Gamma}^{(1)}[v + \phi,\, g_{\mu\nu}] + ...$$

$$= S_{cl}[v,\, g_{\mu\nu}] + \hbar\phi\,\frac{\delta S_{cl}[v,g]}{\delta v} + \hbar\,\bar{\Gamma}^{(1)}[v,\, g_{\mu\nu}] + \mathcal{O}(\hbar^2)\,.$$

Taking into account the equation of motion $\delta S_{cl}(v,g)/\delta v = 0$, we arrive at the useful formula

$$\Gamma[v + \hbar\phi,\, g_{\mu\nu}] = S_{cl}[v,\, g_{\mu\nu}] + \hbar\,\bar{\Gamma}^{(1)}[v,\, g_{\mu\nu}] + \mathcal{O}(\hbar^2)\,.$$

The last relation holds even for the non-minimal scalar field. It shows that at the one-loop level one can derive the effective action as a functional of the classical VEV.

Consider the SSB in the Abelian theory with $\xi = 0$.

Let us define $\varphi = v + h + i\eta$. **At one-loop we can keep the terms of the second order in the quantum fields and disregard higher order terms. In this way we arrive at the expression**

$$S^{(2)} = \int d^4x\sqrt{-g}\Big\{(\partial_\mu h)^2+(\partial_\mu\eta)^2-\frac{1}{4}F_{\mu\nu}^2+2evA_\mu\nabla^\mu\eta+e^2v^2A_\mu A^\mu-4\lambda v^2h^2\Big\}$$

where $(\partial h)^2 = g^{\mu\nu}\partial_\mu h\partial_\nu h$. **Introduce the 'tHooft gauge fixing condition, depending on an arbitrary parameter** α

$$S_{GF} = -\frac{1}{2\alpha}\int d^4x\sqrt{-g}\,(\nabla_\mu A^\mu - 2\,\alpha\,ev\,\eta)^2 .$$

The expression for the action with gauge fixing term is

$$S^{(2)} + S_{GF} = \int d^4x\sqrt{-g}\Big\{ -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + e^2v^2A_\mu A^\mu$$

$$+ (\partial_\mu h)^2 + (\partial_\mu\eta)^2 - 4\lambda v^2h^2 - 2\,\alpha e^2v^2\eta^2\Big\} + \dots,$$

where we kept second order in the quantum fields A^μ, h, η.

The action of the Faddeev-Popov is

$$S_{GH} \,=\, \int d^4x\sqrt{-g}\,\bar{C}\,\left(\Box\,+\,2\,\alpha e^2 v^2\right)\,C\,.$$

After all,

$$\bar{\Gamma}^{(1)}[g_{\mu\nu}] \,=\, \frac{i}{2}\,\mathrm{Tr}\,\ln\,\left[\delta^\mu_\nu\,\Box\,-\,\Big(1-\frac{1}{\alpha}\Big)\nabla^\mu\nabla_\nu\,-\,R^\mu_\nu\,+\,2e^2v^2\,\delta^\mu_\nu\right]$$

$$+\,\frac{i}{2}\,\mathrm{Tr}\,\ln\,(\,\Box+4\lambda v^2\,)+\frac{i}{2}\,\mathrm{Tr}\,\ln\,(\,\Box+2\alpha\,e^2v^2\,)-i\,\mathrm{Tr}\,\ln\,(\Box+2\,\alpha e^2v^2)\,.$$

For an arbitrary α the first term here is related to the functional determinant of a non-minimal massive vector field.

For $\alpha = 0$ there are massless modes, jeopardizing an expected low-energy decoupling.

For all other values of α all the degrees of freedom are massive.

In the particular case $\alpha = 1$ there are only well-known contributions of the minimal massive vector and scalars.

From the works

Ed. Gorbar & I.Sh. JHEP 03,06 (2003); hep-ph/0210388; hep-ph/0303124.

we know that, at least higher derivative contributions of massive modes in curved space suffer the low-energy decoupling.

The last means that the corresponding quantum corrections to the classical higher derivative terms vanish in the IR limit.

According to our consideration, in the theories with SSB the decoupling is guaranteed if we can prove the gauge-fixing independence of the effective action.

The difference between one-loop correction with an arbitrary value of the gauge parameter α and the one with $\alpha = 1$.

$$\bar{\Gamma}^{(1)}[\varphi, g_{\mu\nu};\, \alpha] - \bar{\Gamma}^{(1)}[\varphi, g_{\mu\nu};\, 1].$$

One of the operators is

$$\hat{\mathcal{F}}(\alpha) = \mathcal{F}_\mu^\nu(\alpha) = \delta_\mu^\nu \,\Box - \Big(1 - \frac{1}{\alpha}\Big)\nabla_\mu \nabla^\nu - R_\mu^\nu + m^2\,\delta_\mu^\nu\,,$$

where we denoted $m^2 = 2e^2 v^2$. **Consider the difference**

$$-\frac{1}{2}\,\mathrm{Tr}\,\ln \hat{\mathcal{F}}(\alpha) + \frac{1}{2}\,\mathrm{Tr}\,\ln \hat{\mathcal{F}}(1)$$

$$= -\frac{1}{2}\,\mathrm{Tr}\,\ln\left[\delta_\mu^\nu - \Big(1 - \frac{1}{\alpha}\Big)\nabla_\mu \nabla^\nu \frac{1}{\Box + m^2 - R_{..}}\right].$$

