About unique continuation for a 2D Grushin equation with potential having an internal singularity

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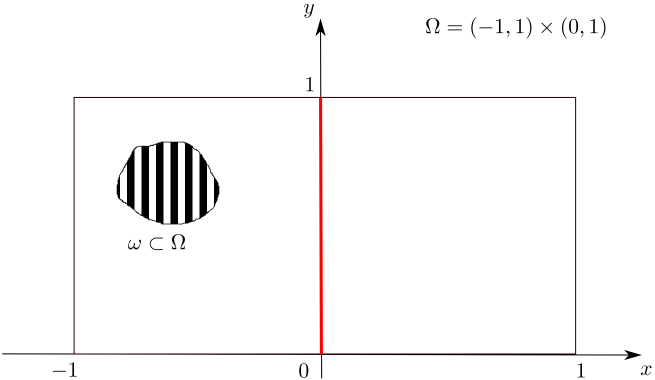
Model studied

$$\partial_t f - \partial^2_{xx} f - |x|^{2\gamma}\partial^2_{yy} f + \frac{c_\nu}{x^2} f = u(t,x,y)\chi_\omega(x,y), \quad (t,x,y) \in (0,T)\times\Omega, \tag{G}$$

with $\gamma > 0$ and $c_\nu := \nu^2 - \frac{1}{4}$, $\nu \in (0,1)$.



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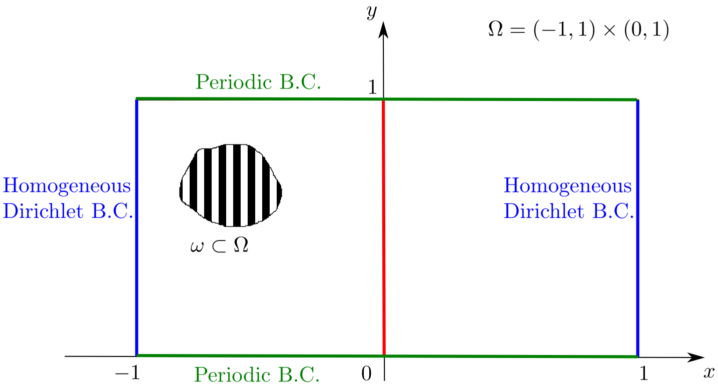

Comments on the model

- **Boscain Laurent** (2011). Laplace-Beltrami operator associated to Grushin-like metric (up to change of variables)

$$Lf = \partial_{xx}^2 f + |x|^{2\gamma}\partial_{yy}^2 f - \frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{f}{x^2}, \text{ on } L^2(\mathbb{R}\times\mathbb{T}).$$

$\frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{1}{x^2} \leftarrow \frac{c_\nu}{x^2}$: decouple effects of degeneracy and singularity.

- $c_\nu := \nu^2 - \frac{1}{4}$. Choice of $\nu < 1$ (i.e. $c_\nu < \frac{3}{4}$) to avoid essential self-adjointness.
- $c_\nu > -\frac{1}{4}$: validity of the Hardy inequality

$$\int_{-1}^{1}\left((f'(x))^2 + \frac{c_\nu}{x^2}f^2(x)\right)\mathrm{d}x \geq 0, \text{ for } f \in H^1(-1,1) \text{ with } f(0) = 0.$$

Crucial tool for inverse square singularity : **Baras Goldstein** (1984), **Vazquez Zuazua** (2000), **Vancostenoble Zuazua** (2008), **Ervedoza** (2008), **Vancostenoble** (2011), **Cazacu** (2013)...

Outline

1. Approximate controllability results
2. Wellposedness issues
3. Study of unique continuation
4. Counterexample to unique continuation for $\nu \in \left(\frac{1}{2}, 1\right)$

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Necessary and sufficient condition for approximate controllability

$(\mathbf{G})$ is approximately controllable in time $T > 0$ if for every $\epsilon > 0$, for every $f^0, f^T \in L^2(\Omega)$ there is $u \in L^2((0,T)\times\Omega)$ such that the associated solution satisfies

$$\|f(T) - f^T\|_{L^2(\Omega)} \leq \epsilon.$$

Main Theorem

Let $T > 0$, $\gamma > 0$ and $\nu \in (0,1)$. $(\mathbf{G})$ is approximately controllable in time $T > 0$ if and only if $\nu \in \left(0, \frac{1}{2}\right]$ i.e. $c_\nu \in \left(-\frac{1}{4}, 0\right]$.

$$\partial_t f - \partial_{xx}^2 f - |x|^{2\gamma}\partial_{yy}^2 f + \frac{c_\nu}{x^2} f = u(t,x,y)\chi_\omega(x,y)$$

- Need to precise the notion of solution
- Different behaviour from non singular and boundary singular cases
- $\omega \cap (\Omega \backslash \{x > 0\}) \neq \emptyset$ and $\omega \cap (\Omega \backslash \{x < 0\}) \neq \emptyset$: ok for $\nu \in (0,1)$.

