Loop corrections in inflation

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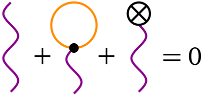
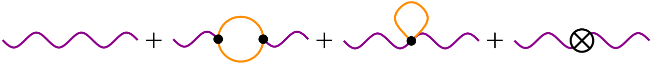
Benasque, September 9th, 2013

work done with and under the supervision of E. Verdaguer and A. Roura,
based on JCAP 08 (2012) 009 (arXiv:1205.3097) and soon-to-be-published work

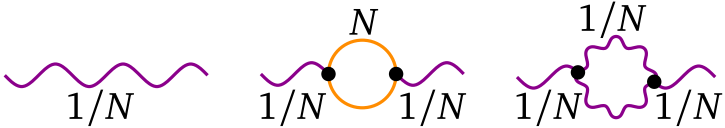
Introduction

- de Sitter (dS) as an idealized model of inflationary geometry (cosmological constant Λ drives inflation)
- Standard perturbation theory is tree level, but there are possibly large quantum corrections from back-reaction and accumulation of small effects
- Back-reaction – semi-classical Einstein equations $G_{ab} + \Lambda g_{ab} = \frac{\kappa^2}{2} \langle T_{ab}[g] \rangle$ (have to be solved self-consistently)
- More back-reaction – fluctuations of the metric $G_{ab}[g + \kappa h] + \Lambda(g_{ab} + \kappa h_{ab}) = \frac{\kappa^2}{2} \langle T_{ab}[g + \kappa h] \rangle$
- Semi-classical background: 

- Metric fluctuations:



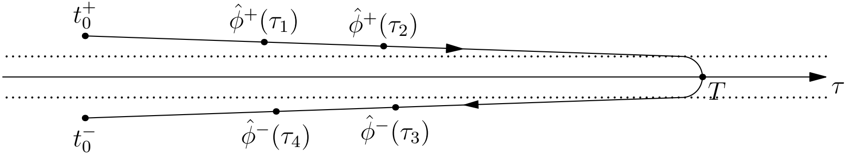
Milestones of the calculation

- Use (almost) standard QFT techniques
- Starting point: $S = \frac{1}{\kappa^2} \int (R[g+\kappa h] - 2\Lambda) \underbrace{\sqrt{-(g+\kappa h)}\, \mathrm{d}^n x}_{\mathrm{d}x} + S_\mathrm{M}[g+\kappa h]$
- Massless, conformally coupled free scalar fields:
  $S_\mathrm{CS}[g,\phi] = -\frac{1}{2} \int \left(\nabla^a \phi \nabla_a \phi + \frac{(n-2)}{4(n-1)} R \phi^2 \right) \mathrm{d}x$
- EFT approach: valid for small curvature and large distance
- How to suppress graviton loops?
- $1/N$-expansion: $1/\kappa^2 = N/(\kappa')^2$, $S_\mathrm{M} = N S_\mathrm{CS}$ and take
  $N \to \infty, \kappa' = \mathrm{const.}$
- 

- Matrix elements via path integral $\langle \text{out}|A|\text{in}\rangle = \int A[\phi]\mathrm{e}^{\mathrm{i}S[\phi]}\mathcal{D}\phi$
  $|\text{in}\rangle$ and $\langle\text{out}|$ are suitable lowest-energy eigenstates ("vacuum") in the infinite past and future (in general: vacuum functionals)
  A is time-ordered
- However, we want expectation values $\langle \text{in}|A|\text{in}\rangle$
- Sum over states at final time T: $\langle \text{in}|A|\text{in}\rangle = \sum_\alpha \langle \text{in}|\alpha(T)\rangle\langle\alpha(T)|A|\text{in}\rangle$
- Double path integral ("doubling of fields")

$$\langle \text{in}|AB|\text{in}\rangle = \int A[\phi^+]B[\phi^-]\mathrm{e}^{\mathrm{i}S[\phi^+]-\mathrm{i}S[\phi^-]}\delta(\phi^+(T)-\phi^-(T))\mathcal{D}\phi^+\mathcal{D}\phi^-$$

  A is time-ordered, B is anti-time-ordered
- Alternatively: closed-time-path (CTP)



- Integrate out matter fields:

$$\langle \mathrm{in}|A[h]B[h]|\mathrm{in}\rangle = \int A[h^+]B[h^-]\mathrm{e}^{\mathrm{i}S[h^+,\phi^+]-\mathrm{i}S[h^-,\phi^-]}\times$$
$$\times\,\delta(\phi^+(T)-\phi^-(T))\delta(h^+(T)-h^-(T))\mathcal{D}\phi^\pm\mathcal{D}h^\pm$$
$$= \int A[h^+]B[h^-]\mathrm{e}^{\mathrm{i}S_{\text{eff}}[h^\pm]}\delta(h^+(T)-h^-(T))\mathcal{D}h^\pm$$