For an arbitrary vector field we can prove (not a simple task)

$$\Big(\nabla^\mu \frac{1}{\Box + m^2 - R_{..}} - \frac{1}{\Box + m^2}\nabla^\mu\Big) A_\mu = 0\,.$$

Using this identity, one can rewrite the difference as an expression involving only the scalar operators

$$-\frac{1}{2}\,\mathrm{Tr}\,\Big[\ln\hat{\mathcal{F}}(\alpha) - \ln\hat{\mathcal{F}}(1)\Big] = -\frac{1}{2}\,\mathrm{Tr}\,\ln\left(\frac{\Box + \alpha m^2}{\Box + m^2}\right). \qquad (2)$$

There is no gauge dependence in the contribution of the Higgs scalar, for its mass M_H does not depend on α.

The gauge dependence in the contribution of the Goldstone scalar is exactly the same as the vector counterpart of (2)

$$\frac{1}{2}\,\mathrm{Tr}\,\ln\left(\frac{\Box + 2\alpha\, e^2 v^2}{\Box + 2\, e^2 v^2}\right) = -\frac{1}{2}\,\mathrm{Tr}\,\ln\left(\frac{\Box + \alpha m^2}{\Box + m^2}\right).$$

Finally, the difference between the two ghost operators contributes as

$$\mathrm{Tr}\,\ln\left(\frac{\Box + \alpha m^2}{\Box + m^2}\right).$$

Hence, the overall gauge fixing dependence cancels out!

$\overline{\text{MS}}$-scheme renormalization in the non-minimal case

Consider first the one-loop case.

Perform the background shift of the scalar according to

$$\varphi = v + h + i\eta ,$$

where h and η are real scalar quantum fields (Higgs and Goldstone).

We face a problem of deriving the divergences in the theory with quantum fields A_μ, h, η, while the background fields include metric and v, which, in turn, also depends on the metric.
The renormalization of the theory looks standard in terms of $g_{\mu\nu}$ and v and very unusual in terms of metric alone.

The part of the action which is bilinear in quantum fields

$$S^{(2)} + S_{GF} = \int d^4x\sqrt{-g}\Big\{ \frac{1}{2}\,A_\mu\Box A^\mu + \frac{1}{2}\left(1-\frac{1}{\alpha}\right)(\nabla_\mu A^\mu)^2 - A^\mu A^\nu R_{\mu\nu}$$

$$+ \frac{1}{2}\,M_A^2 A^2 + (\partial_\mu h)^2 + (\partial_\mu \eta)^2 - M_H^2 h^2 - M_\eta^2 \eta^2 - 2\alpha e\eta\, A^\mu(\partial_\mu v)\Big\}\,,$$

where we introduced new notations

$$M_A^2 = 2e^2v^2\,, \qquad M_h^2 = 6\lambda v^2 - \mu_0^2 - \xi R\,, \qquad M_\eta^2 = 2e^2v^2 + 2\lambda v^2 - \mu_0^2 - \xi R\,.$$

One can rewrite these in a more useful way. We introduce

$$\xi\mathcal{K} = 2\lambda\,\left(v^2 - v_0^2\right) = 2\lambda v^2 - \mu_0^2\,.$$

In the lowest order in curvature we obtain

$$\xi\mathcal{K} \,=\, \frac{2\xi v_0^2}{\Box + 4\lambda v_0^2}\,R \,+\, \mathcal{O}(R^2)\,.$$

At low-energies the derivatives of curvature are very small compared to v_0^2. Then we can expand

$$\frac{1}{\Box + 4\lambda v_0^2} \,=\, \frac{1}{4\lambda v_0^2}\,\left(1 - \frac{\Box}{4\lambda v_0^2} + \,...\right) \,+\, \mathcal{O}(\Box R)\,.$$

In the low-energy approximation we arrive at the representation

$$\xi\mathcal{K} \,=\, \xi R \,+\, \frac{\text{higher derivative terms}}{v_0^2}\,.$$

For the general case

$$\frac{(\Box v)}{v} = \frac{\mu_0^2\,v + \xi R\,v - 2\lambda\,v^3}{v} = \xi R + 2\lambda v_0^2 - 2\lambda v^2 \,=\, \xi R - \xi\mathcal{K}\,.$$

Now the elements of expansion may be written in the form

$$M_A^2 = m^2 \,+\, \frac{e^2}{\lambda}\,\xi\mathcal{K}\,, \qquad\qquad m^2 \,=\, 2e^2 v_0^2\,;$$

$$M_h^2 = m_h^2 \,-\, \xi\,R \,+\, 3\,\xi\,\mathcal{K}\,, \qquad\qquad m_h^2 \,=\, 4\lambda v_0^2\,;$$

$$M_\eta^2 = m^2 \,-\, \xi\,R \,+\, \Big(\frac{e^2}{\lambda} + 1\Big)\,\xi\mathcal{K}\,,$$

where m and m_h are the masses of the fields after SSB.

