From Darmon points to Darmon cycles

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Let G be a Γ_0-type congruence subgroup in $B^\times$ (the indefinite quaternion algebra of discriminant $D > 1$ for simplicity) of level N_G and let $\mathscr{H}(G) = \mathscr{H}^D(G)$ be the Hecke algebra generated by the operators T_l where $l \nmid N_G D$, U_l and W_l where $l \mid N_G$, W_l where $l \mid D$ and W_∞. By a good Hecke operator we mean a T_l for some $l \nmid N_G D$.

Definition 1

Let M be a $\mathscr{H}(G)$-module. We say that M admits an l-Eisenstein/Cuspidal decomposition (of weight k) whenever there exists a decomposition of $\mathscr{H}(G)$-modules $M = M_l^e \oplus M_l^c$ such that $t_l := T_l - l^{k-1} - 1$ vanishes on M_l^e and is invertible on M_l^c for some $l \nmid N_G D$.
We say that it admist an Eisenstein/Cuspidal (of weight k) decomposition if it admits an l-Eisenstein/Cuspidal decomposition for every $l \nmid N_G D$ with $M_{l_1}^e = M_{l_2}^e$ and $M_{l_1}^c = M_{l_2}^c$ for every $l_1, l_2 \nmid N_G D$. In this case we write $M = M^e \oplus M^c$ where $M^e = M_l^e$ and $M^c = M_l^c$.

In the following discussion k will always be fixed and, when the l in the definition of an l-Eisenstein/Cuspidal decomposition is implicit, we simply write $M^e = M_l^e$ and $M^c = M_l^c$.

Consider the exact sequence

$$0 \to C_{har}(\mathscr{E},\mathbb{Z}) \to C_0(\mathscr{E},\mathbb{Z}) \xrightarrow{\delta_s} C(\mathscr{V},\mathbb{Z}) \to 0, \qquad (1)$$

where $\delta_s(c)(v) := \sum_{s(e)=v} c(e)$, which induces, taking Γ-cohomology and applying Shapiro's Lemma, the exact sequence

$$0 \to E \to H^1(\Gamma, C_{har}(\mathscr{E},\mathbb{Z})) \to H^1\left(\Gamma_0\left(pN^+\right),\mathbb{Z}\right)^{p-new} \to 0$$

where we define

$$E := \operatorname{coker}\left(\delta_s : C_0(\mathscr{E},\mathbb{Z})^\Gamma \to C(\mathscr{V},\mathbb{Z})^\Gamma\right),$$

$$H^1\left(\Gamma_0\left(pN^+\right),\mathbb{Z}\right)^{p-new} := \ker(\delta_s) \subset H^1(\Gamma, C_0(\mathscr{E},\mathbb{Z})).$$

The notation is justified by the following lemma and the fact that $H^1(\Gamma_0(pN^+), \mathbb{Z})^{p-new}$ is identified with the p-new part of $H^1(\Gamma_0(pN^+), \mathbb{Z})$, because δ_s corresponds to the degeneracy maps up to Shapiro's isomorphism.

Lemma 2

Consider the exact sequence

$$0 \to E \to H^1(\Gamma, C_{har}(\mathscr{E}, \mathbb{Z})) \to H^1\left(\Gamma_0\left(pN^+\right), \mathbb{Z}\right)^{p-new} \to 0$$

We have that $t_l = 0$ *on* $E \simeq \frac{\mathbb{Z}}{(p+1)\mathbb{Z}} \oplus \mathbb{Z}$ *for every* $l \nmid pN^+$ *while* $t_l : H^1(\Gamma_0(pN^+), \mathbb{Z})^{p-new} \to H^1(\Gamma_0(pN^+), \mathbb{Z})^{p-new}$ *is injective on* $H^1(\Gamma_0(pN^+), \mathbb{Z})^{p-new} \simeq \mathbb{Z}^g$. *There exists* $E \oplus H \subset H^1(\Gamma, C_{har}(\mathscr{E}, \mathbb{Z}))$ *such that* $t_l : H \to H$ *is injective every* $l \nmid pN^+D$, $H \hookrightarrow H^1(\Gamma_0(pN^+), \mathbb{Z})^{p-new}$ *is* $\mathbb{Z}$*-free and with torsion cokernel inducing*

$$\mathbb{Q}H = H^1(\Gamma, C_{har}(\mathscr{E}, \mathbb{Q}))^c = \mathbb{Q} \otimes_{\mathbb{Z}} H \xrightarrow{\sim} H^1\left(\Gamma_0\left(pN^+\right), \mathbb{Q}\right)^{p-new}.$$

Set $\mathscr{H}_p^{ur} := \mathbb{Q}_p^{ur} - \mathbb{Q}_p$ and let $\mathrm{Div}(\mathscr{H}_p)(k_p) \subset \mathrm{Div}\left(\mathscr{H}_p^{ur}\right)$ be the subgroup of divisors that are invariant under the action of $G_{\mathbb{Q}_p^{ur}/k_p}$. We define $\mathrm{Div}^0(\mathscr{H}_p)(k_p) \subset \mathrm{Div}^0\left(\mathscr{H}_p^{ur}\right)$ by means of the following exact sequence

$$0 \to \mathrm{Div}^0(\mathscr{H}_p)(k_p) \to \mathrm{Div}(\mathscr{H}_p)(k_p) \stackrel{\deg}{\to} \mathbb{Z} \to 0. \tag{2}$$

Here we remark that $\mathrm{Div}(\mathscr{H}_p)(\mathbb{Q}_p) \subset \mathrm{Div}(\mathscr{H}_p)(k_p)$ and there is a degree one divisor in $\mathrm{Div}(\mathscr{H}_p)(\mathbb{Q}_p)$, so that the above deg is surjective.

