Spherical symmetry as a test case for unconstrained hyperboloidal evolution

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How to go to future lightlike infinity?

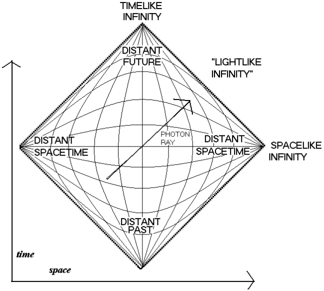
Gravitational waves are only well defined at future null infinity ($\mathscr{I}^+$). This is where the observers of astrophysical events are located.

We want to extract the signal at $\mathscr{I}^+$ - how to get there?

It is useful to conformally compactify our space-time: we conformally rescale the physical metric $\tilde{g}_{\mu\nu}$

$$g_{\mu\nu} \equiv \Omega^2 \tilde{g}_{\mu\nu} \ . \qquad (1)$$

The equations of motion in terms of the rescaled metric $g_{\mu\nu}$ obtain extra terms that diverge at $\mathscr{I}^+$:



$$G_{\mu\nu} = 8\pi\, T_{\mu\nu} - \frac{2}{\Omega}\left(\nabla_\mu \nabla_\nu \Omega - g_{\mu\nu} \nabla^\gamma \nabla_\gamma \Omega\right) - \frac{3}{\Omega^2} g_{\mu\nu} (\nabla_\gamma \Omega) \nabla^\gamma \Omega \ . \qquad (2)$$

What to do?

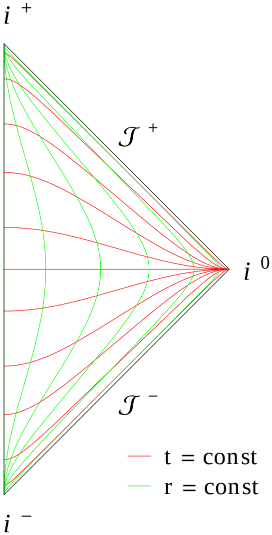
To solve the problem the equations need to be regular and a gauge has to be chosen. This can be done in two ways:

Regularize first - Friedrich's conformal field equations approach
: Here one maintains generality and develop a framework where regularity can be shown. New variables are introduced, which leads to a large system of equations. The gauge can be specified afterwards.

Set gauge condition and then regularize
: One can assume "inertial observers" at $\mathscr{I}^+$ and fixed coordinate location for $\mathscr{I}^+$ from the start. The regularization procedures are adapted to the chosen gauge and the final system is much simpler.

Our aim is a robust code for numerical work and thus we will take the second approach.

How to slice space-time?

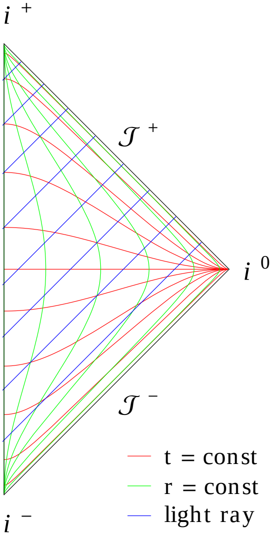


In an initial value formulation we need to slice our space-time.

Spacelike slices are widely used in numerical simulations.

We need a slice that reaches $\mathscr{I}^+$.

How to slice space-time?



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- Use characteristic foliations.

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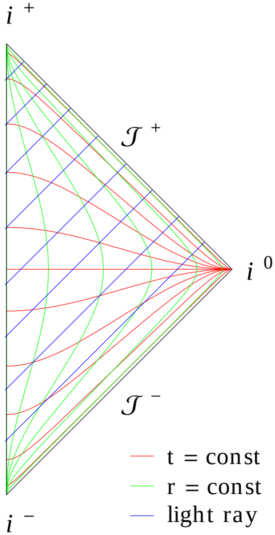

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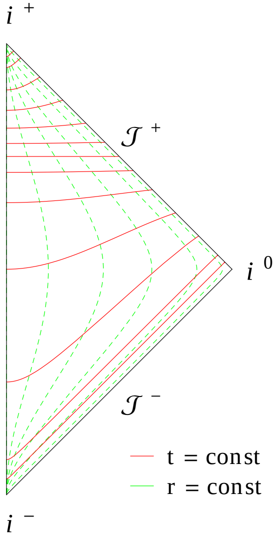
- Use characteristic foliations.
- Match inner spacelike slices with characteristic foliations.

How to slice space-time?



In an initial value formulation we need to slice our space-time.

Spacelike slices are widely used in numerical simulations.

We need a slice that reaches $\mathscr{I}^+$.

- Use characteristic foliations.
- Match inner spacelike slices with characteristic foliations.
- Use hyperboloidal foliations.

Hyperboloidal foliations are spacelike slices that reach $\mathscr{I}^+$.