Finally, the expression for the one-loop divergences is

$$\bar{\Gamma}_1^{(div)} = -\frac{1}{(4\pi)^2(n-4)} \int d^4x\sqrt{-g}\,\Big\{ \frac{1}{2}\,(3m^4 + m_h^4) - \Big(\xi - \frac{1}{6}\Big)m_h^2\,R$$

$$-\frac{1}{2}\,m^2 R + \Big(\frac{e^2}{\lambda}\,m^2 + m_h^2\Big)\cdot 3\,\xi\mathcal{K} + \frac{7}{60}\,C^2_{\mu\nu\alpha\beta} - \frac{8}{45}\,E$$

$$+\Big(\xi - \frac{1}{6}\Big)^2 R^2 + \Big(\frac{3e^4}{2\lambda^2} + 5\Big)\,(\xi\mathcal{K})^2 - \Big[4\,\Big(\xi - \frac{1}{6}\Big) + \frac{e^2}{2\lambda}\Big]\,R\cdot\xi\mathcal{K}\,\Big\}\,,$$

The expression above differs from what is usually expected for the divergencies of the quantum field theory.

Along with the usual local terms, there are many $\mathcal{K}$ - dependent terms, non-local with respect to the background metric $g_{\mu\nu}$.

One can prove that the same types of non-local counterterms, $\mathcal{K}m^2$, $\mathcal{K}R$, $\mathcal{K}^2$ **are sufficient also in higher loops.**

$\langle T_{\mu\nu} \rangle$ of vacuum in the theories with SSB

Let us remember that the effective action is not supposed to be the “final product” of our work.

Such a “product” is the equation of motion for the metric, in other words we need the $\langle T_{\mu\nu} \rangle$ of vacuum. Do we have some surprises in this part?

By definition, the average of the dynamical EMT is

$$\langle T_{\mu\nu}(x) \rangle \;=\; \frac{2}{\sqrt{-g(x)}}\;\frac{\delta \Gamma}{\delta g^{\mu\nu}(x)}\,.$$

At the classical level

$$S_{vac} = S_{EH} \,+\, S_{HD}\,, \qquad \textbf{where}$$

$$S_{EH} \;=\; -\,\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\left(\,R + 2\Lambda\,\right) \qquad \textbf{and}$$

$$S_{HD} = \int d^4x\sqrt{-g}\,\left\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\right\}\,.$$

As far as are interested in the low-energy effects, we concentrate on the part of S_{EH} and related quantum corrections.

At quantum level, the energy-momentum tensor (EMT) is

$$\langle T_{\mu\nu} \rangle \,=\, -\,\frac{2}{\sqrt{-g}}\, g_{\mu\alpha} g_{\nu\beta} \left\langle 0 \left| \frac{\delta S[g, \hat{\phi}]}{\delta g_{\alpha\beta}} \right| 0 \right\rangle ,$$

where $\hat{\phi}$ is quantum field,

$$\hat{\phi} \sim u \hat{a}^{\dagger} + u^{*} \hat{a}$$

and $\quad \hat{a}\,|0\rangle = 0.$

As far as $g_{\mu\alpha}$ is classical external field, so we can take it out of the symbol $\langle|..|\rangle$ freely.

In the functional representation the basic object is generating functional of vertex function, or Effective Action, $\Gamma = \Gamma[g, \phi]$.
For the case of a scalar field

$$\exp\left\{\frac{i}{\hbar}\Gamma[g,\phi]\right\} = \int d\bar{\phi}\, \exp\left\{\frac{i}{\hbar}\Big(S[g,\bar{\phi}+\phi] - \frac{\delta\Gamma[g,\phi]}{\delta\phi}\,\bar{\phi}\Big]\right\}.$$

In the one-loop approximation

$$\Gamma^{(1)}[\phi,\, g_{\mu\nu}] \,=\, S[\phi,\, g_{\mu\nu}] \,+\, \hbar\bar{\Gamma}^{(1)}[\phi,\, g_{\mu\nu}]\,.$$

Then the one-loop EMT of the vacuum is

$$\langle T_{\mu\nu}(x)\rangle^{(1)} = T_{\mu\nu}(x) + \bar{T}^{(1)}_{\mu\nu}(x)\,,$$

where $$T_{\mu\nu} = -\,\frac{2}{\sqrt{-g}}\,g_{\mu\alpha}\,g_{\nu\beta}\,\frac{\delta S}{\delta g_{\alpha\beta}}\Big|_{\phi\to\phi_0}\,.$$

and $$\bar{T}^{(1)}_{\mu\nu} = -\,\frac{2\,\hbar}{\sqrt{-g}}\,g_{\mu\alpha}\,g_{\nu\beta}\,\frac{\delta\bar{\Gamma}^{(1)}}{\delta g_{\alpha\beta}}\Big|_{\phi\to\phi_0}\,.$$

In both cases ϕ_0 is the solution of the equations of motion.