We fix from now on any

$$H \hookrightarrow H^1\left(\Gamma_0\left(pN^+\right), \mathbb{Z}\right)^{p-new}$$

as granted by Lemma 2. Let $C\left(\mathbf{P}^1\left(\mathbb{Q}_p\right), K_p^\times\right)$ be the space of continuous K_p-valued functions on $\mathbf{P}^1\left(\mathbb{Q}_p\right)$. Then we may define a pairing

$$C\left(\mathbf{P}^1\left(\mathbb{Q}_p\right), K_p^\times\right) \otimes C_{har}\left(\mathscr{E}, \mathbb{Z}\right) \to K_p^\times$$

by the rule

$$\Phi\left(f, c\right) := \lim_{\mathscr{T}_0 \subset \mathscr{T}} \left(\prod\nolimits_{e \in \partial \mathscr{T}_0} f\left(t_e\right)^{c(e)}\right), \qquad (3)$$

where $\mathscr{T}_0 \subset \mathscr{T}$ runs over all the net of finite subtrees of $\mathscr{T}$, $\partial \mathscr{T}_0$ denotes the set of boundary edges of $\mathscr{T}_0$ oriented towards $\partial \mathscr{T} = \mathbf{P}^1(\mathbb{Q}_p)$ and $t_e \in U_e$, where $U_e \subset \mathbf{P}^1(\mathbb{Q}_p)$ is the open compact subset attached to the equivalence class of ends starting from e. Here we remark that, given a choice $\{t_e\}_{e\in\mathscr{E}}$, the above limit exists thanks to the boundess of $c \in C_{har}(\mathscr{E}, \mathbb{Z}) \subset C^b_{har}(\mathscr{E}, \mathbb{Z})$ and it does not depend on the choice of $\{t_e\}_{e\in\mathscr{E}}$. This pairing is easily checked to be $\mathbf{GL}_2(\mathbb{Q}_p)$-invariant.

Next we remark that we may form the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Div}^0\left(\mathscr{H}_p\right)\left(k_p\right) & \otimes & C_{har}\left(\mathscr{E},\mathbb{Z}\right) & \xrightarrow{\Phi} & K_p^\times \\
\theta\downarrow & & \| & & \| \\
C\left(\mathbf{P}^1\left(\mathbb{Q}_p\right),K_p^\times\right) & \otimes & C_{har}\left(\mathscr{E},\mathbb{Z}\right) & \xrightarrow{\Phi} & K_p^\times \\
\downarrow & & \| & & \| \\
C\left(\mathbf{P}^1\left(\mathbb{Q}_p\right),K_p^\times\right)/K_p^\times & \otimes & C_{har}\left(\mathscr{E},\mathbb{Z}\right) & \xrightarrow{\Phi} & K_p^\times,
\end{array} \tag{4}$$

where θ is obtained by linear extension of the map

$$\theta : \mathrm{Div}\left(\mathscr{H}_p^{ur}\right) \to C\left(\mathbf{P}^1\left(\mathbb{Q}_p\right), \overline{K_p}^{\times}\right), \ \theta_{\tau_2,\tau_1}(t) := \frac{t-\tau_2}{t-\tau_1}.$$

The fact that the pairing in the second row induces uniquely a pairing in the thirt row follows from the harmonicity of the elemnts of $C_{har}(\mathscr{E},\mathbb{Z})$. In particular this pairing is $\mathbf{GL}_2(\mathbb{Q}_p)$-invariant. Since the composite $\mathrm{Div}(\mathscr{H}_p)(k_p) \to C\left(\mathbf{P}^1(\mathbb{Q}_p), K_p^{\times}\right)/K_p^{\times}$ is $\mathbf{GL}_2(\mathbb{Q}_p)$-invariant it follows that the pairing in the first row is $\mathbf{GL}_2(\mathbb{Q}_p)$-invariant too.

We deduce a Hecke equivariant map

$$\Phi_H : H_1\left(\Gamma, \mathrm{Div}\left(\mathscr{H}_p\right)(k_p)\right) \stackrel{\Phi}{\to} Hom\left(H^1\left(\Gamma, C_{har}\left(\mathscr{E}, \mathbb{Z}\right)\right), K_p^\times\right) \to \mathbf{T}_H\left(K_p\right),$$

where $\mathbf{T}_H\left(K_p^\times\right) := Hom\left(H, K_p^\times\right)$ and the second map is induced by $H \subset H^1\left(\Gamma, C_{har}\left(\mathscr{E}, \mathbb{Z}\right)\right)$. Note that, since H is $\mathbb{Z}$-free, $\mathbf{T}_H$ is indeed a rigid analytic torus, endowed with an action of $\mathbb{T} := \mathscr{H}^D\left(\Gamma_0\left(pN^+\right)\right)^{p-new}$.

Composing with $\log_0 : K_p^\times \to K_p$ and $\mathrm{ord}_p : K_p^\times \to K_p$ we obtain, setting $\log_\lambda := \log_0 - \lambda \mathrm{ord}_p$ for every $\lambda \in \mathbf{P}^1(K_p)$ with $\lambda \neq 0$ and $\log_\infty := \mathrm{ord}_p$,

$$\begin{array}{ccc} H_1(\Gamma, \mathrm{Div}(\mathscr{H}_p)(k_p)) & & \\ \downarrow \Phi & & \\ Hom\left(H^1(\Gamma, C_{har}(\mathscr{E}, \mathbb{Z})), K_p^\times\right) & \to & \mathbf{T}_H(K_p) \\ \downarrow \log_\lambda & & \downarrow \log_\lambda \\ Hom\left(H^1(\Gamma, C_{har}(\mathscr{E}, \mathbb{Z})), K_p\right) & \to & Hom(H, K_p) \\ \| & & \| \\ Hom_{K_p}\left(H^1(\Gamma, C_{har}(\mathscr{E}, K_p)), K_p\right) & \to & Hom_{K_p}\left(H_{K_p}, K_p\right) \end{array}$$

We also remark that we have the factorization

$$\begin{aligned} Hom_{K_p}\left(H^1\left(\Gamma, C_{har}\left(\mathscr{E}, K_p\right)\right), K_p\right) &\rightarrow Hom_{K_p}\left(H^1\left(\Gamma, C_{har}\left(\mathscr{E}, K_p\right)\right)^c, K_p\right) \\ &= Hom_{K_p}\left(H_{K_p}, K_p\right), \end{aligned}$$

where the identification follows from our choice of H in view of Lemma 2 and is induced by $H \subset H^1\left(\Gamma, C_{har}\left(\mathscr{E}, \mathbb{Z}\right)\right)$. Of course, by Lemma 2,

$$H^1\left(\Gamma, C_{har}\left(\mathscr{E}, K_p\right)\right)^c = H^1\left(\Gamma_0\left(pN^+\right), \mathbb{Z}\right)^{p-new}.$$

We define

$$\log_{\lambda}(\Phi_H) : H_1(\Gamma, \mathrm{Div}(\mathscr{H}_p)(k_p)) \stackrel{\log_{\lambda}(\Phi)}{\rightarrow} Hom_{K_p}\left(H^1(\Gamma, C_{har}(\mathscr{E}, K_p)), K_p\right) \rightarrow Hom_{K_p}\left(H_{K_p}, K_p\right)$$

as the morphisms obtained from the above commutative diagram. They are what we will generalized to the higher weight case.

In order to performs such a generalization it will be convenient to redefine these maps in a more convenient way.