1D singular heat equation

Theorem

Let $T > 0$ and $\nu \in (0,1)$. The system

$$\begin{cases} \partial_t f - \partial_{xx}^2 f + \dfrac{c_\nu}{x^2} f = u(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (-1,1), \\ f(t,-1) = f(t,1) = 0, & t \in (0,T), \\ f(0,x) = f^0(x), & x \in (-1,1), \end{cases}$$

is approximately controllable in time T if and only if $\nu \in \left(0, \frac{1}{2}\right]$ i.e. $c_\nu \in \left(-\frac{1}{4}, 0\right]$.

- By-product of the proof of the Main Theorem.

Wellposedness issues

- Limitation : behaviour of functions at the singularity

$$\exists f \in C_0^\infty(\Omega) \text{ such that } -\partial_{xx}^2 f - |x|^{2\gamma}\partial_{yy}^2 f + \frac{c_\nu}{x^2} f \notin L^2(\Omega).$$

- Strategy : for $\nu \in (0,1)$, design a suitable extension $(\mathcal{A},\ D(\mathcal{A}))$ of

$$\left(-\partial_{xx}^2 f - |x|^{2\gamma}\partial_{yy}^2 f + \frac{c_\nu}{x^2} f,\ C_0^\infty(\Omega\backslash\{x=0\})\right)$$

such that

$$\begin{cases} f'(t) = \mathcal{A}f(t) + v(t), \quad t \in [0,T], \\ f(0) = f^0, \end{cases}$$

is well posed.

An auxiliary 1D system

$$\begin{cases} \partial_t f - \partial_{xx}^2 f + \left(\frac{c_\nu}{x^2} + (2n\pi)^2 |x|^{2\gamma}\right) f = 0, & (t,x) \in (0,T) \times (-1,1), \\ f(t,-1) = f(t,1) = 0, & t \in (0,T), \end{cases} \qquad (\mathbf{G_n})$$

- $(\mathbf{G_n})$: system satisfied by the Fourier coefficients in the y variable of the formal solution of $(\mathbf{G})$.
- Study of the operator

$$A_n f := -\partial_{xx}^2 f + \left(\frac{c_\nu}{x^2} + (2n\pi)^2 |x|^{2\gamma}\right) f.$$

Design of $D(A_n)$ such that $(A_n, D(A_n))$ self-adjoint positive on $L^2(-1,1)$.

For any $n \in \mathbb{Z}$, any $f^0 \in L^2(-1,1)$, $(\mathbf{G_n})$ with initial condition f^0 has a unique solution

$$C^0\Big([0,+\infty), L^2(-1,1)\Big) \cap C^0\Big((0,+\infty), D(A_n)\Big) \cap C^1\Big((0,+\infty), L^2(-1,1)\Big).$$

Domain of the 1D operator I

- Regular and singular spaces. $\nu \in (0,1)$.

$$\tilde{H}^2_0(-1,1) := \left\{ f \in H^2(-1,1)\,;\; f(0) = f'(0) = 0 \right\},$$

$$\mathcal{F}_s := \Big\{ f \in L^2(-1,1)\,;\; f = c_1^+ |x|^{\nu+\frac{1}{2}} + c_2^+ |x|^{-\nu+\frac{1}{2}} \text{ on } (0,1)$$
$$\text{and } f = c_1^- |x|^{\nu+\frac{1}{2}} + c_2^- |x|^{-\nu+\frac{1}{2}} \text{ on } (-1,0) \Big\} \subset L^2(-1,1).$$

$$D(A_n) := \Big\{ f = f_r + f_s\,;\; f_r \in \tilde{H}^2_0(-1,1),\ f_s \in \mathcal{F}_s \text{ such that } f(-1) = f(1) = 0,$$
$$c_1^- + c_2^- + c_1^+ + c_2^+ = 0 \text{ and}$$
$$(\nu + \tfrac{1}{2})c_1^- + (-\nu + \tfrac{1}{2})c_2^- = (\nu + \tfrac{1}{2})c_1^+ + (-\nu + \tfrac{1}{2})c_2^+ \Big\},$$

