Bell's Theorem on arbitrary causal structures

Follow-up to
Beyond Bell's Theorem: Correlation Scenarios
arXiv:1206.5115

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joint work (not) in progress with Rob Spekkens

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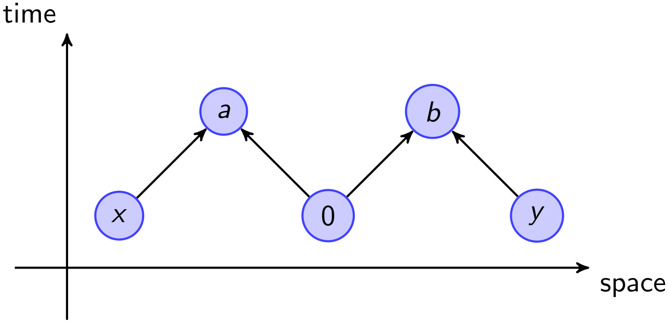
Causal structure of Bell's Theorem I

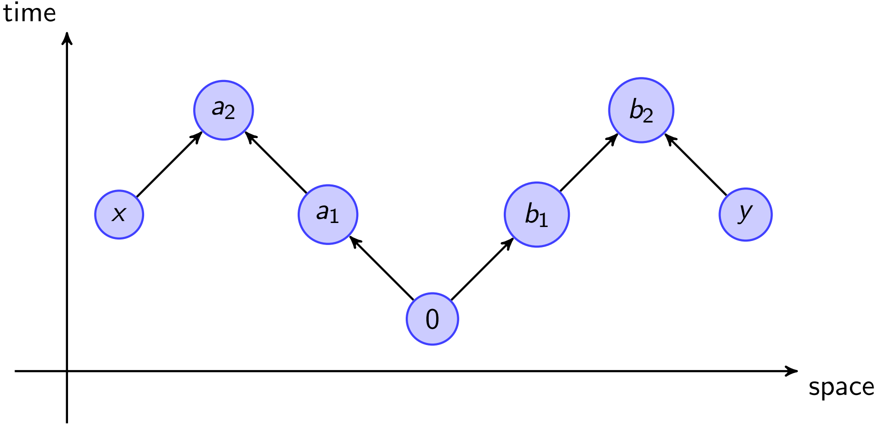
Spacetime diagram of bipartite Bell scenario:



- A network of **events** connected by **causal links**.
- Every event has an associated **outcome**. In practice, even the source will have a non-trivial outcome!
- Hidden variables propagate along the causal links.
- Repeated trials give measurement statistics $P(a, b, x, y, 0)$.

Causal structure of Bell's Theorem II

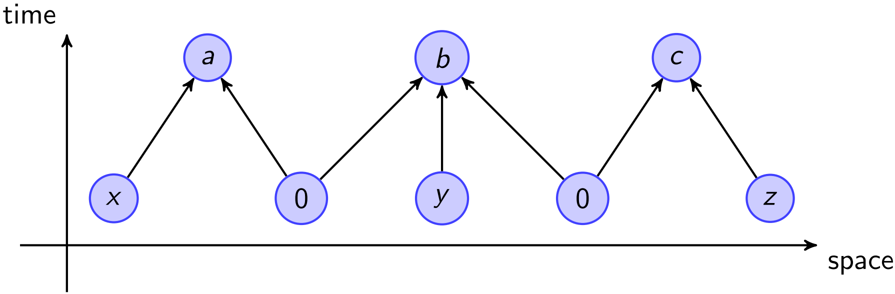
Spacetime diagram of Popescu's "hidden nonlocality" scenario:



- Same story! A network of outcome-producing events connected by hidden-variable-carrying causal links.

Causal structure of Bell's Theorem III

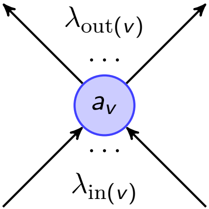
Spacetime diagram of Branciard–Gisin–Pironio's "bilocality" scenario:



- What's going on here, generally?