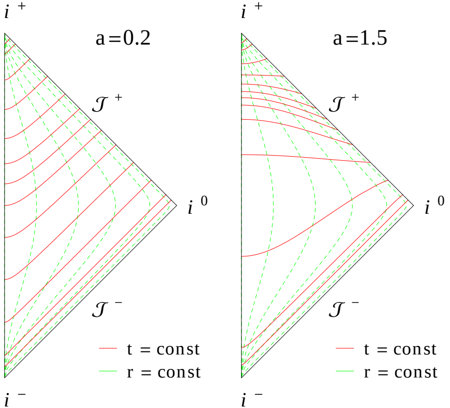
Family of hyperboloidal foliations

They can be described by the new coordinate t

$$t = \tilde{t} - h(\tilde{r}) \ , \qquad (3)$$

related to the old $\tilde{t}$ by a height function

$$h(\tilde{r}) = \sqrt{a^2 + \tilde{r}^2} \ . \quad (4)$$



To reach $\mathscr{I}^+$ we compactify the radial coordinate, introducing the new compactified r and the position of null infinity $r_{\mathscr{I}}$:

$$\tilde{r} = \frac{r}{\Omega} \ , \qquad \Omega = \frac{r_{\mathscr{I}}^2 - r^2}{2\, a\, r_{\mathscr{I}}} \ . \qquad (5)$$

Implementation of set-gauge-then-regularize

To evolve the Einstein equations as an initial value formulation on a hyperboloidal foliation first setting the gauge and then regularizing, we have mainly two options:

Elliptic hyperbolic problem

- constrained evolution
- ✗ global equations, hard to avoid singularities
- ✓ easier to stabilize
- ✓ can impose boundary conditions
- ✗ slower
- Moncrief, Rinne, ...

Purely hyperbolic problem

- free evolution
- ✓ easier to avoid singularities (using excision or punctures)
- ✗ instabilities
- ✗ cannot impose boundary conditions (have to evolve)
- ✓ faster
- Zenginoğlu, Bardeen et al., ...

Evolution variables

We use either the Generalized BSSN formulation or a similar conformal version of the Z4 formulation of the Einstein equations.

The line element with our spherically symmetric metric variables is

$$ds^2 = -\alpha^2 dt^2 + \chi^{-1} \left[\gamma_{rr}(dr + \beta^r dt)^2 + \gamma_{\theta\theta} r^2 \left(d\theta^2 + \sin^2\theta d\phi\right)\right] \ . \quad (6)$$

Also the trace of the extrinsic curvatue K, a component A_{rr} of its trace-free part and the contracted difference of Christoffel symbols Λ^r. And for the Z4 formulation its variables Θ and Z_r.

We add a massless scalar field with evolution equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - 2g^{\mu\nu}\nabla_\mu\Phi\frac{\nabla_\nu\Omega}{\Omega} = 0 \ , \quad (7)$$

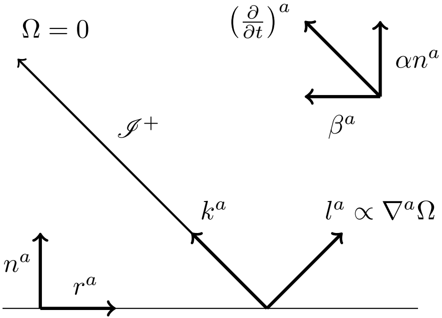
and its spherically symmetric variables are Φ and $\Pi = \dot{\Phi}$.

Scri-fixing condition

To fix the location of $\mathscr{I}^+$ the time vector has to flow along $\mathscr{I}^+$, that is, $\left(\frac{\partial}{\partial t}\right)^a = \alpha n^a + \beta^a$ has to become null at $\mathscr{I}^+$.

Conditions:

$$\left.\left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)_a\right|_{\mathscr{I}} = 0,$$
$$\left.(\nabla^a\Omega)(\nabla_a\Omega)\right|_{\mathscr{I}} = 0,$$
$$\left.\left(\frac{\partial}{\partial t}\right)^a (\nabla_a\Omega)\right|_{\mathscr{I}} = 0.$$



The final expression for β^r is

$$\beta^r_{\mathscr{I}} = a\,\alpha \frac{\chi}{\gamma_{rr}} \partial_r\Omega\bigg|_{\mathscr{I}} . \tag{9}$$

Generalized Bona-Massó slicing condition

The generalized Bona-Massó equation that we use takes the form

$$\dot{\alpha} = \beta^r \alpha' - f(\alpha)\,(K - K_0) + L_0 \ , \tag{10}$$

where we have the freedom to choose the two functions K_0 and L_0.

- The presence of K_0 is necessary to prevent the equation from growing exponentially, as the extrinsic curvature K is negative for flat space-time.
- The source function L_0 is calculated from flat space-time initial data on the hyperboloidal foliation.