Comments :

- $$\int_{-1}^{1} \left(f_r'(x)^2 + \frac{c_\nu}{x^2} f_r(x)^2 \right) \mathrm{d}x \geq 0, \quad \forall f_r \in \tilde{H}^2_0(-1,1).$$

Domain of the 1D operator II

- $$-\partial_{xx}^2 f_s + \frac{c_\nu}{x^2} f_s = 0, \quad \forall f_s \in \mathcal{F}_s.$$
- For every $f = f_r + f_s \in D(A_n)$,

$$A_n f = \left(-\partial_{xx}^2 + \frac{c_\nu}{x^2}\right) f_r + (2n\pi)^2 |x|^{2\gamma} f \in L^2(-1,1).$$

$(A_n, D(A_n))$ is self-adjoint and satisfies

$$\langle A_n f, f \rangle \geq m_\nu \int_{-1}^{1} \partial_x f_r(x)^2 \mathrm{d}x + (2n\pi)^2 \int_{-1}^{1} |x|^{2\gamma} f(x)^2 \mathrm{d}x,$$

with $m_\nu := \min\{1, 4\nu^2\}$.

$\Longrightarrow$ wellposedness of $(\mathbf{G_n})$

- Construction inspired by general theory of self-adjoint extensions of Sturm-Liouville operators : **Zettl** (2005).
- Construction impossible for $\nu \geq 1$: $x^{-\nu+\frac{1}{2}} \notin L^2(0,1)$

Semigroup associated to the 2D problem

- Periodic Fourier basis

$$\varphi_n(y) := \sqrt{2}\sin(2n\pi y),\ n \in \mathbb{N}^*, \quad \varphi_0(y) := 1,$$
$$\varphi_{-n}(y) := \sqrt{2}\cos(2n\pi y),\ n \in \mathbb{N}^*$$

- Definition of a C^0 semigroup of contraction : $f^0 \in L^2(\Omega)$.

  - $$f^0(x,y) = \sum_{n\in\mathbb{Z}} f_n^0(x)\varphi_n(y),$$

  - f_n solution of $(\mathbf{G_n})$ with initial condition f_n^0,

  - $$(S(t)f^0)(x,y) := \sum_{n\in\mathbb{Z}} f_n(t,x)\varphi_n(y).$$

Generator of the semigroup

- $\mathcal{A}$ infinitesimal generator of $S(t)$.

$$D(\mathcal{A}) = \left\{ f \in L^2(\Omega)\,;\, f_n \in D(A_n),\, \sum_{n\in\mathbb{Z}} ||A_n f_n||^2_{L^2(-1,1)} < +\infty \right\},$$

$$\mathcal{A}f = -\sum_{n\in\mathbb{Z}} (A_n f_n)(x)\varphi_n(y).$$

- Extension of the singular Grushin operator

$$\mathcal{A}f = \partial^2_{xx} f + |x|^{2\gamma}\partial^2_{yy} f - \frac{c_\nu}{x^2} f, \quad \forall f \in C_0^\infty(\Omega\backslash\{x=0\}).$$

- $(\mathbf{G})$ to be understood in the sense

$$\begin{cases} f'(t) = \mathcal{A}f(t) + v(t), & t \in [0,T], \\ f(0) = f^0, \end{cases}$$

with $v(t) := (x,y) \mapsto u(t,x,y)\chi_\omega(x,y)$. Unique mild solution.

Unique continuation

- Duality : approximate controllability $\Longleftrightarrow$ unique continuation of the adjoint system i.e.

$$S(t)g^0 \equiv 0 \text{ on } (0,T)\times\omega \quad \Longrightarrow \quad S(t)g^0 \equiv 0 \text{ on } (0,T)\times\Omega.$$

- $\nu \in \left(0, \frac{1}{2}\right]$.

$$S(t)g^0 \equiv 0 \text{ on } (0,T)\times\omega \quad \Longrightarrow \quad S(t)g^0 \equiv 0 \text{ on } (0,T)\times\Omega.$$

- $\nu \in \left(\frac{1}{2}, 1\right)$.

$$\exists g^0 \in L^2(\Omega)\backslash\{0\};\ S(t)g^0 \equiv 0 \text{ on } (0,T)\times\omega.$$