- Obtain divergences and introduce counter-terms quadratic in curvature tensors $S_{R^2}[h] = \alpha \int \left(R^{abcd}R_{abcd} - R^{ab}R_{ab}\right)\mathrm{d}x + \beta \int R^2\,\mathrm{d}x$
- Conformal fields renormalize only α (in dimensional regularization)
- Finite effective action

$$S_{\text{eff}}[h^\pm] = -\mathrm{i}\lim_{n\to 4}\ln\left\langle \mathrm{e}^{\mathrm{i}(S[h^+,\phi^+]+S_{R^2}[h^+])-\mathrm{i}(S[h^-,\phi^-]+S_{R^2}[h^-])}\right\rangle_\phi$$

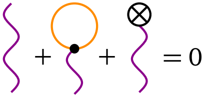
- explicitly calculated in Poincaré patch of dS — Campos, Verdaguer '94,'96

Semiclassical background

- $1/N$ expansion equivalent to truncating $S_{\text{eff}}[h^\pm]$ to $\mathcal{O}\left(h^2\right)$ – h_{ab} is free field with "quantum-corrected" action
- Quantum equations of motion by functional derivative $\delta S_{\text{eff}}/\delta h^\pm_{ab} = 0$
- Vanishing perturbation $h^\pm = 0$ gives the semi-classical Einstein equations

$$G_{ab} + \Lambda g_{ab} + \alpha\kappa^2 A_{ab} + \beta\kappa^2 B_{ab} = \kappa^2/2\langle T_{ab}[\phi]\rangle_\phi$$

with $T_{ab}[\phi] = -2\delta S_{\text{M}}[\phi]/\delta g^{ab}$



- Self-consistent de Sitter solution with $\Lambda_{\text{eff}} = \Lambda\left(1 + \#N\kappa^2\Lambda\right)$ (long known: for dS-invariant state, $\langle T_{ab}[\phi]\rangle_\phi \propto g_{ab}$)

Metric fluctuations

- For correlation functions $\mathcal{G}^{MN}_{abcd}(x,y) = -\mathrm{i}\langle 0|h^M_{ab}(x)h^N_{cd}(x')|0\rangle$, use perturbative expansion in κ^2
- Split metric perturbation in scalar, vector, tensor parts and fix gauge $h_{ab} = \delta^0_a\delta^0_b\psi + \delta^0_{(a}\nabla_{b)}\chi + \nabla_{(a}v^{\mathrm{T}}_{b)} + h^{\mathrm{TT}}_{ab}$, equations decouple
- h^{TT}_{ab} dynamical (hyperbolic equation), ψ, χ and v^{T}_a constrained by $T_{ab}[\phi]$ (elliptic equation – algebraic in spatial Fourier space)
- Tensor correlation function

$$\mathcal{G}^{AB}_{abcd}(x,y) = G^{AB}_{abcd}(x,y)$$
$$- \kappa^2 \int G^{AM}_{abmn}(x,x') \left(\frac{\delta^2 S^{(2)}_{\mathrm{eff}}[h^\pm]}{\delta h^M_{mn}(x') h^N_{pq}(y')} \right) G^{NB}_{pqcd}(y',y) \,\mathrm{d}x' \,\mathrm{d}y' + \mathcal{O}\left(\kappa^4\right)$$

- Scalar, vector correlation function

$$\mathcal{G}^{AB}_{abcd}(x,y) \propto \frac{1}{\triangle} \left(\frac{\delta^2 S^{(2)}_{\mathrm{eff}}[h^\pm]}{\delta h^A_{ab}(x') h^B_{cd}(y')} \right)$$