Generalized harmonic slicing condition

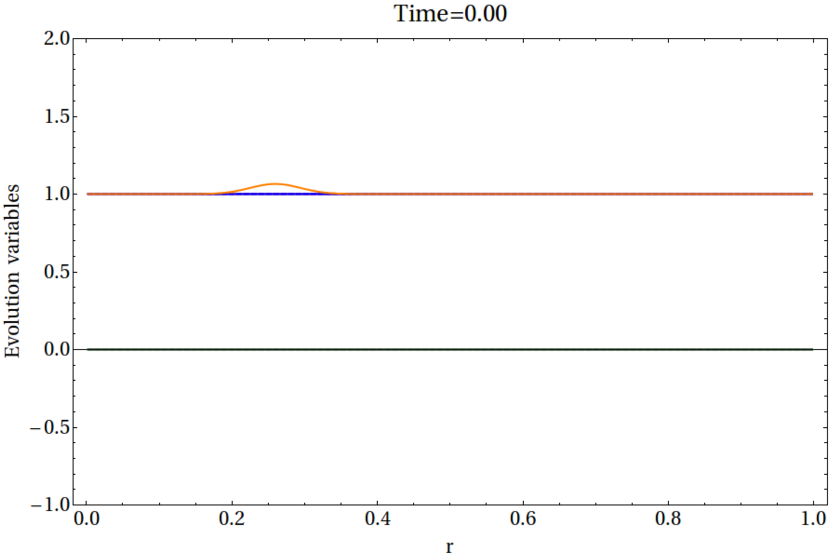
The generalized harmonic equation that we use takes the form

$$\dot{\alpha} = \beta^r \alpha' - \alpha^2 (K - K_0) + L_0 \ , \tag{10}$$

where we have the freedom to choose the two functions K_0 and L_0.

- The presence of K_0 is necessary to prevent the equation from growing exponentially, as the extrinsic curvature K is negative for flat space-time.
- The source function L_0 is calculated from flat space-time initial data on the hyperboloidal foliation.

Evolution: χ, γ_{rr}, $\gamma_{\theta\theta}$, A_{rr}, K, Λ^r, α



Analysis by subsystems

The conformal Einstein equations are 3+1 decomposed into a GBSSN/conformal Z4 form and reduced to spherical symmetry. However, the system is unstable when evolved numerically.

Divide the original system of equations (GBSSN) to find instabilities:

1. Single equations: useful for detecting exponential growths. Make sure all terms like $\dot{A} = \lambda A$ are controlled.

- To further simplify the (sub)systems of equations we can eliminate γ_{rr} using $\gamma_{\theta\theta}$ as $\gamma_{rr} \equiv \gamma_{\theta\theta}^{-2}$.

2. $\chi\,\gamma_{\theta\theta}\,A_{rr}\,\Lambda^r$ subsystem: is the largest hyperbolic subsystem that can be constructed with these variables and it is stable.

3. $\alpha\,K$ subsystem: exponential growth caused by a term $\dot{K} = \frac{\lambda}{\Omega}K$ with $\lambda > 0$. Solved with a variable transformation for K.

Constraint damping

Many of the final equations possess a damping term of the form

$$\dot{A} = \frac{\lambda}{\Omega} A \; . \tag{11}$$

The values of λ for each variable evaluated at $\mathscr{I}^+$ $(r = r_{\mathscr{I}})$ are:

$\dot{\chi}$	$\dot{\gamma}_{rr}$	$\dot{\gamma}_{\theta\theta}$	$\dot{A}_{rr}$	$\dot{K}$	$\dot{\Lambda}^r$	$\dot{\alpha}$
0	0	0	$-\frac{2r_{\mathscr{I}}}{a^2}$	$-\frac{r_{\mathscr{I}}}{a^2}$	0	$-\frac{3r_{\mathscr{I}}}{a^2}$

The variable Λ^r does not have a damping term $\rightarrow$ add it using the constraint

$$\mathcal{C}^r = \Lambda^r + \frac{2}{\gamma_{rr} r} - \frac{2}{\gamma_{\theta\theta} r} - \frac{\gamma'_{rr}}{2\gamma_{rr}^2} + \frac{\gamma'_{\theta\theta}}{\gamma_{\theta\theta}\gamma_{rr}} \tag{12}$$

and the system becomes stable!

Spherically symmetric equations

$$\dot{\chi} = \beta^r \chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r \gamma_{rr}' \chi}{3\gamma_{rr}} - \frac{2\beta^r \gamma_{\theta\theta}' \chi}{3\gamma_{\theta\theta}} - \frac{2\beta^{r\prime}\chi}{3} - \frac{4\beta^r\chi}{3r} + \frac{2\beta^r \chi \Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}, \quad (13a)$$

Spherically symmetric equations

$$\dot{\chi} = \beta^r \chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r \gamma'_{rr}\chi}{3\gamma_{rr}} - \frac{2\beta^r \gamma'_{\theta\theta}\chi}{3\gamma_{\theta\theta}} - \frac{2\beta^{r\prime}\chi}{3} - \frac{4\beta^r\chi}{3r} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}, \quad (13a)$$

$$\dot{\gamma_{rr}} = -2A_{rr}\alpha + \frac{2\beta^r\gamma'_{rr}}{3} - \frac{2\gamma_{rr}\beta^r\gamma'_{\theta\theta}}{3\gamma_{\theta\theta}} + \frac{4\gamma_{rr}\beta^{r\prime}}{3} - \frac{4\gamma_{rr}\beta^r}{3r}, \quad (13b)$$