Definition 3

Let $\mathscr{A}_n := \mathscr{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ be the space of K_p-valued locally analytic functions on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most n at ∞. More precisely, an element $f \in \mathscr{A}_n$ is a locally analytic function $f : \mathbb{Q}_p \to K_p$ for which there exists an integer N such that f is locally analytic on $\{z \in \mathbb{Q}_p : ord_p(x) \geq N\}$ and admits a convergent expansion

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 + \sum_{r \geq 1} a_{-r} z^{-r}$$

on $\{z \in \mathbb{Q}_p : ord_p(z) < N\}$.

The space $\mathscr{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ carries a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ defined by the rule $f \cdot \gamma = \frac{(cx+d)^n}{\det(\gamma)^{n/2}} \cdot f(\frac{ax+b}{cx+d})$, for any $f \in \mathscr{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$. Note that $\mathbf{P}_k(K_p)$ is a natural $\mathrm{GL}_2(\mathbb{Q}_p)$-submodule of it, where $\mathbf{P}_n := \mathbf{P}_n(K_p)$ is the space of K_p-valued polynomials on of degree $\leq n$.

We have

$$0 \to \mathbf{P}_n \to \mathscr{A}_n \to \mathscr{A}_n/\mathbf{P}_n \to 0$$

and define $\mathscr{D}_n := \mathscr{D}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ and $\mathscr{D}_n^0 := \mathscr{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ by taking the continuous K_p-duals:

$$0 \to \mathscr{D}_n^0 \to \mathscr{D}_n \to \mathbf{V}_n \to 0$$

Put $n := k-2$ from now on.

Definition 4

$\mathscr{D}_n^{0,b} := \mathscr{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b$ is the subset of those $\mu \in \mathscr{D}_k^0$ for which there is a constant A such that, for all $i \geq 0$, $j \geq 0$, and all $a \in \mathbb{Z}_p$,

$$|\mu((x-a)^i | a + p^j \mathbb{Z}_p)| \leq p^{A - j(i-1-k/2)}.$$

There is an epimorphism of $\mathrm{GL}_2(\mathbb{Q}_p)$-modules

$$r : \mathscr{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p) \twoheadrightarrow C_{har}(\mathscr{E}, \mathbf{V}_n(K_p)),$$
$$r(\mu)(e)(P) = \int_{U_e} P(t) d\mu(t) := \mu(P \cdot \chi_{U_e})$$

restricting to an inclusion

$$\mathscr{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b \hookrightarrow C_{har}(\mathscr{E}, \mathbf{V}_n(K_p))$$

and we set $C_{har}(\mathscr{E}, \mathbf{V}_n(K_p))^b := r\left(\mathscr{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b\right)$.

Noticing that we have

$$\log_\lambda(\theta_{\tau_2,\tau_1}) = \log_\lambda\left(\frac{t-\tau_2}{t-\tau_1}\right) =: \theta_{\tau_2,\tau_1}^{\log_\lambda}(t) \in \mathscr{A}_0(\mathbb{P}^1(\mathbb{Q}_p), K_p)$$

we may consider the following analogue of (4)

$$\begin{array}{ccccc}
\mathrm{Div}^0(\mathscr{H}_p)(k_p)\otimes \mathbf{P}_n(K_p) & \otimes & \mathscr{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b & \stackrel{\Phi^{\log_\lambda}}{\rightarrow} & K_p^\times \\
\theta^{\log_\lambda}\downarrow & & \| & & \| \\
\mathscr{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p) & \otimes & \mathscr{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b & \stackrel{\Phi^{\log_\lambda}}{\rightarrow} & K_p^\times \\
\downarrow & & \| & & \| \\
\mathscr{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p)/\mathbf{P}_n(K_p) & \otimes & \mathscr{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b & \stackrel{\Phi^{\log_\lambda}}{\rightarrow} & K_p^\times,
\end{array}$$

where $\theta^{\log_\lambda}$ is obtained by K_p-linear extension of the map

$$\theta^{\log_\lambda} : \mathrm{Div}\left(\mathscr{H}_p^{ur}\right) \otimes \mathbf{P}_n\left(K_p\right) \to \mathscr{A}_n(\mathbb{P}^1(\mathbb{Q}_p), K_p),$$
$$\theta^{\log_\lambda}_{\tau_2,\tau_1,P}\left(t\right) := \log_\lambda\left(\frac{t-\tau_2}{t-\tau_1}\right) P\left(t\right).$$

A similar remark as in the multiplicative setting implies the $\mathbf{GL}_2\left(\mathbb{Q}_p\right)$-invariance.

The key fact allowing us to extend the p-adic integration theory to the higher weight setting is the following.

Theorem 5

The above map r induces an Hecke equivariant isomorphism

$$H^1\left(\Gamma,\mathscr{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p),K_p)^b\right)\stackrel{\sim}{\to}H^1\left(\Gamma,C_{har}(\mathscr{E},\mathbf{V}_n(K_p))\right).$$

By means of Theorem 5 the pairing $\Phi^{\log_\lambda}$ induces a Hecke equivariant morphism:

$$\Phi_c^{\log_\lambda}:H_1\left(\Gamma,\mathrm{Div}^0\left(\mathscr{H}_p\right)(k_p)\otimes\mathbf{P}_n\left(K_p\right)\right)\stackrel{\Phi^{\log_\lambda}}{\to}$$
$$Hom_{K_p}\left(H^1\left(\Gamma,C_{har}(\mathscr{E},\mathbf{V}_n(K_p))\right),K_p\right)\to\mathbf{H}_k^\vee\left(K_p\right),$$

where we set $\mathbf{H}_k^\vee(K_p):=Hom_{K_p}\left(H^1\left(\Gamma,C_{har}(\mathscr{E},\mathbf{V}_n(K_p))^c\right),K_p\right)$.

The claimed generalization is a consequence of the following proposition.

Proposition 6

When $k = 2$ *we have* $\log_\lambda(\Phi) = \Phi^{\log_\lambda} \circ i$ *and* $\log_\lambda(\Phi_H) = \Phi_c^{\log_\lambda} \circ i$, *where*

$$i : H_1\left(\Gamma, \mathrm{Div}^0(\mathscr{H}_p)(k_p)\right) \to H_1\left(\Gamma, \mathrm{Div}^0(\mathscr{H}_p)(k_p) \otimes K_p\right).$$

Proof.