If we deal with purely classical theory, then one has to replace value $\phi_0 = \phi_{0c}$. At the one-loop level we have

$$\frac{\delta S[g, \phi_0]}{\delta \phi} + \hbar \, \frac{\delta \bar{\Gamma}^{(1)}[g, \phi_0]}{\delta \phi} = 0 \, ,$$

This equation can be solved by iterations in $\hbar$.

$$\phi_0 = \phi_{0c} + \hbar \phi_1 \, ,$$

where ϕ_{0c} is the classical solution.

In the first order in $\hbar$ we meet the equation

$$\frac{\delta^2 S[g, \phi_{0c}]}{\delta \phi \delta \phi} \phi_1 + \frac{\delta \bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta \phi} = 0 \, ,$$

and obtain the solution in the form

$$\phi_1 = - \left(\frac{\delta^2 S[g, \phi_{0c}]}{\delta \phi \, \delta \phi} \right)^{-1} \frac{\delta \, \bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta \phi} \, .$$

So, at one-loop order we have

$$\phi_0 = \phi_{0c} - \hbar \left(\frac{\delta^2 S[g, \phi_{0c}]}{\delta\phi\,\delta\phi} \right)^{-1} \frac{\delta\,\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta\phi}\,.$$

One has to replace this formula into the expression for EMT,

$$\langle T_{\mu\nu} \rangle = -\frac{2}{\sqrt{-g}}\, g_{\mu\alpha} g_{\nu\beta}\, \frac{\delta}{\delta g_{\alpha\beta}} \Big\{ S[g, \phi_0] + \hbar\,\bar{\Gamma}^{(1)}[g, \phi_0] \Big\}\,.$$

In this way we arrive at the general expression for the EMT in the scalar theory with SSB:

$$\langle T_{\mu\nu} \rangle = -\frac{2}{\sqrt{-g}}\, g_{\mu\alpha}\, g_{\nu\beta} \left\{ \frac{\delta\, S[g, \phi_{0c}]}{\delta g_{\alpha\beta}} + \hbar\, \frac{\delta\,\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta g_{\alpha\beta}} \right.$$

$$\left. -\,\hbar\, \frac{\delta^2 S[g, \phi_{0c}]}{\delta g_{\alpha\beta} \delta\phi} \left(\frac{\delta^2 S[g, \phi_{0c}]}{\delta\phi\,\delta\phi} \right)^{-1} \frac{\delta\,\bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta\phi} \right\}.$$

The first two terms are pretty well known.

The last term is qualitatively new one. It is there because we deal with an interacting theory with SSB.

M. Asorey, P.M. Lavrov, B.J. Ribeiro, I.Sh., Phys.Rev. D85 (2012) 104001; arXive: 1202.4235.

Looking at the expression

$$\langle T_{\mu\nu}\rangle = -\,\frac{2}{\sqrt{-g}}\,g_{\mu\alpha}\,g_{\nu\beta}\left\{\frac{\delta\,S[g,\phi_{0c}]}{\delta g_{\alpha\beta}} + \hbar\,\frac{\delta\,\bar{\Gamma}^{(1)}[g,\phi_{0c}]}{\delta g_{\alpha\beta}}\right.$$

$$\left. -\,\hbar\,\frac{\delta^2 S[g,\phi_{0c}]}{\delta g_{\alpha\beta}\delta\phi}\left(\frac{\delta^2 S[g,\phi_{0c}]}{\delta\phi\,\delta\phi}\right)^{-1}\frac{\delta\,\bar{\Gamma}^{(1)}[g,\phi_{0c}]}{\delta\phi}\right\}$$

$$= \langle T_{\mu\nu}(\phi_{0c})\rangle_v + \langle T_{\mu\nu}(\phi_{0c})\rangle_i\,.$$

the two following questions are in order:

- **Is there a relation between the two quantum contributions?**
- **Does the new quantum term violate the conservation law?**

$$\nabla_\mu\langle T^\mu_\nu\rangle = 0$$

The last question is especially important for the low-energy physics. The reason is as follows:

Low energies for the gravitational field means metric close to the flat one.

Then it is appropriate to perform the curvature expansion.
Up to the first order in such a expansion we have only local expressions in the Effective Action.