- Full result

$$
\begin{aligned}
\mathcal{G}^{-+}_{abcd}(\eta,\eta',\mathbf{p}) = -\mathrm{i}\,(P_{ac}P_{bd}+P_{ad}P_{bc}-P_{ab}P_{cd})\Bigg[& \frac{(|\mathbf{p}|\eta-\mathrm{i})(|\mathbf{p}|\eta'+\mathrm{i})}{2H^2\eta^2(\eta')^2|\mathbf{p}|^3}\mathrm{e}^{-\mathrm{i}|\mathbf{p}|(\eta-\eta')}\left(1+6\alpha\kappa^2H^2\ln\left(\frac{\bar{\mu}}{H}\right)-(5\alpha-2\beta)\,\kappa^2H^2\right) \\
& + \frac{3}{2}\alpha\kappa^2\left(\frac{2}{|\mathbf{p}|\eta\eta'}\mathrm{e}^{-\mathrm{i}|\mathbf{p}|(\eta-\eta')}+S(\eta-\eta',\mathbf{p})\right) \\
& - \frac{3}{2}\alpha\kappa^2\frac{1}{\eta^2(\eta')^2}\left(I_2(\eta,\eta',\mathbf{p})+I_2^*(\eta',\eta,\mathbf{p})-I_3(\eta,\eta',\mathbf{p})+I_4(\eta,\eta',\mathbf{p})\right)\Bigg] \\
+ \frac{\mathrm{i}}{2}\alpha\kappa^2\Bigg[& 12\delta^0_{(a}P_{b)(c}\delta^0_{d)}\frac{1}{\mathbf{p}^2}\left(\partial_\eta^2+\mathbf{p}^2\right)+9\delta^0_{(a}p_{b)}\delta^0_{(c}p_{d)}\frac{1}{(\mathbf{p}^2)^2}\partial_\eta^2+3\delta^0_{(a}p_{b)}\delta^0_{(c}p_{d)}\frac{\eta-\eta'}{\mathbf{p}^2}\partial_\eta \\
& - \delta^0_{(a}p_{b)}\delta^0_{(c}p_{d)}\eta\eta' - 3\mathrm{i}\delta^0_a\delta^0_b\delta^0_{(c}p_{d)}\frac{\eta}{\mathbf{p}^2}\partial_\eta^2+3\mathrm{i}\delta^0_{(a}p_{b)}\delta^0_c\delta^0_d\frac{\eta'}{\mathbf{p}^2}\partial_\eta^2 \\
& + \mathrm{i}\delta^0_{(a}\left(\delta^0_{b)}p_{(c}+p_{b)}\delta^0_{(c}\right)\delta^0_{d)}\eta\eta'\partial_\eta+\delta^0_a\delta^0_b\delta^0_c\delta^0_d\eta\eta'\partial_\eta^2\Bigg]S(\eta-\eta',\mathbf{p})\,,
\end{aligned}
$$

with

$$
P_{ab} = \eta_{ab}+\delta^0_a\delta^0_b-\frac{p_ap_b}{\mathbf{p}^2}\,,\quad S(\eta-\eta',\mathbf{p}) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}|\mathbf{p}|(\eta-\eta')}\mathcal{P}\frac{1}{\eta-\eta'}+\pi\delta(\eta-\eta')\,,
$$

and

$$
\begin{aligned}
I_2(\eta,\eta',\mathbf{p}) &= |\mathbf{p}|^{-3}\mathrm{e}^{\mathrm{i}|\mathbf{p}|(\eta+\eta')}\,(|\mathbf{p}|\eta+\mathrm{i})\left(|\mathbf{p}|\eta'+\mathrm{i}\right)\left[\mathrm{Ein}\,(-2\mathrm{i}|\mathbf{p}|\eta)+\ln\,(2\mathrm{i}|\mathbf{p}|\eta)+\gamma\right] \\
&\quad + |\mathbf{p}|^{-3}\mathrm{e}^{-\mathrm{i}|\mathbf{p}|(\eta-\eta')}\,(|\mathbf{p}|\eta-\mathrm{i})\left(|\mathbf{p}|\eta'+\mathrm{i}\right)\ln\,(-2|\mathbf{p}|\eta) \\
I_3(\eta,\eta',\mathbf{p}) &= |\mathbf{p}|^{-3}\mathrm{e}^{-\mathrm{i}|\mathbf{p}|(\eta-\eta')}\,(|\mathbf{p}|\eta-\mathrm{i})\left(|\mathbf{p}|\eta'+\mathrm{i}\right)\left[\ln\left(2\mathrm{i}|\mathbf{p}|(\eta-\eta')\right)+\gamma\right] \\
I_4(\eta,\eta',\mathbf{p}) &= |\mathbf{p}|^{-3}\mathrm{e}^{\mathrm{i}|\mathbf{p}|(\eta-\eta')}\,(|\mathbf{p}|\eta+\mathrm{i})\left(|\mathbf{p}|\eta'-\mathrm{i}\right)\left[\mathrm{Ein}\left(-2\mathrm{i}|\mathbf{p}|(\eta-\eta')\right)+\ln\left(2\mathrm{i}|\mathbf{p}|(\eta-\eta')\right)+\gamma\right]\,.
\end{aligned}
$$