(Sketch) If $\mu \in \mathscr{D}_k^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b$ and $f \in \mathscr{A}_k(\mathbb{P}^1(\mathbb{Q}_p), K_p)$ then

$$\mu(f) = \lim_{\mathscr{T}_0 \subset \mathscr{T}} \left(\sum_{e \in \partial \mathscr{T}_0} r(\mu)(e) f(t_e) \right)$$

where $t_e \neq \infty$ for every $e \in \mathscr{E}$. Furthemore, $\mathscr{D}_k^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b \xrightarrow{\sim} C_{har}(\mathscr{E}, K_p)^b \supset C_{har}(\mathscr{E}, \mathbb{Z})$. Hence, if $r(\mu) = c \in C_{har}(\mathscr{E}, \mathbb{Z})$ with $\mu \in \mathscr{D}_k^0(\mathbb{P}^1(\mathbb{Q}_p), K_p)^b$ and $f \in C\left(\mathbf{P}^1(\mathbb{Q}_p), K_p^\times\right)$ is such that $\log_\lambda(f) \in \mathscr{A}_k(\mathbb{P}^1(\mathbb{Q}_p), K_p)$, then

$$\log_\lambda \Phi(f, c) \overset{(3)}{=} \log_\lambda \left(\lim_{\mathscr{T}_0 \subset \mathscr{T}} \left(\prod_{e \in \partial \mathscr{T}_0} f(t_e)^{c(e)} \right) \right)$$

$$= \lim_{\mathscr{T}_0 \subset \mathscr{T}} \left(\sum_{e \in \partial \mathscr{T}_0} r(\mu)(e) \log_\lambda(f(t_e)) \right) = \mu(\log_\lambda(f)).$$

The claim is then easily deduced. □

Consider the morphisms

$$\Phi_{H,\partial} : H_2(\Gamma,\mathbb{Z}) \xrightarrow{\partial} H_1\left(\Gamma, \mathrm{Div}^0(\mathscr{H}_p)(k_p)\right) \xrightarrow{\Phi_H} \mathbf{T}_H(K_p),$$

$$\Phi_{c,\partial}^{\log_\lambda} : H_2(\Gamma,\mathbf{P}_n(K_p)) \xrightarrow{\partial} H_1\left(\Gamma, \mathrm{Div}^0(\mathscr{H}_p)(k_p)\otimes \mathbf{P}_n(K_p)\right) \to \mathbf{H}_k^\vee(K_p)$$

where ∂ is obtained from (2).

Theorem 7

The morphism $\Phi_{c,\partial}^{\mathrm{ord}} : H_2(\Gamma,\mathbf{P}_n(K_p)) \to \mathbf{H}_k^\vee(K_p)$ is surjective and induces an isomorphism $H_2(\Gamma,\mathbf{P}_n(K_p))^c \xrightarrow{\sim} \mathbf{H}_k^\vee(K_p)$. There exists a unique $\mathscr{L} \in \mathbb{T}_{\mathbb{Q}_p}$ such that

$$\Phi_{c,\partial}^{\log_0} = \mathscr{L} \circ \Phi_{c,\partial}^{\mathrm{ord}} = \Phi_{c,\partial}^{\mathrm{ord}} \circ \mathscr{L}.$$

Define

$$\Phi_c^{\log} := -\Phi_c^{\log_0} \oplus \Phi_c^{\mathrm{ord}} : H_1\left(\Gamma, \mathrm{Div}^0(\mathscr{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)\right) \to \mathbf{D}_k(K_p),$$
$$\mathbf{D}_k(K_p) := \mathbf{H}_k^{\vee}(K_p)^2,\ \Phi_{c,\partial} := \Phi_c \circ \partial.$$

We also set

$$L_H := \mathrm{Im}\left(\Phi_{H,\partial}\right) \subset \mathbf{T}_H(\mathbb{Q}_p)\ \text{(when } k=2\text{)},$$
$$F := \mathrm{Im}\left(\Phi_{c,\partial}\right) \subset \mathbf{D}_k(\mathbb{Q}_p) =: \mathbf{D}_k.$$

Corollary 8

If $k = 2$, then $L_H \subset \mathbf{T}_H(\mathbb{Q}_p) =: Hom(H, \mathbb{Q}_p^\times)$ is a Hecke stable and $\mathbb{Z}$-free subgroup and $\mathrm{ord}(L_H) \subset Hom(H, \mathbb{Q}_p)$ is a $\mathbb{Z}$-lattice. In particular $A_H(K_p) := \mathbf{T}_H(K_p)/L_H$ is represented by a rigid analytic abelian variety $A_H/\mathbb{Q}_p$ with multiplication by $\mathbb{T}$.
For an arbitrary k we have $\mathbf{D}_k = \mathbf{D}_k^+ \oplus \mathbf{D}_k^-$ where $\mathbf{D}_k^\pm$ has a natural structure of $\mathbb{T}_{\mathbb{Q}_p}$-monodromy module structure over $\mathbb{Q}_p$ with Fontaine-Mazur $\mathscr{L}$-invariant $\mathscr{L}^\pm$ such that $\mathscr{L} = \mathscr{L}^+ \oplus \mathscr{L}^-$ and $\mathbf{D}_2$ is the $\mathbb{T}_{\mathbb{Q}_p}$-monodromy module attached to the $G_{\mathbb{Q}_p}$ representation $V_p(A_H)$.

For arbitrary k, the structure of monodromy module on $\mathbf{D}_k^{\pm}$ is given in such a way that $F \subset \mathbf{D}_k$ is the only non-trivial step in the filtration. Hence we may consider

$$\overline{\Phi}_c^{\log} : H_1\left(\Gamma, \mathrm{Div}^0\left(\mathscr{H}_p\right)(k_p) \otimes \mathbf{P}_n\left(K_p\right)\right) \stackrel{\Phi_c^{\log}}{\to} \mathbf{D}_k \to \frac{\mathbf{D}_k}{F}.$$

Note also that there is a unique isomorphism $\frac{\mathbf{D}_k}{F} \stackrel{\sim}{\to} \mathbf{H}_k^{\vee}\left(K_p\right)$ making the following diagram commutative:

$$\begin{array}{ccccc} \mathbf{D}_k\left(K_p\right) = & \mathbf{H}_k^{\vee}\left(K_p\right) \oplus \mathbf{H}_k^{\vee}\left(K_p\right) & \to & \frac{\mathbf{D}_k}{F} \\ (x,y) \mapsto -(x + \mathscr{L} y) & \downarrow & & \downarrow \\ & \mathbf{H}_k^{\vee}\left(K_p\right) & = & \mathbf{H}_k^{\vee}\left(K_p\right). \end{array} \qquad (5)$$