Bayesian networks with **classical** hidden variables I

- The events and causal links form a **DAG (directed acyclic graph)** $G = (\mathcal{V}, \mathcal{E})$ with a node set $\mathcal{V}$ and edge set $\mathcal{E}$.
- Every $v \in \mathcal{V}$ carries a random variable a_v. Notation: when $S \subseteq \mathcal{V}$, write a_S for $(a_v)_{v \in S}$.
- For $v \in \mathcal{V}$, write
  - $\mathrm{in}(v)$ = set of ingoing edges at v,
  - $\mathrm{out}(v)$ = set of outgoing edges at v.
- Intended local structure of a node v:



Bayesian networks with classical hidden variables II

Definition

A joint distribution $P(a_{\mathcal{V}})$ is a **classical correlation** with respect to G if there exist random variables $(\lambda_e)_{e\in\mathcal{E}}$ and for all $v \in \mathcal{V}$ a conditional distribution

$$P(a_v, \lambda_{\mathrm{out}(v)} | \lambda_{\mathrm{in}(v)})$$

such that

$$P(a_{\mathcal{V}}) = \int \prod_{v\in\mathcal{V}} P(a_v, \lambda_{\mathrm{out}}(v) | \lambda_{\mathrm{in}}(v))\, \mathrm{d}\lambda_{\mathcal{V}}$$

Computational interpretation:

- **information flow** λ_e along every edge e,
- an **information processing gate** $P(a_v, \lambda_{\mathrm{out}(v)} | \lambda_{\mathrm{in}(v)})$ at every node.

Bayesian networks with classical hidden variables III

Correspondence with the usual notions:

- If $\mathrm{in}(v) = \emptyset$ and $a_v = 0$, then v acts like a source.
- If $\mathrm{in}(v) = \emptyset$ and $|\mathrm{out}(v)| = 1$, then v acts like a choice of measurement setting.
- If $\mathrm{out}(v) = \emptyset$, then v acts like a measurement.
- In general, a node combines all these things!

The emphasis on the causal structure helps to clarify the assumptions made on the hidden variables:

- Realism/Factorizability/Separability: usage of classical probability theory.
- Locality: only the given causal links can create dependencies among variables.

Bayesian networks with classical hidden variables IV

Theorem (Properties of classical correlations)

- In the above definition, one can assume that all $P(a_v, \lambda_{\mathrm{out}(v)} | \lambda_{\mathrm{in}(v)})$ with $\lambda_{\mathrm{in}(v)} \neq \emptyset$ are **deterministic**:

$$a_v = f(\lambda_{\mathrm{in}(v)}), \qquad \lambda_{\mathrm{out}(v)} = g(\lambda_{\mathrm{in}(v)}).$$

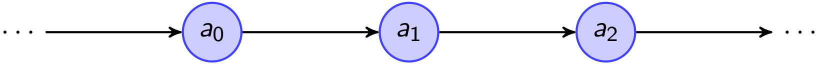
- $P(a_{\mathcal{V}})$ is classical if and only if there exists a **hidden Bayesian network** $P(\omega_{\mathcal{V}})$ on G with a variable ω_v at each $v \in \mathcal{V}$ and functions h_v such that

$$a_v = h_v(\omega_v).$$

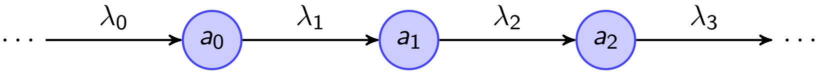
- The set of classical correlations only depends on the **causal set** (=partial order) induced by G.

Example: hidden Markov models

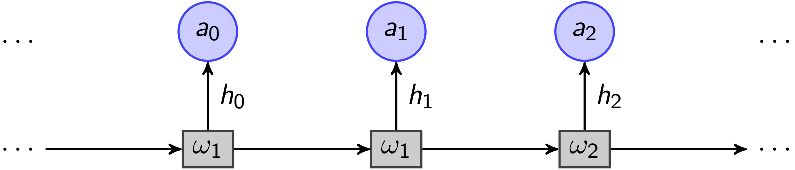
Stochastic process $P(a_{\mathbb{Z}})$:



Edge-emitting hidden Markov model:



State-emitting hidden Markov model:

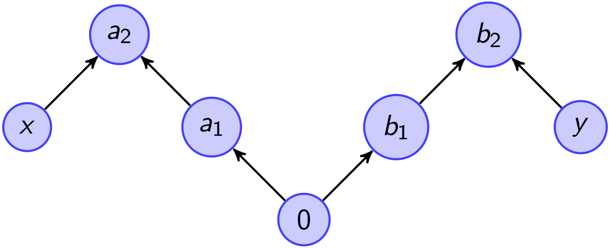


Applications to speech recognition and many other things!