- Cosmological observable: power spectrum
  $\delta^2(\eta, \mathbf{p}) = \eta^{ac}\eta^{bd}\mathcal{G}^{-+}_{abcd}(\eta, \eta, \mathbf{p})$
- Problem: One-loop correction is genuine distribution → smear with test function (measurement resolution: Gaussian of width $\sigma \ll 1$)
- super-horizon modes: $\delta^2 \to \frac{\kappa^2 H^2}{16\pi^3}(1 + \kappa^2 H^2 \text{const.}) + \mathcal{O}(|\mathbf{p}|\eta)$
  (no logarithmic running $\sim \ln|\mathbf{p}|$, invariant under subset of dS isometries) Weinberg '05,'10, Chaicherdsakul '06, Adshead, Easther, Lim '09
- sub-horizon modes:
  $\delta^2 \to \frac{\kappa^2 H^2}{16\pi^3}\mathbf{p}^2\eta^2[1 - 3\alpha\kappa^2 H^2 \mathbf{p}^2\eta^2(1 + (|\mathbf{p}|\sigma)^{-1})] + \mathcal{O}(1/(|\mathbf{p}|\eta))$
- Corrections much too small to be observable even by Planck 1303.5075, 1303.5076

Riemann tensor correlator

- Problem: results obtained for a specific gauge which needs boundary conditions at spatial infinity to be defined, but we don't know anything outside of our horizon
- Riemann tensor is gauge-invariant (to linear order) and local (infrared-safe observable)
- $R^{ab}{}_{cd} = 2H^2 \delta^a_{[c} \delta^b_{d]} + 2\kappa H^4 \eta^2 P^{[ab]}{}_{[cd]}{}^{mn} h_{mn}$

$$P^{ab}{}_{cd}{}^{mn} h_{mn} = \eta^2 \eta^{ma} \eta^{bn} \partial_n \partial_c h_{dm} + 2\eta \eta^{ma} \eta^{bn} \partial_n \delta^0_c h_{dm} - 2\delta^b_0 \eta^{am} \left(\delta^0_c h_{dm} + \eta \partial_c h_{dm} \right) + \delta^b_d \left(\delta^a_c h_{00} - 2\eta^{an} h_{cn} - 2\delta^a_0 h_{c0} + 2\delta^0_c \eta^{an} h_{n0} \right) + \eta \delta^b_d \eta^{an} \left(2\partial_{(c} h_{n)0} - \partial_0 h_{cn} \right)$$

- (Connected) two-point function $\left\langle R^{ab}{}_{cd}(x) R^{m'n'}{}_{p'q'}(x') \right\rangle = 4\kappa^2 H^8 \eta^2 (\eta')^2 P^{[ab]}{}_{[cd]}{}^{kl} P^{[m'n']}{}_{[p'q']}{}^{s't'} \left\langle h_{kl}(x) h_{s't'}(x') \right\rangle$
- $\mathcal{O}\left(10^4\right)$ terms – Mathematica and xAct — www.xact.es
- dS-invariant result, expressed in terms of $Z = \cos(H\mu)$ and its derivatives
- First and second Bianchi identities satisfied

- Decomposition in Weyl-Weyl, Weyl-Ricci and Ricci-Ricci correlator
- Typical term: $\left\langle C^2 \right\rangle = -518400 N\kappa^4 H^8/\pi^4 (1+Z)^{-3} \ln\left[\frac{1}{2}(1-Z)\right] - 43200 N\kappa^4 H^8/\pi^4 (7 - 35Z + 29Z^2 - 25Z^3)(1+Z)^{-2}(1-Z)^{-4}$
- Only divergent at $Z = 1$ (coincidence), singularity at $Z = -1$ (antipodal points) only apparent
- Fall-off at large separation $Z \to \infty$:
  $\sim \kappa^2 H^6 |Z|^{-2} + \kappa^4 H^8 |Z|^{-2} + \kappa^4 H^8 |Z|^{-3} \ln |Z|$
  (exponential fall-off in proper time)
- Fall-off of Weyl-Ricci correlator: $\sim \kappa^4 H^8 |Z|^{-2}$
- Fall-off of Ricci-Ricci correlator: $\sim \kappa^4 H^8 |Z|^{-4}$
- Ricci-Ricci correlator related to stress tensor correlator:
  $\langle R_{ab}(x) R_{c'd'}(y) \rangle = \frac{\kappa^4}{4} \langle T_{ab}(x) T_{c'd'}(y) \rangle$
  (Correlator of Ricci scalar vanishes for conformal fields: conformal anomaly cancels out in connected two-point function)

Conclusions and Outlook

- Corrections to the power spectrum due to conformal scalars (unfortunately too small to be observable)
- dS-invariant Riemann tensor correlator (infrared-safe observable)
- Generalization to non-perturbative calculation (all orders in κ^2)
- Generalization to other types of matter (main ingredient: stress tensor two-point function)

Thank you for your attention!

Questions?