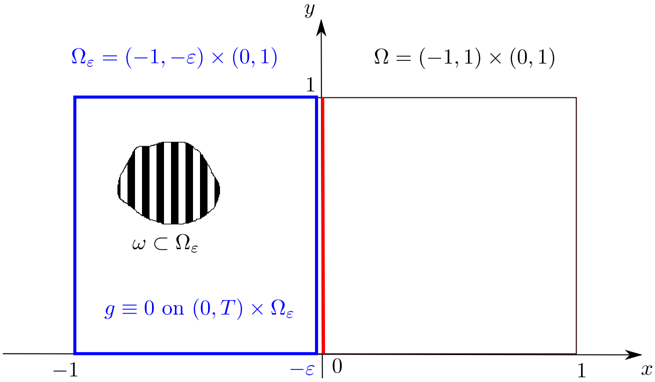
Strategy for unique continuation

$\nu \in (0,1)$. $g^0 \in L^2(\Omega)$; $S(t)g^0 \equiv 0$ on $(0, T) \times \omega$.

Strategy for unique continuation

$\nu \in (0,1)$. $g^0 \in L^2(\Omega)$; $S(t)g^0 \equiv 0$ on $(0,T) \times \omega$.

- unique continuation for uniformly parabolic operators : $S(t)g^0 \equiv 0$ on $(-1,0) \times (0,1)$



Strategy for unique continuation

$\nu \in (0,1)$. $g^0 \in L^2(\Omega)$; $S(t)g^0 \equiv 0$ on $(0,T) \times \omega$.

- unique continuation for uniformly parabolic operators : $S(t)g^0 \equiv 0$ on $(-1,0) \times (0,1)$
- reduction to 1D problem with boundary singularity : transmission conditions

$$S(t)g^0 = \sum_{n\in\mathbb{Z}} g_n(t)\varphi_n \equiv 0 \text{ on } (0,T) \times (-1,0) \times (0,1).$$

Then, for any $n \in \mathbb{Z}$

- $g_{n,r} \equiv 0$, $g_{n,s} \equiv 0$ on $(0,T) \times (-1,0)$.
- $c_1^-(g_n) = c_2^-(g_n) = 0$ + transmission conditions $\implies$ $c_1^+(g_n) = c_2^+(g_n) = 0$
- $g_{n,s} \equiv 0$: $g_n = g_{n,r}\chi_{(0,1)}$.

$$\begin{cases} \partial_t g_n - \partial_{xx}^2 g_n + \left(\frac{c_\nu}{x^2} + (2n\pi)^2|x|^{2\gamma}\right) g_n = 0, & (t,x) \in (0,T) \times (0,1) \\ g_n(t,0) = g_n(t,1) = 0, \\ \partial_x g_n(t,0) = 0, \end{cases}$$

$\overset{??}{\Longrightarrow}$ $g_n \equiv 0$ for any $n \in \mathbb{Z}$.

Strategy for unique continuation

$\nu \in (0,1)$. $g^0 \in L^2(\Omega)$; $S(t)g^0 \equiv 0$ on $(0,T) \times \omega$.

- unique continuation for uniformly parabolic operators : $S(t)g^0 \equiv 0$ on $(-1,0) \times (0,1)$
- reduction to 1D problem with boundary singularity : transmission conditions
- Unique continuation for any $n \in \mathbb{Z}$, for $\boxed{\nu \in \left(0, \frac{1}{2}\right]}$: 1D Carleman estimate

  Key points :

  - $\nu \in \left(0, \frac{1}{2}\right]$: $c_\nu \leq 0$.
  - $g'(0) = 0$: $\displaystyle\int_0^1 \frac{g^2(x)}{x^3} \mathrm{d}x < +\infty$

Details

1. Approximate controllability results

2. Wellposedness issues

3. Study of unique continuation

4. Counterexample to unique continuation for $\nu \in \left(\frac{1}{2}, 1\right)$

Explicit counterexample

- Bessel function of first kind

$$J_\nu(x) := \left(\frac{x}{2}\right)^\nu \sum_{k\in\mathbb{N}} \frac{(-1)^k}{2^{2k}k!\,\Gamma(k+\nu+1)}x^{2k}, \quad x\in[0,+\infty)$$