Then the following diagram is commutative:

$$\begin{array}{ccccc}
H_1\left(\Gamma, \mathrm{Div}^0(\mathscr{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)\right) & \stackrel{\Phi_c^{\log}}{\rightarrow} & \mathbf{D}_k & \rightarrow & \frac{\mathbf{D}_k}{F} \\
\| & & \downarrow & & \|\downarrow \\
H_1\left(\Gamma, \mathrm{Div}^0(\mathscr{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)\right) & \stackrel{\Phi_c^{\log_0} - \mathscr{L}\Phi_c^{\mathrm{ord}}}{\rightarrow} & \mathbf{H}_k^{\vee}(K_p) & = & \mathbf{H}_k^{\vee}(K_p)
\end{array}$$

When $k=2$, writing
$i: H_1\left(\Gamma, \mathrm{Div}^0\left(\mathscr{H}_p\right)\left(k_p\right)\right) \to H_1\left(\Gamma, \mathrm{Div}^0\left(\mathscr{H}_p\right)\left(k_p\right) \otimes K_p\right)$ we deduce, thanks to Proposition 6, that $\Phi_c \circ i$ is identified with

$$\begin{aligned}\left(\Phi_c^{\log_0} - \mathscr{L} \circ \Phi_c^{\mathrm{ord}}\right) \circ i &= \left(\log_0\left(\Phi_H\right) \circ i\right) - \left(\mathscr{L} \circ \mathrm{ord}\left(\Phi_H\right) \circ i\right) \\ &= \log_{A_H} \circ \Phi_H \circ i, \qquad (6)\end{aligned}$$

where

$$\log_{A_H} := \log_0 - \mathscr{L} \circ \mathrm{ord}\left(\Phi_H\right) : \mathbf{A}_H\left(K_p\right) \to Hom_{K_p}\left(H_{K_p}, K_p\right)$$

is the logarithm of the rigid analytic abelian variety $\mathbf{A}_H\left(K_p\right)$.

It is convenient to single out one of the two copies $\mathbf{D}_k^{\pm}$, which is obtained by means of the involution W_∞. This is the manifestation, in the weight 2 setting, of a degree 2 isogeny $\mathbf{A}_H \to \mathbf{A}_H^+ \times \mathbf{A}_H^-$.

Theorem 9

The above defined $\mathscr{L}$-invariant $\mathscr{L}^{\pm}$ is equal to the Fontain-Mazur $\mathscr{L}$-invariant attached to the $\mathbb{Q}_p$-adic representation $V_k := V_k(\Gamma_0(pN^+))^{p-new}$. In particular, $\mathscr{L}^+ = \mathscr{L}^-$.

As an application of Theorem 9 one may find an isomorphism of monodromy modules $\mathbf{D}_k^{\pm} \simeq D_k(\Gamma_0(pN^+))^{p-new} =: D_k$, where $D_k(\Gamma_0(pN^+))^{p-new}$ is the monodromy module attached to $V_k(\Gamma_0(pN^+))^{p-new}$. It follows that we have

$$\overline{\Phi}_c^{\log} : H_1\left(\Gamma, \mathrm{Div}^0(\mathscr{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)\right) \xrightarrow{\Phi_c^{\log}} D_k \to \frac{D_k}{F}.$$

Since $H_1(\Gamma, \mathbf{P}_n(K_p)) = 0$, we have from (2)

$$\frac{H_1\left(\mathrm{Div}^0(\mathscr{H}_p)(k_p) \otimes \mathbf{P}_n(K_p)\right)}{\partial(H_2(\Gamma, \mathbf{P}_n(K_p)))} \stackrel{\sim}{\to} H_1(\Gamma, \mathrm{Div}(\mathscr{H}_p)(k_p))$$

and, noticing that $\overline{\Phi}_c(\partial(H_2(\Gamma, \mathbf{P}_n(K_p)))) = F$ by construction, we may consider

$$AJ_c^{\log} : H_1(\Gamma, \mathrm{Div}(\mathscr{H}_p)(k_p)) \to \frac{D_k}{F}$$

induced by $\overline{\Phi}_c$.

In the weight 2 case Theorem 9 implies that there is an isogeny $\mathbf{A}_H^{\pm} \to A =: A(\Gamma_0(pN^+))^{p-new}$ (inducing the isomorphism between the associated monodromy modules). Here $H_1(\Gamma, \mathbb{Z})$ is a finite group because it is finitely generated and $H_1(\Gamma, K_p) = 0$, say of order h. It follows that we have

$$h : H_1(\Gamma, \mathrm{Div}(\mathscr{H}_p)(k_p)) \stackrel{\widetilde{h}}{\to} \frac{H_1\left(\mathrm{Div}^0(\mathscr{H}_p)(k_p)\right)}{\partial(H_2(\Gamma, \mathbb{Z}))} \subset H_1(\Gamma, \mathrm{Div}(\mathscr{H}_p)(k_p))$$

and we may define

$$AJ_{c,h} : H_1(\Gamma, \mathrm{Div}(\mathscr{H}_p)(k_p)) \stackrel{h}{\to} \frac{H_1\left(\mathrm{Div}^0(\mathscr{H}_p)(k_p)\right)}{\partial(H_2(\Gamma, \mathbb{Z}))} \to \mathbf{A}_H^{\pm}(K_p) \to A(K_p)$$

because $\Phi_H(\partial(H_2(\Gamma, \mathbb{Z}))) = L_H$.

Let $K/\mathbb{Q}$ be a quadratic field such that we may write $pN = pN^+N^-$ where $(pN, D_K) = 1$, the primes dividing N^+ are split in K, those dividing pN^- are inert in K and pN^- is squarefree and divisible by an odd numer of primes. Then $L(f/K, s)$ vanish at the central criticl point for a new weight k modular form of level $\Gamma_0(pN)$, whose p-adic representation may be realized in the Shimura curve of disctiminant $D = N^-$ taking a $\Gamma_0(pN^+)$-level structure.

It is possible define a period maps

$$\Gamma\backslash\mathscr{E}\left(\mathscr{O}_K, R_0\left(pN^+\right)\right) \to H_1\left(\Gamma, \mathrm{Div}\left(\mathscr{H}_p\right)(k_p)\right),$$
$$\Gamma\backslash\mathscr{E}\left(\mathscr{O}_K, R_0\left(pN^+\right)\right) \to H_1\left(\mathrm{Div}^0\left(\mathscr{H}_p\right)(k_p) \otimes \mathbf{P}_k\left(K_p\right)\right)$$

compatible with
$i : H_1\left(\Gamma, \mathrm{Div}^0\left(\mathscr{H}_p\right)(k_p)\right) \to H_1\left(\Gamma, \mathrm{Div}^0\left(\mathscr{H}_p\right)(k_p) \otimes K_p\right)$ in the weight 2 case. Here we remark that $\mathscr{E}\left(\mathscr{O}_K, R_0\left(pN^+\right)\right) \neq \phi$ thanks to our assumptions.