Spherically symmetric equations

$$\dot{\chi} = \beta^r\chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r\gamma_{rr}'\chi}{3\gamma_{rr}} - \frac{2\beta^r\gamma_{\theta\theta}'\chi}{3\gamma_{\theta\theta}} - \frac{2\beta^{r\prime}\chi}{3} - \frac{4\beta^r\chi}{3r} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}, \quad (13a)$$

$$\dot{\gamma_{rr}} = -2A_{rr}\alpha + \frac{2\beta^r\gamma_{rr}'}{3} - \frac{2\gamma_{rr}\beta^r\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{4\gamma_{rr}\beta^{r\prime}}{3} - \frac{4\gamma_{rr}\beta^r}{3r}, \quad (13b)$$

$$\dot{\gamma_{\theta\theta}} = \frac{A_{rr}\gamma_{\theta\theta}\alpha}{\gamma_{rr}} - \frac{\gamma_{\theta\theta}\beta^r\gamma_{rr}'}{3\gamma_{rr}} + \frac{\beta^r\gamma_{\theta\theta}'}{3} - \frac{2\gamma_{\theta\theta}\beta^{r\prime}}{3} + \frac{2\gamma_{\theta\theta}\beta^r}{3r}, \quad (13c)$$

Spherically symmetric equations

$$\dot{\chi} = \beta^r \chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r \gamma_{rr}' \chi}{3\gamma_{rr}} - \frac{2\beta^r \gamma_{\theta\theta}' \chi}{3\gamma_{\theta\theta}} - \frac{2{\beta^r}'\chi}{3} - \frac{4\beta^r \chi}{3r} + \frac{2\beta^r \chi \Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}, \quad (13a)$$

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$$\dot{\gamma_{\theta\theta}} = \frac{A_{rr}\gamma_{\theta\theta}\alpha}{\gamma_{rr}} - \frac{\gamma_{\theta\theta}\beta^r \gamma_{rr}'}{3\gamma_{rr}} + \frac{\beta^r \gamma_{\theta\theta}'}{3} - \frac{2\gamma_{\theta\theta}{\beta^r}'}{3} + \frac{2\gamma_{\theta\theta}\beta^r}{3r}, \quad (13c)$$

$$\begin{aligned}
\dot{A_{rr}} = {} & \beta^r A_{rr}' + \frac{2}{3}\gamma_{rr}\alpha\chi{\Lambda^r}' - \frac{\alpha\chi\gamma_{rr}''}{3\gamma_{rr}} + \frac{\alpha\chi\gamma_{\theta\theta}''}{3\gamma_{\theta\theta}} - \frac{2\chi\alpha''}{3} + \frac{\alpha\chi''}{3} + \alpha A_{rr}\left[K + 2(1-C_{Z4c})\Theta\right] \\
& - \frac{2\alpha A_{rr}^2}{\gamma_{rr}} + \frac{4{\beta^r}' A_{rr}}{3} - \frac{4\beta^r A_{rr}}{3r} + \frac{\alpha\chi\left(\gamma_{rr}'\right)^2}{2\gamma_{rr}^2} - \frac{2\alpha\chi\left(\gamma_{\theta\theta}'\right)^2}{3\gamma_{\theta\theta}^2} - \frac{\alpha\left(\chi'\right)^2}{6\chi} - \frac{2\gamma_{rr}\alpha\Lambda^r\chi}{3r} \\
& + \frac{2\alpha\chi}{r^2}\left(\frac{\gamma_{rr}}{\gamma_{\theta\theta}} - 1\right) - \frac{A_{rr}\beta^r\gamma_{rr}'}{3\gamma_{rr}} - \frac{2A_{rr}\beta^r\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{\alpha\Lambda^r\chi\gamma_{rr}'}{3} - \frac{\alpha\Lambda^r\chi\gamma_{rr}\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} - \frac{2\alpha\chi\gamma_{rr}'}{3\gamma_{\theta\theta}r} \\
& + \frac{2\alpha\chi\gamma_{rr}\gamma_{\theta\theta}'}{\gamma_{\theta\theta}^2 r} - \frac{4\alpha\chi\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}r} + \frac{\chi\alpha'\gamma_{rr}'}{3\gamma_{rr}} + \frac{\chi\alpha'\gamma_{\theta\theta}'}{3\gamma_{\theta\theta}} + \frac{2\chi\alpha'}{3r} - \frac{2\alpha'\chi'}{3} - \frac{\alpha\chi'\gamma_{rr}'}{6\gamma_{rr}} - \frac{\alpha\chi'\gamma_{\theta\theta}'}{6\gamma_{\theta\theta}} \\
& - \frac{\alpha\chi'}{3r} + \frac{4}{3}Z_r\alpha\chi' - \frac{2\alpha\chi\gamma_{rr}'\Omega'}{3\gamma_{rr}\Omega} - \frac{2\alpha\chi\gamma_{\theta\theta}'\Omega'}{3\gamma_{\theta\theta}\Omega} + \frac{4\alpha\chi'\Omega'}{3\Omega} + \frac{A_{rr}\beta^r\Omega'}{\Omega} - \frac{3\alpha A_{rr}}{a\Omega} \\
& + \frac{4\alpha\chi\Omega''}{3\Omega} - \frac{4\alpha\chi\Omega'}{3r\Omega} + \frac{8Z_r\alpha\chi\Omega'}{3\Omega} - \frac{16}{3}\pi\alpha\chi\left(\Phi'\right)^2, \quad (13d)
\end{aligned}$$