solution of

$$x^2y''(x)+xy'(x)+(x^2-\nu^2)y=0.$$

$\lambda\in(0,+\infty)$ such that $J_\nu(\sqrt{\lambda})=0$. The function $b_\lambda : x\mapsto x^{\frac{1}{2}}J_\nu(x\sqrt{\lambda})$ satisfies

$$\begin{cases} -b_\lambda''(x)+\dfrac{c_\nu}{x^2}b_\lambda(x)=\lambda b_\lambda(x), \\ b_\lambda(0)=b_\lambda(1)=0. \end{cases}$$

$$b_\lambda'(x) \underset{x\to 0}{\sim} C(\lambda,\nu)x^{\nu-\frac{1}{2}}.$$

Conclusion. $\boxed{\nu\in\left(\frac{1}{2},1\right)}$: $g(t,x,y):=e^{-\lambda t}b_\lambda(x)\chi_{(0,1)}(x)$ solution vanishing on ω.

Adaptation to the 1D singular heat equation

$$\begin{cases} \partial_t f - \partial_{xx}^2 f + \frac{c_\nu}{x^2} f = u(t,x)\chi_\omega(x), & (t,x) \in (0,T)\times(-1,1), \\ f(t,-1) = f(t,1) = 0, & t \in (0,T), \end{cases} \qquad \textbf{(H)}$$

with $\omega = (a,b)$, $-1 \leq a < b \leq 0$, $\gamma > 0$ and $c_\nu := \nu^2 - \frac{1}{4}$, $\nu \in (0,1)$.

Rewritten in terms of A_0. $(A_0, D(A_0))$ self-adjoint positive on $L^2(0,1)$.

- $e^{-A_0 t} g^0 = g_r(t) + g_s(t)$.
- $e^{-A_0 t} g^0 \equiv 0$ on $(0,T)\times\omega$: $g_r \equiv 0$ on $(0,T)\times(-1,0)$ and $g_s \equiv 0$ on $(0,T)\times(-1,1)$.
- $\nu \in \left(0, \frac{1}{2}\right]$: Carleman inequality for $\mathcal{P}_0 \implies$ unique continuation.
- $\nu \in \left(\frac{1}{2}, 1\right)$: explicit counterexample to unique continuation (Bessel function).

Open Problems and perspectives

Conclusion

- Design of a suitable self-adjoint extension of the operator
$$-\partial_{xx}^2 - |x|^{2\gamma}\partial_{yy}^2 + \frac{c}{x^2}.$$
- Necessary and sufficient condition for unique continuation
  - Classical results for regular parabolic operators + Carleman estimate for 1D heat equation with boundary singularity
  - Explicit counterexample

Perspectives

- Null controllability
- Other extensions Details

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- Other extensions Details

Thank you for your attention.

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Carleman inequality

$\boxed{\nu \in \left(0, \frac{1}{2}\right]}$: $c_\nu = \nu^2 - \frac{1}{4} \leq 0.$

$$\mathcal{P}_n := \partial_t - \partial_{xx}^2 + \left(\frac{c_\nu}{x^2} + (2n\pi)^2 |x|^{2\gamma}\right)$$

Weights : $\sigma(t,x) := \theta(t)p(x)$ where $\theta : t \mapsto \frac{1}{t(T-t)}$, $p \in C^4([0,1], \mathbb{R})$ such that on $[0,1]$

$$p(x) \geq m_0 > 0, \quad p_x(x) \geq m_1 > 0, \quad -p_{xx}(x) \geq m_2 > 0.$$

Let $T > 0$ and $Q_T := (0,T) \times (0,1)$. There exist $R_0, C_0 > 0$ such that for any $R \geq R_0$, for any $g \in C^1((0,T], L^2(0,1)) \cap C^0((0,T], H^2 \cap H_0^1(0,1))$ with $\partial_x g(t,0) = 0$ on $(0,T)$,

$$C_0 \iint_{Q_T} \left(R^3\theta^3 g^2 + R\theta g_x^2\right) e^{-2R\sigma} \mathrm{d}x\mathrm{d}t \leq \iint_{Q_T} |\mathcal{P}_n g|^2 e^{-2R\sigma} \mathrm{d}x\mathrm{d}t.$$

Conclusion :

$$\nu \in \left(0, \tfrac{1}{2}\right] \implies \text{unique continuation of the adjoint system.}$$

Heuristic of the Carleman strategy

Key points : $z := e^{-R\sigma} g$

- Boundary term

$$R \int_0^T \theta p_x z_x(t,1)^2 \mathrm{d}t \geq 0.$$

- Potentials

$$R \iint_{Q_T} \left(-2\frac{c_\nu}{x^3} + 2\gamma(2n\pi)^2 x^{2\gamma-1} \right) \theta p_x z^2 \mathrm{d}x\mathrm{d}t \geq 0.$$

  - $c_\nu \leq 0$
  - Supplementary Neumann condition : $h \in H^2 \cap H_0^1(0,1)$

$$h'(0) = 0 \implies \int_0^1 \frac{h(x)^2}{x^3} \mathrm{d}x < +\infty.$$

- End with a classical proof.