Morever, the conservation law

$$\nabla_\mu \langle T^\mu_\nu \rangle \,=\, 0$$

is fixing the EMT to be

$$\langle T_{\mu\nu} \rangle \,=\, C_1\, g_{\mu\nu} \,+\, C_2\, G_{\mu\nu}\,,$$

where

$$C_1 = k_4\Omega^4 + k_2\Omega^2 + k_L \ln(\Omega/\mu_0) + k_{fin} \quad \textbf{and} \quad C_2 = l_2\Omega^2 + l_L \ln(\Omega/\mu_0) + l_{fin}$$

with $k_4,\, k_2,\, k_L,\, k_{fin}$ **and** $l_2,\, l_L,\, l_{fin}$ **being numerical constants.**

The identity corresponding to diffeomorphism invariance is

$$\int d^4x\sqrt{-g}\left\{\frac{2}{\sqrt{-g}}\,\frac{\delta\Gamma[g,\phi]}{\delta g_{\mu\nu}}\,\nabla_\mu\xi_\nu + \frac{1}{\sqrt{-g}}\,\frac{\delta\Gamma[g,\phi]}{\delta\phi}\,\xi^\mu\nabla_\mu\phi\right\} = 0\,.$$

One can take into account that

$$\frac{\delta\Gamma[g,\phi_0]}{\delta\phi} = 0 \qquad \textbf{and arrive at} \qquad \nabla_\mu\langle T^\mu_\nu\rangle = 0\,.$$

At zero order in $\hbar$ we obviously have

$$\frac{\delta S[g,\phi_{0c}]}{\delta\phi} = 0 \qquad \textbf{and} \qquad \nabla^\mu T_{\mu\nu}\Big|_{\phi_{0c}} = 0\,.$$

Next, at the first order in $\hbar$ the solution is $\phi_0 = \phi_{0c} + \hbar\phi_1$.
One should expect that neither one of the two quantum terms separately satisfy the conservation law and only for their sum this equation must be valid,

$$\nabla^\mu\langle T_{\mu\nu}(\phi_{0c})\rangle_v + \nabla^\mu\langle T_{\mu\nu}(\phi_{0c})\rangle_i = 0\,.$$

Practical calculation: Classical Part

Keeping only terms linear in curvature tensors, we arrive at

$$T_{\mu\nu}(\phi_{0c}) \,=\, \xi v_0^2 \left(R_{\mu\nu} - \frac{1}{2}\,Rg_{\mu\nu}\right) \,-\, \frac{\lambda v_0^4}{12}\,g_{\mu\nu}\,.$$

This is the induced contribution to the Einstein equations, which can be written as

$$\left(\frac{1}{8\pi G_{vac}} + \frac{1}{8\pi G_{ind}}\right)\left(R_{\mu\nu} - \frac{1}{2}\,Rg_{\mu\nu}\right) \,-\, \left(\rho_\Lambda^{vac} \,+\, \rho_\Lambda^{ind}\right) g_{\mu\nu} \,=\, T_{\mu\nu}^{\rm matter}\,.$$

where

$$\frac{1}{8\pi G_{ind}} = -\xi v_0^2 \qquad \textbf{and} \qquad \rho_\Lambda^{ind} = -\frac{\lambda v_0^4}{12}\,.$$

G_{vac} **and** ρ_Λ^{vac} **are the vacuum Newton constant and the cosmological constant density - independent parameters.**

G_{ind} **and** ρ_Λ^{ind} **are induced quantities which depend on the details of the quantum theory of matter fields.**

The induced and vacuum cosmological constant densities are, at least, 55 orders of magnitude greater than their sum

$$\rho_\Lambda^{obs} = \rho_\Lambda^{vac} + \rho_\Lambda^{ind}\,.$$

This gives rise to the cosmological constant problem.

On the contrary, the relative magnitude of G_{ind} ,

$$\frac{G_{ind}}{G_{vac}} = -\,\frac{8\pi\xi v_0^2}{M_P^2}\,,$$

is small for the SM case when $v_0^2 \approx 10^5 GeV^2$. Even if the value of ξ corresponds to the Higgs inflation, $\xi \approx 40000$, the Planck suppression is strong due to the $M_P^2 \approx 10^{38} GeV^2$ and hence the induced contribution is irrelevant.

The situation can be quite different in GUT's,

$v_0^2 \approx 10^{32} GeV^2\,,$ **then it can be even** $\dfrac{G_i}{G_v} \approx 1\,.$

.

Practical calculation: Quantum Part

Quantum calculations are not too easy and have been actually performed only in $\mathcal{O}(\hbar)$ and $\mathcal{O}(R)$ approximations.

Because of the $\mathcal{O}(R)$ approximation we need only zero order in the derivative expansion,

$$\bar{\Gamma}^{(1)}(g,\phi) = \int d^4x\sqrt{-g}\,\Big\{ -\bar{V}_{eff}(\phi) + \frac{1}{2}Z(\phi)(\nabla\phi)^2 + ...\Big\}\,,$$

so we need just

$$\bar{\Gamma}^{(1)}(g,\phi) \,\approx\, \int d^4x\sqrt{-g}\,\Big\{ -\bar{V}_{eff}(\phi)\Big\}\,.$$

The reason is that

$$\nabla_\mu\phi_{0c} \;=\; \nabla_\mu v_0 + \nabla_\mu v_1 \;=\; \nabla_\mu v_1 \;=\; \frac{\xi v_0}{2m^2}\,\nabla_\mu R \,+\, \mathcal{O}(\nabla^3 R)\,.$$

The renormalized effective potential is

$$\bar{V}_{eff}^{ren}(g_{\mu\nu},\,\varphi)\;=\;V_0^{ren}\;+\;V_1^{ren}R$$

$$=\;\frac{1}{2(4\pi)^2}\Big[\frac{1}{2}\big(V''-m^2\big)^2-\Big(\xi-\frac{1}{6}\Big)\,R\,\big(V''-m^2\big)\Big]\ln\Big(\frac{V''-m^2}{\mu^2}\Big)\,.$$

where $V=V(\varphi)=\lambda\varphi^4/4$.