Spherically symmetric equations

$$\dot{\chi} = \beta^r\chi' + \frac{2\alpha\chi(K+2\Theta)}{3} - \frac{\beta^r\gamma'_{rr}\chi}{3\gamma_{rr}} - \frac{2\beta^r\gamma'_{\theta\theta}\chi}{3\gamma_{\theta\theta}} - \frac{2\beta^{r\prime}\chi}{3} - \frac{4\beta^r\chi}{3r} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}\ , \tag{13a}$$

$$\dot{\gamma_{rr}} = -2A_{rr}\alpha + \frac{2\beta^r\gamma'_{rr}}{3} - \frac{2\gamma_{rr}\beta^r\gamma'_{\theta\theta}}{3\gamma_{\theta\theta}} + \frac{4\gamma_{rr}\beta^{r\prime}}{3} - \frac{4\gamma_{rr}\beta^r}{3r}\ , \tag{13b}$$

$$\dot{\gamma_{\theta\theta}} = \frac{A_{rr}\gamma_{\theta\theta}\alpha}{\gamma_{rr}} - \frac{\gamma_{\theta\theta}\beta^r\gamma'_{rr}}{3\gamma_{rr}} + \frac{\beta^r\gamma'_{\theta\theta}}{3} - \frac{2\gamma_{\theta\theta}\beta^{r\prime}}{3} + \frac{2\gamma_{\theta\theta}\beta^r}{3r}\ , \tag{13c}$$

$$\begin{aligned}\dot{A_{rr}} = {} & \beta^r A'_{rr} + \frac{2}{3}\gamma_{rr}\alpha\chi\Lambda^{r\prime} - \frac{\alpha\chi\gamma''_{rr}}{3\gamma_{rr}} + \frac{\alpha\chi\gamma''_{\theta\theta}}{3\gamma_{\theta\theta}} - \frac{2\chi\alpha''}{3} + \frac{\alpha\chi''}{3} + \alpha A_{rr}\left[K+2(1-C_{Z4c})\Theta\right] \\
& - \frac{2\alpha A_{rr}^2}{\gamma_{rr}} + \frac{4\beta^{r\prime}A_{rr}}{3} - \frac{4\beta^r A_{rr}}{3r} + \frac{\alpha\chi\left(\gamma'_{rr}\right)^2}{2\gamma_{rr}^2} - \frac{2\alpha\chi\left(\gamma'_{\theta\theta}\right)^2}{3\gamma_{\theta\theta}^2} - \frac{\alpha\left(\chi'\right)^2}{6\chi} - \frac{2\gamma_{rr}\alpha\Lambda^r\chi}{3r} \\
& + \frac{2\alpha\chi}{r^2}\left(\frac{\gamma_{rr}}{\gamma_{\theta\theta}} - 1\right) - \frac{A_{rr}\beta^r\gamma'_{rr}}{3\gamma_{rr}} - \frac{2A_{rr}\beta^r\gamma'_{\theta\theta}}{3\gamma_{\theta\theta}} + \frac{\alpha\Lambda^r\chi\gamma'_{rr}}{3} - \frac{\alpha\Lambda^r\chi\gamma_{rr}\gamma'_{\theta\theta}}{3\gamma_{\theta\theta}} - \frac{2\alpha\chi\gamma'_{rr}}{3\gamma_{\theta\theta}r} \\
& + \frac{2\alpha\chi\gamma_{rr}\gamma'_{\theta\theta}}{\gamma_{\theta\theta}^2 r} - \frac{4\alpha\chi\gamma'_{\theta\theta}}{3\gamma_{\theta\theta}r} + \frac{\chi\alpha'\gamma'_{rr}}{3\gamma_{rr}} + \frac{\chi\alpha'\gamma'_{\theta\theta}}{3\gamma_{\theta\theta}} + \frac{2\chi\alpha'}{3r} - \frac{2\alpha'\chi'}{3} - \frac{\alpha\chi'\gamma'_{rr}}{6\gamma_{rr}} - \frac{\alpha\chi'\gamma'_{\theta\theta}}{6\gamma_{\theta\theta}} \\
& - \frac{\alpha\chi'}{3r} + \frac{4}{3}Z_r\alpha\chi' - \frac{2\alpha\chi\gamma'_{rr}\Omega'}{3\gamma_{rr}\Omega} - \frac{2\alpha\chi\gamma'_{\theta\theta}\Omega'}{3\gamma_{\theta\theta}\Omega} + \frac{4\alpha\chi'\Omega'}{3\Omega} + \frac{A_{rr}\beta^r\Omega'}{\Omega} - \frac{3\alpha A_{rr}}{a\Omega} \\
& + \frac{4\alpha\chi\Omega''}{3\Omega} - \frac{4\alpha\chi\Omega'}{3r\Omega} + \frac{8Z_r\alpha\chi\Omega'}{3\Omega} - \frac{16}{3}\pi\alpha\chi\left(\Phi'\right)^2\ ,\end{aligned} \tag{13d}$$