We define, setting $\mathscr{E} := \mathscr{E}(\mathscr{O}_K, R_0(pN^+))$,

$$P_K := \sum_{\Psi\in\mathscr{E}} AJ_{c,h}(\Psi) \in A(K_p)$$

and

$$\log(y_K) := \sum_{\Psi\in\mathscr{E}} AJ_c^{\log}(\Psi) \in \frac{D_k}{F}.$$

Recall that, writing X for the free group of degree zero divisors supported at the supersingular points of the reduction of X_{pN^+,N^-}, we have $A(K_p) = \frac{Hom(X, K_p^\times)}{X}$. We may consider the following commutative diagram:

$$\begin{array}{ccccccc} & \mathbb{Q} \otimes \mathbf{A}_H^{\pm}(K) & \rightarrow & \mathbb{Q} \otimes A(K) & \hookrightarrow & Sel_p(K, V_2) \\ & \cap & & \cap & & \downarrow \\ & \mathbb{Q} \otimes \mathbf{A}_H^{\pm}(K_p) & \rightarrow & \mathbb{Q} \otimes A(K_p) & \rightarrow & H^1_f(K_p, V_2) \\ & \| \downarrow \log_{\mathbf{A}_H^{\pm}} & & \| \downarrow \log_A & & \| \downarrow \log \\ \mathbf{H}_k^{\vee,\pm}(K_p) = & Hom_{K_p}(H^{\pm}, K_p) & \stackrel{\sim}{\rightarrow} & Hom_{K_p}(X, K_p) & = & \frac{D_2}{F}, \end{array}$$

where $\log := \exp^{-1}$ is the inverse of the Bloch-Kato exponential map, which is easily checked to be an isomorphism in this case, and we are considering the Kummer morphisms. Here the identifcation $Hom_{K_p}(X, K_p) = \frac{D_2}{F}$ can be choosen to be compatible with $\mathbf{H}_k^{\vee,\pm}(K_p) = \frac{\mathbf{D}_k^{\pm}}{F}$ appearing in (5), thanks to Theorem 5. Then (6) implies that

$$\begin{aligned} \log_A(P_K) &= \sum_{\Psi\in\mathscr{E}} \log_A\left(AJ_{c,h}(\Psi)\right) \\ &= \sum_{\Psi\in\mathscr{E}} \left(\Phi_c^{\log_0} - \mathscr{L}\Phi_c^{\mathrm{ord}}\right)\left(AJ_{c,h}(\Psi)\right) \\ &= h\sum_{\Psi\in\mathscr{E}} \left(\Phi_c^{\log_0} - \mathscr{L}\Phi_c^{\mathrm{ord}}\right)\left(AJ_c(\Psi)\right) \simeq h\sum_{\Psi\in\mathscr{E}} \Phi_c(\Psi) \\ &= h\log(y_K). \end{aligned}$$

Hence we define, for an arbitrary k,

$$y_K := \exp(\log(y_K)) \in H_f^1(K_p, V_k).$$

CONJECTURE. We have that y_K (resp. P_K when $k=2$) comes from a global cohomology class in $Sel_p(K, V_k)$ (resp. a global point in $A(K)$). The global classes from which the points/cycles come "should explain low rank instances of the Birch and Swinnerton-Dyer conjecture".

As an evidence towards this conjecture we may state the following result. Let W_N be the Atkin-Lehner involution acting on the space of modular forms and define A^{w_N}, $V_k^{w_N}$ and $D_k^{w_N}$ as the quotients where $W_N = w_N$.

Theorem 10

On the quotient $D_k^{w_N}$ such that $w_N = (-1)^{k/2}$ we have that $y_K^{w_N}$ (resp. $P_K^{w_N}$ when $k = 2$) comes from a global cohomology class in $Sel_p(K, V_2)$ (resp. a global point in $\mathbb{Q} \otimes A(K)$).

Remark 11

We remark that the weight $k = 2$ case follows form the statement about $y_K^{w_N}$, in light of $\log_A(P_K) = \text{hlog}(y_K)$ and the proof relative to $y_K^{w_N}$, showing that this class comes from a global cycle and therefore an element of $\mathbb{Q} \otimes A(K)$. We will return on this fact in the subsequent proposals.

...and related Proposals

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2013 Effective Methods for Darmon Points

First Proposal

Let T be a $\mathbb{Z}_p$-adic representatin of G_K such that $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ is semistable and define $A := \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} T$, so that we have the exact sequence

$$0 \to T \xrightarrow{\iota} V \xrightarrow{\pi} A \to 0.$$

For every place v of K we define $H^1_{st}(K_v, V)$ by means of the following exact sequence

$$0 \to H^1_{st}(K_v, V) \to H^1(K_v, V) \to \begin{cases} H^1(K_v^{ur}, V) & \text{if } p \nmid v, \\ H^1\left(K_v, \mathbf{B}_{st} \otimes_{\mathbb{Q}_p} V\right) & \text{if } p \mid v. \end{cases}$$

We also define

$$Sel\left(K,V\right) := \ker\left(H^1\left(K,V\right) \stackrel{\mathrm{res}_v}{\to} \prod_v \frac{H^1\left(K_v,V\right)}{H^1_{st}\left(K_v,V\right)}\right).$$

Consider the exact sequence

$$H^1\left(K_v,T\right) \stackrel{\iota}{\to} H^1\left(K_v,V\right) \stackrel{\pi}{\to} H^1\left(K_v,A\right)$$

and define

$$\begin{aligned} H^1_{st}\left(K_v,T\right) &:= \iota^{-1}\left(H^1\left(K_v,V\right)\right) \subset H^1\left(K_v,T\right), \\ H^1_{st}\left(K_v,A\right) &:= \pi\left(H^1\left(K_v,V\right)\right) \subset H^1\left(K_v,A\right). \end{aligned}$$

Next we define

$$Sel(K,T) \;:\; = \ker\left(H^1(K,T) \overset{\text{res}_v}{\to} \prod_v \frac{H^1(K_v,T)}{H^1_{st}(K_v,T)}\right),$$

$$Sel(K,A) \;:\; = \ker\left(H^1(K,A) \overset{\text{res}_v}{\to} \prod_v \frac{H^1(K_v,A)}{H^1_{st}(K_v,A)}\right).$$

Then we find the exact sequence:

$$Sel(K,T) \to Sel(K,V) \to Sel(K,A).$$

In particular we may applying this construction to $T = T_k$. In the weight 2 setting we find $P_K \in A(K_p)$ which gives an element of $\mathbb{Z}_p \otimes_{\mathbb{Z}} A(K_p)$ and then a cohomology class $c_K \in H^1_{st}(K_v, T)$.