Bayesian networks with **quantum** hidden variables

What are **quantum correlations** on G?

Answer: Interpret G as a diagram in categorical quantum mechanics!



- Each wire represents a Hilbert space $\mathcal{H}_e$,
- each node represents **quantum instrument**, i.e. a completely positive map

$$\Phi_{a_v} : \mathcal{B}(\mathcal{H}_{\mathrm{in}(v)}) \longrightarrow \mathcal{B}(\mathcal{H}_{\mathrm{out}(v)})$$

  such that $\sum_{a_v} \Phi_{a_v}$ is trace-preserving.
- Intuition: quantum computer with classical outcome at each gate.

Generalizing No-Signaling

Definition

$P(a_{\mathcal{V}})$ is a **correlation** if for any number of subsets $X_1, \ldots, X_n \subseteq \mathcal{V}$ with pairwise disjoint causal past,

$$P(a_{X_1}, \ldots, a_{X_n}) = \prod_i P(a_{X_i}).$$

Examples:

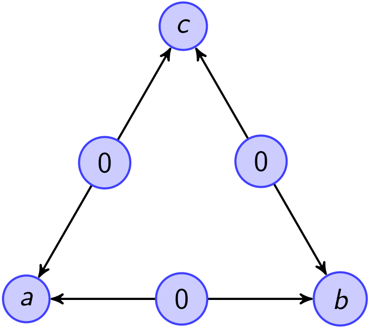
- Any quantum correlation is a correlation.
- In a Bell scenario, $P(a, b, x, y)$ is a correlation if and only if

$$P(a, x, y) = P(a, x) \cdot P(y), \qquad P(b, x, y) = P(b, y) \cdot P(x).$$

$\rightarrow$ equivalent to standard no-signaling equations!

Results on new scenarios (arXiv:1206.5115)

The triangle scenario:



- One possible implementation: three parties and three sources. Space-like separation between: any two parties; every party and the opposite source.

Results on new scenarios II (arXiv:1206.5115)

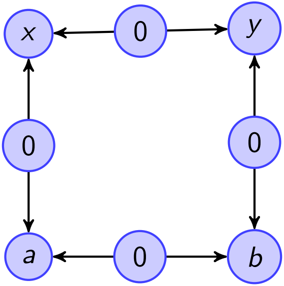
Theorem

In the triangle scenario, there are non-classical quantum correlations $P(a, b, c)$.

- Proof is simple, but not obvious. Idea: let c "simulate" the measurement settings of a bipartite Bell scenario. Use entropic inequalities to reduce to Bell's Theorem.
- Characterizing the set of classical correlations is very challenging!
- It's not even clear how the bound the number of values needed for each hidden variable.

Results on new scenarios III (arXiv:1206.5115)

The square scenario:



Theorem

In the square scenario, there are non-classical correlations $P(a, b, c, d)$.

- Proof is simple, but not obvious. Idea: take $P(a, b, c, d)$ to be given by the $P(a, b, x, y)$ of a PR-box, and find a Hardy-like paradox.
- The existence of non-classical quantum correlations is open.

Beyond quantum information

The present framework is an approach to **causal inference** in the presence of **hidden variables**. This is applicable very generally! One does not need to assume G to be known: test several possibilities for G and see which ones turn the given data into a classical correlation.

One should expect such hidden variables to naturally occur in many fields of science! Whenever there is a system which gets probed at different locations, repeated trials reveal correlations between these locations, but the underlying variables and processes are partially or completely unknown.
$\rightarrow$ E.g.: microbiology, meteorology.

Converse application: use a concrete hidden variable model as a computational paradigm to **generate** a desired complicated pattern of classical correlations.
$\rightarrow$ Example: some existing applications of hidden Markov models.

Lots of open problems!

For example:

- Relation to **quantum information processing protocols**: which protocols are secretly based on these ideas?
  Is the idea of causal hidden variables helpful for the development of new protocols?
- Find bounds on classical and quantum correlations!
- Classify the scenarios: which graphs display non-classical correlations?

Summary

- Bell scenarios are a very special subclass of correlation scenarios!
- The general approach unifies the notions of source, choice of measurement setting and measurement into the notion of **event**: information processing of hidden variables together with output of a classical outcome.
- These **hidden Bayesian networks** are a general idea for doing causal inference in the presence of hidden variables.
- Lots of challenging open problems!