Spherically symmetric equations

$$\dot{\chi} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{2\beta^r\chi\Omega'}{\Omega} - \frac{2\alpha\chi}{a\Omega}\,, \tag{13a}$$

$$\dot{\gamma_{rr}} = \text{Principal part} + \ldots\,, \tag{13b}$$

$$\dot{\gamma_{\theta\theta}} = \text{Principal part} + \ldots\,, \tag{13c}$$

$$\dot{A_{rr}} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{A_{rr}\beta^r\Omega'}{\Omega} - \frac{3\alpha A_{rr}}{a\Omega} + \Omega\text{ terms} + \text{Matter terms}\,, \tag{13d}$$

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$$\begin{aligned}\dot{K} = {} & \beta^r K' - \frac{\chi\alpha''}{\gamma_{rr}} + \frac{\alpha}{3}(K+2\Theta)^2 + \frac{3\alpha A_{rr}^2}{2\gamma_{rr}^2} + \frac{\alpha'\gamma_{rr}'\chi}{2\gamma_{rr}^2} - \frac{\alpha'\gamma_{\theta\theta}'\chi}{\gamma_{rr}\gamma_{\theta\theta}} - \frac{2\alpha'\chi}{\gamma_{rr}r} + \frac{\alpha'\chi'}{2\gamma_{rr}} \\ & + \frac{\kappa_1(1-\kappa_2)\alpha\Theta}{\Omega} + \frac{2C_{Z4c}Z_r\chi\alpha'}{\gamma_{rr}} - \frac{\alpha\chi\gamma_{rr}'\Omega'}{2\gamma_{rr}^2\Omega} + \frac{\alpha\chi\gamma_{\theta\theta}'\Omega'}{\gamma_{rr}\gamma_{\theta\theta}\Omega} + \frac{3\chi\alpha'\Omega'}{\gamma_{rr}\Omega} - \frac{\alpha\chi'\Omega'}{2\gamma_{rr}\Omega} \\ & - \frac{2Z_r\alpha\chi\Omega'}{\gamma_{rr}\Omega} + \frac{\left[K+2(1-C_{Z4c})\Theta\right]\beta^r\Omega'}{\Omega} - \frac{2\alpha(K+2\Theta)}{a\Omega} + \frac{2\alpha\chi\Omega'}{\gamma_{rr}r\Omega} + \frac{\alpha\chi\Omega''}{\gamma_{rr}\Omega} \\ & + \frac{3\alpha}{a^2\Omega^2} - \frac{3\alpha\chi\left(\Omega'\right)^2}{\gamma_{rr}\Omega^2} + \frac{8\pi(\Pi-\beta^r\Phi')^2}{\alpha} , \end{aligned} \quad (13e)$$

Spherically symmetric equations

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$$\dot{\gamma_{\theta\theta}} = \text{Principal part} + \ldots\ , \tag{13c}$$

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$$\dot{K} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{K\beta^r\Omega'}{\Omega} - \frac{2\alpha K}{a\Omega} + \Omega\ \text{terms} + \text{Matter terms}\ , \tag{13e}$$

Spherically symmetric equations

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$$\dot{K} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{K\beta^r\Omega'}{\Omega} - \frac{2\alpha K}{a\Omega} + \Omega \text{ terms} + \text{Matter terms} \ , \quad (13e)$$

$$\dot{\Lambda^r} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{\xi_{\Lambda^r}\Lambda^r\beta^{r\prime}}{a\Omega} + \Omega \text{ terms} + \text{Matter terms} \ , \quad (13f)$$

$$\dot{\alpha} = \text{Principal part} + \ldots - \frac{3\alpha\beta^r\Omega'}{\Omega} + \Omega \text{ terms} \ , \quad (13g)$$

$$\dot{\Theta} = \text{Princ.} + \ldots + \frac{C_{Z4c}\Theta\beta^r\Omega'}{\Omega} + \frac{(3C_{Z4c}-4)\alpha\Theta}{a\Omega} - \frac{\kappa_1(2+\kappa_2)\alpha\Theta}{\Omega} + \Omega \text{ terms} + \text{Mat.} \ , \quad (13h)$$

$$\dot{Z}_r = \text{Principal part} + \ldots + \frac{2\alpha Z_r}{a\Omega} - \frac{\kappa_1\alpha Z_r}{\Omega} + \frac{2\xi_{Z_r}Z_r\beta^{r\prime}}{a\gamma_{rr}\Omega} + \Omega \text{ terms} + \text{Matter terms} \ , \quad (13i)$$

$$\dot{\Phi} = \text{Principal part} \ , \quad (13j)$$

$$\dot{\Pi} = \text{Principal part} + \ldots + \text{Z4 terms} + \frac{3\Pi\beta^r\Omega'}{\Omega} - \frac{3\alpha\Pi}{a\Omega} + \Omega \text{ terms} \ . \quad (13k)$$