The divergent part of the non-renormalized effective potential is

$\bar{V}_{eff}^{div}(g_{\mu\nu},\,\varphi)\;=\;V_0^{div}\;+\;V_1^{div}R$, **where**

$$\bar{V}_0^{div}=\frac{1}{32\pi^2}\Big\{\Omega^2V''-\frac{1}{2}\big(V''-m^2\big)^2\ln\frac{\Omega^2}{m^2}\Big\}\,,$$

and $$\bar{V}_1^{div}=\frac{1}{32\pi^2}\Big(\xi-\frac{1}{6}\Big)\Big\{-\,\Omega^2\,+\,\big(V''-m^2\big)\,\ln\frac{\Omega^2}{m^2}\Big\}\,,$$

It proves useful to introduce a notation for the one-loop contributions to the equations of motion for a scalar field,

$$\bar{\varepsilon}^{(1)} = \bar{\varepsilon}^{(1)}_{div} + \bar{\varepsilon}^{(1)}_{fin} \;=\; \frac{1}{\sqrt{-g}}\,\frac{\delta\bar{\Gamma}^{(1)}}{\delta\phi}\Big|_{\phi_{0c}} \;=\; -\,\frac{\partial\,\bar{V}^{(1)}_{eff}}{\partial\phi}\Big|_{\phi_{0c}}\,.$$

After adding the corresponding counterterm, we meet $\bar{\varepsilon}^{(1)}_{ren}$**.**

Let us now remember that $\bar{V}_{eff} = V_0(\phi) + V_1(\phi)\,R$ **and** $\phi_{0c} = v_0 + v_1$,

Then,

$$\bar{\varepsilon}^{(1)} = -\,\frac{\partial\,\bar{V}^{(1)}_0}{\partial\phi}\Big|_{\phi_{0c}} -\; R\,\frac{\partial\,\bar{V}^{(1)}_1}{\partial\phi}\Big|_{\phi_{0c}}$$

$$=\; -\,\frac{\partial\,\bar{V}^{(1)}_0}{\partial\phi}\Big|_{v_0} \;-\; \frac{\partial^2\,\bar{V}^{(1)}_0}{\partial\phi^2}\Big|_{v_0} v_1 \;-\; R\,\frac{\partial\,\bar{V}^{(1)}_1}{\partial\phi}\Big|_{v_0} \;=\; \bar{\varepsilon}^{(1)}_0 \;+\; \bar{\varepsilon}^{(1)}_1\,.$$

The calculation of “vacuum” quantum part is easy

$$
\langle \bar{T}_{\mu\nu} \rangle_v \,=\, -\,\frac{2\,\hbar}{\sqrt{-g}}\, g_{\mu\alpha}\, g_{\nu\beta}\, \frac{\delta\, \bar{\Gamma}^{(1)}[g, \phi_{0c}]}{\delta g_{\alpha\beta}}
$$

$$
=\, -\,2\,\hbar V_1(v_0)\Big(R_{\mu\nu} - \frac{1}{2}\, R g_{\mu\nu} \Big) \,+\, \hbar V_0(v_0)\, g_{\mu\nu} \,+\, \hbar\, v_1\, g_{\mu\nu}\, \frac{\partial\, \bar{V}_0^{(1)}}{\partial \phi}\Big|_{v_0}
$$

$$
=\, -\,2\,\hbar V_1(v_0)\, G_{\mu\nu} \,+\, \hbar V_0(v_0)\, g_{\mu\nu} \,-\, \frac{\hbar\,\xi\, v_0}{2m^2}\, R\, \bar{\varepsilon}_0^{(1)}\, g_{\mu\nu}\,.
$$

This formula confirms what we have anticipated above, namely:

- **The first two terms are quantum contributions to the Einstein tensor and cosmological constant part.**

- **However, the last term is odd: it violates conservation law and can not be derived from the action principle.**

“Induced” quantum part is more involved.