General theory for self-adjoint extensions

Proposition

Let u and v in $\tilde{H}_0^2 \oplus \mathcal{F}_s$ be such that their restriction on $(0,1)$ (resp. $(-1,0)$) are linearly independent modulo $H_0^2(0,1)$ (resp. $H_0^2(-1,0)$) and

$$[u,v](-1) = [u,v](0^-) = [u,v](0^+) = [u,v](1) = 1.$$

Let $M_1, \dots, M_4$ be 4×2 complex matrices. Then every self-adjoint extension of the minimal operator is given by the restriction to the functions f satisfying the boundary conditions

$$M_1 \begin{pmatrix} [f,u](-1) \\ [f,v](-1) \end{pmatrix} + M_2 \begin{pmatrix} [f,u](0^-) \\ [f,v](0^-) \end{pmatrix} + M_3 \begin{pmatrix} [f,u](0^+) \\ [f,v](0^+) \end{pmatrix} + M_4 \begin{pmatrix} [f,u](1) \\ [f,v](1) \end{pmatrix} = 0,$$

where the matrices satisfy $(M_1\, M_2\, M_3\, M_4)$ has full rank and

$$M_1 E M_1^* - M_2 E M_2^* + M_3 E M_3^* - M_4 E M_4^* = 0, \text{ with } E := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Conversely, every choice of such matrices defines a self-adjoint extension.

- Definition of u and v.
  Solutions of
$$-f''(x) + \frac{c_\nu}{x^2} f(x) = 0, \quad x \in (0,1)$$
  with $(u(1) = 0, u'(1) = 1)$ and $(v(1) = -1, v'(1) = 0)$ i.e.
$$u(x) = \frac{1}{2\nu} x^{\nu+1/2} - \frac{1}{2\nu} x^{-\nu+1/2},$$
$$v(x) = -\frac{\nu - 1/2}{2\nu} x^{\nu+1/2} - \frac{\nu + 1/2}{2\nu} x^{-\nu+1/2}.$$
  Similar construction on $(-1,0)$ i.e.
$$u(x) = -\frac{1}{2\nu} |x|^{\nu+1/2} + \frac{1}{2\nu} |x|^{-\nu+1/2},$$
$$v(x) = -\frac{\nu - 1/2}{2\nu} |x|^{\nu+1/2} - \frac{\nu + 1/2}{2\nu} |x|^{-\nu+1/2}.$$

Application to Grushin operator II

- Choice of matrices $M_1, \dots, M_4$.
  For any $f \in D_{max}$,

$$[f,u](1) = f(1), \qquad [f,v](1) = f'(1),$$
$$[f,u](-1) = f(-1), \qquad [f,v](-1) = f'(-1).$$

  Dirichlet boundary conditions at ± 1

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 & 0 \\ \multicolumn{2}{c}{\tilde{M}_2} \\ 0 & 0 \end{pmatrix}, \; M_3 = \begin{pmatrix} 0 & 0 \\ \multicolumn{2}{c}{\tilde{M}_3} \\ 0 & 0 \end{pmatrix}, \; M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Application to Grushin operator III

- Conditions of Proposition : $(\tilde{M}_2\, \tilde{M}_3)$ has rank 2 and $\det(\tilde{M}_2) = \det(\tilde{M}_3)$.

$$[f,u](0^+) = c_1^+ + c_2^+, \qquad [f,v](0^+) = \left(\nu + \frac{1}{2}\right) c_1^+ + \left(-\nu + \frac{1}{2}\right) c_2^+,$$
$$[f,u](0^-) = c_1^- + c_2^-, \qquad [f,v](0^-) = -\left(\nu + \frac{1}{2}\right) c_1^- - \left(-\nu + \frac{1}{2}\right) c_2^-.$$

Thus, the choice $\tilde{M}_2 = \tilde{M}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ lead to the definition of $D(A_n)$.