Simulation setup

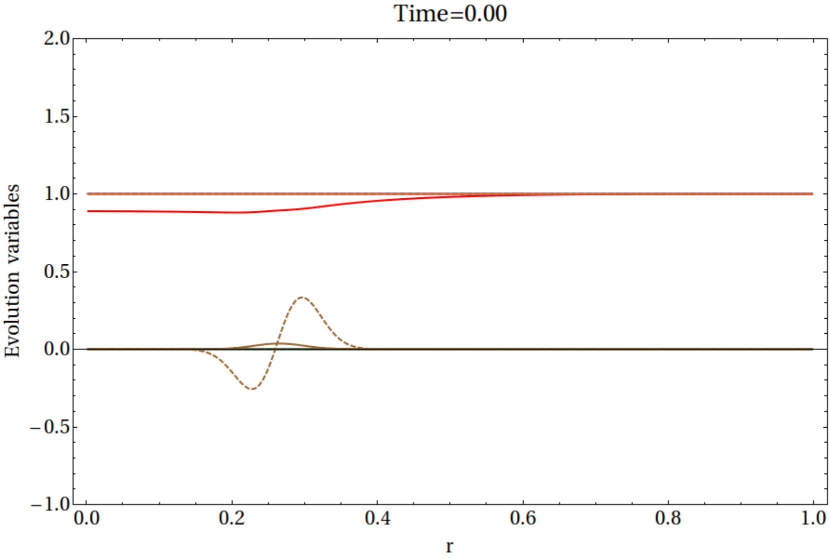
The numerical code uses:

- Method of Lines
    - 4th order Runge-Kutta time integrator
    - Finite Differences of orders 2, 4, 6 or 8
- Kreiss-Oliger dissipation
- Staggered grid

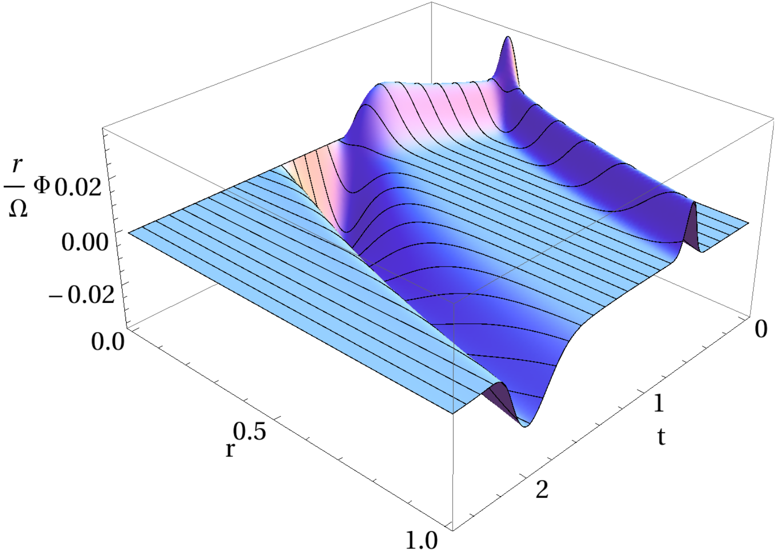
The equations have been tested for:

- Gauge waves: vacuum solution with perturbation set on α.
- Scalar field: initial perturbation set on the scalar field.

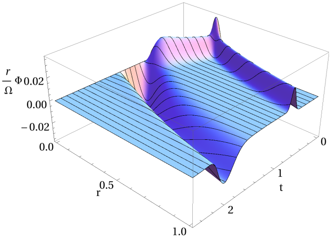
Evolution: χ, γ_{rr}, $\gamma_{\theta\theta}$, A_{rr}, K, Λ^r, α, $\frac{r}{\Omega}\Phi$, Π

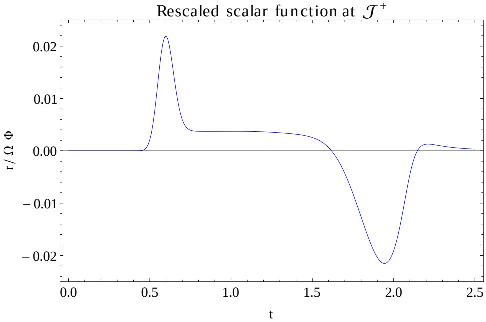


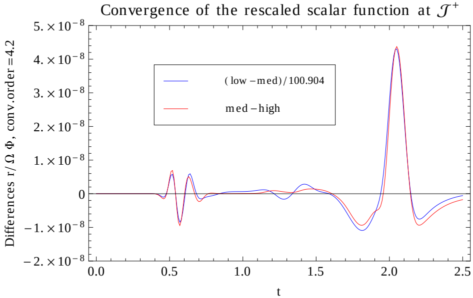
Scalar field



Scalar field







Summary

- We set up the problem in spherical symmetry using the conformally compactified equations on hyperboloidal foliations.
- The appropriate gauge conditions were set and, after finding out the system was unstable, the equations were successfully regularized.
- We have been able to evolve the scalar field coupled to the Einstein equations, as well as to extract its signal at $\mathscr{I}^+$.
- Next steps: BHs, collapse of scalar field, use different gauge conditions in the interior of the integration domain and on $\mathscr{I}^+$.