First we rewrite $\langle \bar{T}_{\mu\nu}\rangle_i$ in a more useful form, $\int_y \equiv \int d^4y\sqrt{-g(y)}$

$$\langle T_{\mu\nu}(x)\rangle_i \,=\, 2\hbar\, g_{\mu\alpha}(x)\, g_{\nu\beta}(x)\, \int_y \int_z \left(\frac{1}{\sqrt{-g}}\, \frac{\delta^2 S[g,\phi_{0c}]}{\delta g_{\alpha\beta}(x)\delta\phi(y)} \right)$$

$$\times \left(\frac{1}{\sqrt{-g(y)}}\, \frac{\delta^2 S[g,\phi_{0c}]}{\delta\phi(y)\,\delta\phi(z)} \right)^{-1} \times \left(\frac{1}{\sqrt{-g(z)}}\, \frac{\delta\,\bar{\Gamma}^{(1)}[g,\phi_{0c}]}{\delta\phi(z)} \right).$$

The metric-dependent quantities are always understood through the normal coordinate expansions, e.g.,

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, y^\alpha y^\beta + \,\dots\,,$$

$$\nabla_\mu \nabla_\nu \,=\, \partial_\mu \partial_\nu + \frac{2}{3}\, R^\lambda{}_{(\mu\nu)\,\tau}\; y^\tau \partial_\lambda \,+\, \dots\,.$$

The next step is to rewrite the expressions in a more useful form,

$$\frac{1}{\sqrt{-g}}\frac{\delta^2 S}{\delta\phi(y)\delta g_{\mu\nu}(x)} = \xi\phi\big(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\big) + (2\xi-1)(\nabla_\mu\phi)\nabla_\nu$$

$$+\Big(\frac{1}{2}-2\xi\Big)g_{\mu\nu}(\nabla^\lambda\phi)\nabla_\lambda + \xi\big(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\big)$$

$$-\xi\phi\Big(R_{\mu\nu}-\frac{1}{2}R\,g_{\mu\nu}\Big)+\frac{1}{2}m^2\phi\,g_{\mu\nu}-\frac{\lambda}{6}\phi^3 g_{\mu\nu}$$

$$= \Big(\xi v_0 + \frac{\xi^2 v_0}{2m^2}R\Big)\big(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\big) - \xi v_0 R_{\mu\nu}$$

$$+\frac{1}{3}\xi v_0\Big[2\big(R^\lambda{}_{(\mu\nu)\ \tau} + \eta_{\mu\nu}R^\lambda_\tau\big)y^\tau\partial_\lambda + R_{\mu\alpha\nu\beta}\,y^\alpha y^\beta\partial^2 + \eta_{\mu\nu}R^\rho{}_{\alpha\beta}{}^\sigma\,y^\alpha y^\beta\,\partial_\rho\partial_\sigma\big)\Big].$$

The second factor is

$$\left(\frac{1}{\sqrt{-g}} \, \frac{\delta S^2[g, \phi_{0c}]}{\delta\phi \, \delta\phi} \right)^{-1}_{y,z} = G(y, z;\, \phi_{0c})\,,$$

This is the propagator of the scalar excitations near the point of the minima,

$$\left(\frac{1}{[-g(y)]^{1/4}\,[-g(z)]^{1/4}} \times \frac{\delta S^2[g, \phi_{0c}]}{\delta\phi(y)\,\delta\phi(z)} \right)^{-1} = \bar{G}(y, z;\, \phi_{0c})\,.$$

Using $\phi_{oc} = v_0 + v_1$ **we arrive at**

$$\frac{1}{\sqrt{-g}} \, \frac{\delta^2 S}{\delta\phi \, \delta\phi} = -\Box + 2m^2 + \xi\Big(1 - \frac{6m^2}{\Box + 2m^2}\Big) R$$

The Euclidean version of the second factor is

$$\bar{G}(z-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(z-y)} \left[\frac{1}{k^2 + 2m^2} - \Big(2\xi - \frac{1}{6}\Big) \frac{R}{(k^2 + 2m^2)^2} \right] .$$

The third factor is the effective equation of motion. One can write it as a sum of classical and quantum parts,

$$\varepsilon = \bar{\varepsilon}^{(0)} + \hbar\bar{\varepsilon}^{(1)}\,, \quad \textbf{where} \quad \bar{\varepsilon}^{(1)} = \bar{\varepsilon}_0^{(1)} + \bar{\varepsilon}_1^{(1)}$$

In the flat-space limit we have

$$\langle T_{\mu\nu}(x)\rangle_i^0 \,=\, 2\hbar\,\xi v_0 \int d^4z\,d^4y\;\delta^4(x-y)\big(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\big)_y$$

$$\times \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(y-z)}}{k^2+2m^2}\;\bar{\varepsilon}_0^{(1)}(z),$$

$$=\, 2\hbar\xi\,v_0\,\bar{\varepsilon}_0^{(1)} \int d^4k\;\delta^4(k)\;\frac{k_\mu k_\nu - k^2\eta_{\mu\nu}}{k^2+2m^2}\;e^{ikx} \,=\, 0\,.$$

Thus, the contribution of the new term to the induced cosmological constant is zero.