- Is it possible to define, more generally, $c_K \in H^1_{st}(K_v, T_k)$ such that $\imath(c_K) = y_K \in H^1_{st}(K_v, V_k)$?
  - An integral version of the p-adic integration theory should be needed.
  - Also, an understanding of those representation T such that V is a monodromy module should be needed by means of an "integral Fontaine theory".

When $T_p = T_k$, the representation comes from a geometric setting. In this case we have a family T_l of $\mathbb{Z}_l$-adic representation which manifest a coherence, due to the fact that they are realizations of a motive (eventually except for a finite set of primes). There is a set of primes S such that T_l is unramified at $S \cup \{l\}$ and, writing Fr_q for the grometric Frobenius at $q \notin S \cup \{l\}$,

$$L_q(T, s) := \det\left(1 - Fr_q q^{-s} : V_l\right)^{-1}$$

is such that $\det(1 - Fr_v X : V_l) \in \mathbb{Q}_l[X]$ has rational coefficients and does not depend on the choice of $l \notin S$. In the weight 2 setting

$$P_K \in A(K_p) \to H^1_{st}(K_p, T_l)$$

for ant l. The same is true in the higher weight setting replacing P_K by the Heegner cycle.

- What about the misssing l-adic components of Darmon cycles? As we remarked the proof of the rationality statements produces a global cycle in $CH^*_{\mathbb{Q}}$ mapping to y_K. Hence, up to the finite primes appearing in the denominator one could take the l-components of the Darmon cycle as those defined by this cycle; but here we are looking for a refined theory allowing an a priori definition which applies to the more general y_Ψs. Furthermore, we do not know if this gives the correct definition in the weight 2 setting!

Second Proposal

Let B be the quaternion algebra of discriminant D, definite or indefinite. Then there are $B^\times$-representations V_k^B with coefficients in $\mathbb{Q}$ such that $F \otimes_{\mathbb{Q}} V_k^B = V_k(F)$ for every splitting field. They can be endowed with invariant $\mathbb{Z}$-lattices L_k. Suppose p is a prime and N^+ is a integer with $(pN^+, D) = (p, N^+) = 1$.

Assume now that B is definite and let $\Gamma \subset B_p^\times = \mathbf{GL}_2(\mathbb{Q}_p)$ be an arithmetic group obtained from a $\Gamma_0(N^+)$-level structure and no integral condition at p. More precisely we take $K_0(N^+) \subset B^\times\left(\mathbb{A}^{f,p}\right)$, set $\widetilde{\Gamma} := i_p(K_0(N^+) \cap B^\times) \subset B_p^\times$ and then take the norm one elements Γ.
Suppose that $K_0(N^+)$ is small enought, so that Γ acts on the Bruhat-Tits tree without fixed points. After inverting p we fix a non-degenerate Γ-invariant pairing

$$(\cdot,\cdot) : L_k \otimes L_k \to \mathbb{Z}[1/p]$$

which naturally induces non-degenerates pairings

$$(\cdot,\cdot)_{\mathscr{E}} : C_0(\mathscr{E}, L_k)^\Gamma \otimes C_0(\mathscr{E}, L_k)^\Gamma \to \mathbb{Z}[1/p],$$
$$(\cdot,\cdot)_{\mathscr{V}} : C(\mathscr{V}, L_k)^\Gamma \otimes C(\mathscr{V}, L_k)^\Gamma \to \mathbb{Z}[1/p].$$

We note that we have:

$$C_0(\mathscr{E}, L_k)^{\Gamma} \leftrightarrow \text{weight } k \text{ and } \Gamma_0\left(pN^+\right)\text{-level modular forms on } B,$$
$$C(\mathscr{V}, L_k)^{\Gamma} \leftrightarrow \text{two copies of weight } k \text{ and } \Gamma_0\left(N^+\right)\text{-level modular forms on } B.$$

We have exact sequences

$$\begin{aligned} 0 &\rightarrow L_k \rightarrow C(\mathscr{V}, L_k) \xrightarrow{d} C_0(\mathscr{E}, L_k) \rightarrow 0, \\ 0 &\rightarrow C_{har}(\mathscr{E}, L_k) \rightarrow C_0(\mathscr{E}, L_k) \xrightarrow{\delta} C(\mathscr{V}, L_k) \rightarrow 0, \end{aligned}$$

where $d(c)(e) := c(t(e)) - c(s(e))$ and $\delta(c)(v) := \sum_{s(e)=v} c(e)$. They induces

$$C(\mathscr{V}, L_k)^\Gamma \xrightarrow{d} C_0(\mathscr{E}, L_k)^\Gamma \text{ and } C_0(\mathscr{E}, L_k)^\Gamma \xrightarrow{\delta} C(\mathscr{V}, L_k)^\Gamma$$

which are adjoint:

$$(x, dy)_{\mathscr{E}} = (\delta x, y)_{\mathscr{V}}, \; x \in C_0(\mathscr{E}, L_k) \text{ and } y \in C(\mathscr{V}, L_k).$$

Define

$$\Delta_{\mathscr{E}} \; : \; = d\delta : C(\mathscr{E}, L_k) \to C(\mathscr{E}, L_k),$$
$$\Delta_{\mathscr{V}} \; : \; = \delta d : C(\mathscr{V}, L_k) \to C(\mathscr{V}, L_k).$$

They induce

$$\Delta_{\mathscr{E}}^{\Gamma} \; : \; C(\mathscr{E}, L_k)^{\Gamma} \to C(\mathscr{E}, L_k)^{\Gamma},$$
$$\Delta_{\mathscr{V}}^{\Gamma} \; : \; C(\mathscr{V}, L_k)^{\Gamma} \to C(\mathscr{V}, L_k)^{\Gamma}$$

such that

$$C_{har}(\mathscr{E}, L_k)^{\Gamma} = \ker\left(\Delta_{\mathscr{E}}^{\Gamma}\right) \text{ and } L_k^{\Gamma} = \ker\left(\Delta_{\mathscr{V}}^{\Gamma}\right).$$

Following Jordan and Livné, define

$$\Phi_{\mathscr{V}}(L_k) := \frac{\ker\left(\Delta_{\mathscr{V}}^{\Gamma}\right)^{\perp}}{\operatorname{Im}\left(\Delta_{\mathscr{V}}^{\Gamma}\right)} = \frac{\left(L_k^{\Gamma}\right)^{\perp}}{(\delta d)(C(\mathscr{V}, L_k))}.$$