Thank you for your attention!

Questions?

3+1 decomposed equations

$$\partial_\perp \bar\gamma_{ab} = -2\alpha \bar K_{ab} \,, \quad (14a)$$

$$\partial_\perp \bar K_{ab} = \alpha\left[R[\bar D]_{ab} - 2\bar K^c_a \bar K_{bc} + \bar K_{ab}(K-2C_{Z4c}\Theta)+2\bar D_{(a}Z_{b)} - \frac{\kappa_1(1+\kappa_2)\bar\gamma_{ab}\Theta}{\Omega}\right] - \bar D_b \bar D_a \alpha$$
$$+\frac{3\bar\gamma_{ab}\left[(\partial_\perp\Omega)^2-\alpha^2\bar D^c\Omega\bar D_c\Omega\right]}{\alpha\Omega^2} + \frac{4\alpha Z_{(a}\bar D_{b)}\Omega}{\Omega} + \frac{2\alpha\bar D_b\bar D_a\Omega}{\Omega} - \frac{2\alpha\bar\gamma_{ab}Z^c\bar D_c\Omega}{\Omega}$$
$$+\frac{\bar\gamma_{ab}\bar D^c\alpha\bar D_c\Omega}{\Omega} + \frac{\alpha\bar\gamma_{ab}\bar\triangle\Omega}{\Omega} + \frac{2\bar K_{ab}\partial_\perp\Omega}{\Omega} + \frac{\bar\gamma_{ab}(K-2C_{Z4c}\Theta)\partial_\perp\Omega}{\Omega}$$
$$+\frac{\bar\gamma_{ab}\partial_\perp\alpha\partial_\perp\Omega}{\alpha^2\Omega} - \frac{\bar\gamma_{ab}\partial_\perp\partial_\perp\Omega}{\alpha\Omega} + 4\pi\alpha\left[\bar\gamma_{ab}(S-\rho)-2S_{ab}\right], \quad (14b)$$

$$\partial_\perp\Theta = \frac{\alpha}{2}\left[R[\bar D] - \bar K_{ab}\bar K^{ab} + K(K-2C_{Z4c}\Theta)+2\bar D_a Z^a - \frac{2\kappa_1(2+\kappa_2)\Theta}{\Omega}\right] - C_{Z4c}Z^a\bar D_a\alpha$$
$$+\frac{3\left[(\partial_\perp\Omega)^2-\alpha^2\bar D^a\Omega\bar D_a\Omega\right]}{\alpha\Omega^2} + \frac{2\alpha\bar\triangle\Omega}{\Omega} + \frac{2(K-2C_{Z4c}\Theta)\partial_\perp\Omega}{\Omega} - 8\pi\alpha\rho \,, \quad (14c)$$

$$\partial_\perp Z_a = \alpha\left[\bar D_b\bar K^b_a - \bar D_a K + \bar D_a\Theta - 2\bar K_{ab}Z^b - \frac{\kappa_1 Z_a}{\Omega}\right] - C_{Z4c}\Theta\bar D_a\alpha + \frac{2\alpha\Theta\bar D_a\Omega}{\Omega}$$
$$-\frac{2\bar D_a\partial_\perp\Omega}{\Omega} - \frac{2\alpha\bar K_a{}^b\bar D_b\Omega}{\Omega} - \frac{2Z_a\partial_\perp\Omega}{\Omega} + \frac{2\bar D_a\alpha\partial_\perp\Omega}{\alpha\Omega} - 8\pi\alpha J_a \,. \quad (14d)$$

$$\mathcal{H} = R[\bar D] - \bar K_{ab}\bar K^{ab} + K^2 + \frac{6\left[(\partial_\perp\Omega)^2-\alpha^2\bar D^a\Omega\bar D_a\Omega\right]}{\alpha^2\Omega^2} + \frac{4\bar\triangle\Omega}{\Omega} + \frac{4K\partial_\perp\Omega}{\alpha\Omega} - 16\pi\rho \,, \quad (15a)$$

$$\mathcal{M}^a = \bar D_b\bar K^{ab} - \bar\gamma^{ab}\bar D_b K - \frac{2\bar K^{ab}\bar D_b\Omega}{\Omega} - \frac{2\bar\gamma^{ab}\bar D_b\partial_\perp\Omega}{\alpha\Omega} + \frac{2\bar\gamma^{ab}\bar D_b\alpha\partial_\perp\Omega}{\alpha^2\Omega} - 8\pi J^a \,. \quad (15b)$$