The last step is to perform the curved - space calculation in the $\mathcal{O}(R)$ approximation.

$$\langle T_{\mu\nu}(x)\rangle_i^1 \;=\; 2\hbar\,\bar{\varepsilon}_0^{(1)} \int d^4y d^4z \int \frac{d^4k}{(2\pi)^4} \sum_{i=1}^{5} O_{\mu\nu}^{(i)}(y)\delta^4(x-y)\,\frac{e^{ik(y-z)}}{k^2+2m^2}\,,$$

where

$$O_{\mu\nu}^{(1)} \;=\; -\,\xi v_0\, R_{\mu\nu}\,,$$

$$O_{\mu\nu}^{(2)} \;=\; \frac{\xi^2 v_0}{2m^2}\, R\,(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\,,$$

$$O_{\mu\nu}^{(3)} \;=\; \frac{2}{3}\,\xi v_0\,\big[R^\lambda{}_{(\mu\nu)\tau} + \eta_{\mu\nu}R^\lambda_\tau\big]\,y^\tau\,\partial_\lambda\,,$$

$$O_{\mu\nu}^{(4)} \;=\; \frac{1}{3}\xi v_0\, R_{\mu\alpha\nu\beta}\, y^\alpha y^\beta\,\partial^2\,,$$

$$O_{\mu\nu}^{(5)} \;=\; \frac{1}{3}\,\xi v_0\,\eta_{\mu\nu}\, R_\alpha{}^{\rho\sigma}{}_\beta\; y^\alpha y^\beta\,\partial_\rho\partial_\sigma\,.$$

After certain (not small) algebra we come to the result

$$\langle T_{\mu\nu}\rangle_i^1 \,=\, \frac{\hbar\,\xi\,v_0}{m^2}\,\bar{\varepsilon}_0^{(1)}\Big(-\,R_{\mu\nu}\,+\,R\eta_{\mu\nu}\Big)\,.$$

Obviously, this expression is different from $G_{\mu\nu}$ and therefore it violates covariance and conservation law.

However, if we sum up with the previous result for $\langle T_{\mu\nu}\rangle_v^1$ we arrive at the expression which agrees with our expectations,

$$\langle T_{\mu\nu}\rangle^1 \,=\, \langle T_{\mu\nu}\rangle_i^1 \,+\, \langle \bar{T}_{\mu\nu}\rangle_v^1$$

$$=\,-\,2\,\hbar V_1(v_0)\,G_{\mu\nu}\,+\,\hbar V_0(v_0)\,g_{\mu\nu}\,-\,\frac{\hbar\,\xi\,v_0}{m^2}\,\Big(R_{\mu\nu}\,-\,\frac{1}{2}\,Rg_{\mu\nu}\Big)\bar{\varepsilon}_0^{(1)}\,,$$

$$=\,-\,\hbar\Big[2\,V_1(v_0)\,+\,\frac{\xi\,v_0}{m^2}\,\bar{\varepsilon}_0^{(1)}\Big]\,G_{\mu\nu}\,+\,\hbar V_0(v_0)\,g_{\mu\nu}\,.$$

Now we can use

$$V_0^{ren}(v_0) = \frac{1}{(4\pi)^2}\, m^4 \,\ln\left(\frac{2m^2}{\mu^2}\right)$$

and

$$V_1^{ren}(v_0) = -\,\frac{m^2}{(4\pi)^2}\,\ln\left(\frac{2m^2}{\mu^2}\right),$$

to obtain

$$\langle T_{\mu\nu}\rangle_{ren} = \frac{\hbar\, m^4}{(4\pi)^2}\,\ln\left(\frac{2m^2}{\mu^2}\right) g_{\mu\nu} - \frac{m^2}{(4\pi)^2}\Big[2(1+3\xi)\,\ln\left(\frac{2m^2}{\mu^2}\right)+3\xi\Big]\,G_{\mu\nu}$$

For the divergent part we meet $\langle T_{\mu\nu}\rangle_{div} =$

$$= \frac{\hbar\, m^2}{32\,\pi^2}\,\Big[3\Omega^2-2m^2\,a\Big(\frac{\Omega^2}{m^2}\Big)\Big]\,g_{\mu\nu}+\frac{\hbar}{16\,\pi^2}\,\Big(4\xi-\frac{1}{6}\Big)\,\Big\{\Omega^2-2m^2\,\ln\frac{\Omega^2}{m^2}\Big\}\,G_{\mu\nu}$$

Conclusions

- The SSB is highly non-trivial issue in curved space-time, leading to classical and quantum non-localities in the induced action of gravity.

- The renormalization can be performed in a consistent way even in the broken phase, however it becomes more complex.

- There is a qualitatively new contribution to the vacuum stress tensor, even after we have the Effective Action of vacuum.

- Finally, the conservation law still controls well the quantum terms and we meet only usual vacuum terms, in the linear in curvature approximation.

$$V_0^{div}(v_0) = \frac{m^2}{32\,\pi^2}\left[3\Omega^2 - 2m^2\,\ln\left(\frac{\Omega^2}{m^2}\right)\right].$$

$$V_1^{div}(v_0) = \frac{1}{32\,\pi^2}\left(\xi - \frac{1}{6}\right)\left[-\,\Omega^2 + 2m^2\,\ln\left(\frac{\Omega^2}{m^2}\right)\right].$$