Note that we have

$$0 \to L_k^{\Gamma} \to C(\mathscr{V}, L_k)^{\Gamma} \xrightarrow{d} C_0(\mathscr{E}, L_k)^{\Gamma} \xrightarrow{\partial} H^1(\Gamma, L_k) \to 0.$$

We define

$$\Phi_{\mathscr{E}}(L_k) \quad : \quad = \frac{H^1(\Gamma, L_k)}{\partial\left(\ker\left(\Delta_{\mathscr{E}}^{\Gamma}\right)\right)} = \frac{H^1(\Gamma, L_k)}{\partial\left(C_{har}(\mathscr{E}, L_k)^{\Gamma}\right)}$$

$$\stackrel{\sim}{\leftarrow} \frac{C_0(\mathscr{E}, L_k)^{\Gamma}}{C_{har}(\mathscr{E}, L_k)^{\Gamma} + d\left(C(\mathscr{V}, L_k)^{\Gamma}\right)}.$$

Suppose $k = 2$ and let Φ be the group of connected components of the Néron model of the Picard variety of the Mumford curve attached to Γ. Then

$$\Phi_{\mathscr{V}}(L_k) \simeq \Phi \simeq \Phi_{\mathscr{E}}(L_k)$$

by Raynaud and Grothendieck respectively.

Note that, if $x \in C_0(\mathscr{E}, L_k)^\Gamma$ and $y \in \ker(\Delta_{\mathscr{V}}^\Gamma)$, then $(\delta x, y)_{\mathscr{V}} = (x, dy)_{\mathscr{E}} = 0$, so that

$$\delta : C_0(\mathscr{E}, L_k)^\Gamma \to \ker\left(\Delta_{\mathscr{V}}^\Gamma\right)^\perp \subset C(\mathscr{V}, L_k)^\Gamma.$$

By definition $\delta\left(C_{har}(\mathscr{E}, L_k)^\Gamma\right) = 0$ and $\delta\left(d\left(C(\mathscr{V}, L_k)^\Gamma\right)\right) = \mathrm{Im}(\Delta_{\mathscr{V}}^\Gamma)$. It follows that δ induces

$$\overline{\delta} : \Phi_{\mathscr{E}}(L_k) \to \Phi_{\mathscr{V}}(L_k).$$

It can be proved that there is an exact sequence

$$0 \to \Phi_{\mathscr{E}}(L_k) \xrightarrow{\overline{\delta}} \Phi_{\mathscr{V}}(L_k) \to \frac{\left(L_k^{\Gamma}\right)^{\perp}}{\delta\left(C_0(\mathscr{E}, L_k)^{\Gamma}\right)} \to 0.$$

It is proved by Jordan and Livné that the $\Phi_{\mathscr{E}}(L_k)$ detects congruences between p-new and p-old modular forms.

On the other hand, there is a natural l-adic sheaf $\mathscr{L}_{k,l}$ attached to L_k (on the Mumford curve attached to Γ) and the theory of vanishing cycles allows us to define the analogue of the l-component of the group of connected components, that we denote by $\Phi\left(\mathscr{L}_{k,l}\right)$, extending the definition in the weigth 2 case.

- Is it possible to define an explicit identification $\Phi\left(\mathscr{L}_{k,l}\right) \simeq \Phi_{\mathscr{V}}\left(L_k\right)_l$ or $\Phi\left(\mathscr{L}_{k,l}\right) \simeq \Phi_{\mathscr{E}}\left(L_k\right)$? (M. Chida's suggestion: look at H. Carayol's paper "Sur les représentations l-adiques associées aux forms modulaires de Hilbert").

Suppose now that B is indefinite (non-split to avoid some "Eisenstein type" consideration). We have in this case, with Γ as in the previous talk (let N be large enough so that no elliptic points appears),

$$H^1(\Gamma, C_0(\mathscr{E}, L_k)) \simeq H^1\left(\Gamma_0\left(pN^+\right), L_k\right),$$
$$H^1(\Gamma, C(\mathscr{V}, L_k)) \simeq H^1\left(\Gamma_0\left(N^+\right), L_k\right)^2.$$

Then we may replace $(\cdot,\cdot)_{\mathscr{E}}$ and $(\cdot,\cdot)_{\mathscr{V}}$ above by the cup products induced by $(\cdot,\cdot)$: unfortunately d and δ are not adjoint each other.

The definition of $\Phi_{\mathscr{E}}(L_k)$ does not require taking orthogonal complement and has a formal analogue:

$$\begin{aligned}\Phi_{\mathscr{E}}(L_k) \quad : \quad &= \frac{\overline{H}^2(\Gamma, L_k)}{\partial\left(H^1(\Gamma, C_{har}(\mathscr{E}, L_k))\right)} \\ &\overset{\sim}{\leftarrow} \frac{H^1(\Gamma, C_0(\mathscr{E}, L_k))}{H^1(\Gamma, C_{har}(\mathscr{E}, L_k)) + d\left(H^1(\Gamma, C(\mathscr{V}, L_k))\right)}.\end{aligned}$$

Here we consider the "shifted" exact sequence:

$$0 \to H^1(\Gamma, L_k) \to H^1(\Gamma, C(\mathscr{V}, L_k)) \overset{d}{\to} H^1(\Gamma, C_0(\mathscr{E}, L_k)) \overset{\partial}{\to} \overline{H}^2(\Gamma, L_k) \to 0,$$

where $\overline{H}^2(\Gamma, L_k)$ denotes the image of ∂.

We remark that, setting

$$\Phi_{\mathscr{V}}(L_k) := \frac{H^1(\Gamma, C(\mathscr{V}, L_k))}{(\delta d)(H^1(\Gamma, C(\mathscr{V}, L_k)))}$$

we have the exact sequence, induced by δ,

$$0 \to \Phi_{\mathscr{E}}(L_k) \to \Phi_{\mathscr{V}}(L_k) \to \frac{H^1(\Gamma, C(\mathscr{V}, L_k))}{\delta(H^1(\Gamma, C_0(\mathscr{E}, L_k)))} \to 0.$$

Then $\Phi_{\mathscr{V}}(L_k)$ is well known to detect primes of congruence.

- Let $\mathscr{L}_{k,l}$ be the l-adic sheaf attached to L_k on the indefinite Shimura curve attached to $\Gamma_0(pN^+)$. Can we identify $\Phi\left(\mathscr{L}_{k,l}\right) \simeq \Phi_{\mathscr{V}}(L_k)_l$ or $\Phi\left(\mathscr{L}_{k,l}\right) \simeq \Phi_{\mathscr{E}}(L_k)$ or $\Phi\left(\mathscr{L}_{k,l}\right) \simeq \Phi_?(L_k)_l$?