Relaxation Techniques for Switching Control of the Flux Function in Conservation Laws

(work in progress)

Falk Hante

hante@math.fau.de

U Erlangen-Nuremberg, Germany

FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE
FAKULTÄT

Department
MATHEMATIK

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A Traffic-Flow Model with Switching Speed Limit Control

Consider the LWR traffic flow model, for a link $[0, L]$ given by

$$\rho_t + (f^j(\rho))_x = 0$$

with a finite family $\{f^j\}_{j=1}^N$ of Greenshield flux functions

$$f^j(\rho) = \rho\, u^j_{\max}(1 - \frac{\rho}{\rho_{\max}})$$

where

- ρ models the traffic density at time t on $[0, L]$
- $u^j_{\max}$ is the prescribed maximum velocity in mode j
- $\rho_{\max}$ is the critical (congestion) density.

A Dynamic (Mixed-)Integer Optimization Problem

- Minimize the cost
$$\int_0^{t_f} \int_0^L \rho^2(x,t)\,dx\,dt$$
by choosing $j(t) \in \{1,\ldots,N\}$ for all $t \in [0,t_f]$ s. t. known initial data
$$\rho(0,x) = \rho_0(x)$$
and boundary data
$$\rho(t,0) = \rho_L(t), \quad \rho(t,L) = \rho_R(t).$$
- We assume that the initial and boundary data is piecewise $W^{1,1}$.
- Cost function motivated by [Fuegenschuh, Herty, Martin: 2006].

Some Remarks on the Problem

- Total discretization and dynamic programming are too expensive to solve the problem accurately in reasonable time (real time?)
- Will consider relaxation techniques which can provide ε-optimal solutions by solving only one classical optimal control problem.
- Similar problem and approach in [Bayen, Raffard, Tomlin: HSCC 2004], but
  - the control is boundary inflow (on/off ramp metering)
  - no analysis!
- We aim at
  - a-priori estimates on the quality of the solution
  - a numerical method to provide the controls
  - proving convergence of the method.

Outline

Hyperbolic Relaxation [Jin-Xin: 1995]

- For $\kappa > 0$, a constant $a > 0$ satisfying

$$-a \leq \frac{d}{d\rho} f^j(\rho) \leq a, \qquad \text{for all } \rho,\ j$$

and a new variable η defined on $[0, t_f] \times [0, L]$ consider

$$\begin{cases} \rho_t + \eta_x = 0 \\ \eta_t + a^2 \rho_x = -\frac{1}{\kappa}(\eta - f^j(\rho)) =: g^j(\eta, \rho) \end{cases}$$

with the initial and boundary data

$$\begin{aligned} \rho(0,x) &= \rho_0(x), \quad \eta(0,x) = \eta_0(x) := f^{j(0)}(\rho_0(x)), \qquad x \in [0, L] \\ \rho(t,0) &= \rho_L(t), \quad \eta(t,0) = \eta_R(t) := f^{j(t)}(\rho_L(t)), \qquad t \in [0, t_f] \\ \rho(t,L) &= \rho_R(t), \quad \eta(t,L) = \eta_L(t) := f^{j(t)}(\rho_R(t)), \qquad t \in [0, t_f]. \end{aligned}$$

- Resulting system is a switching semilinear hyperbolic system [H., Leugering, Seidman: Appl. Math. Opt. 2009]

Control Variable Transformation

Introducing new binary control functions

$$\alpha_j \colon [0, t_f] \to \{0, 1\}, \; j = 1, \ldots, N$$

satisfying

$$\sum_{j=1}^{N} \alpha_j(t) = 1, \text{ for a. e. } t \in (0, t_f)$$

the control problem is equivalent to

$$\begin{cases} \rho_t + \eta_x = 0 \\ \eta_t + a^2 \rho_x = \sum_{j=1}^{N} \alpha_j g^j(\eta, \rho) \end{cases}$$

subject to appropriate boundary conditions, with

$$j(t) = \sum_{j=1}^{N} \alpha_j(t) j.$$

L^1-Solution of the Semilinear System

Recall that the solution $y = (\eta, \rho)$ is given by the fixed point of a transformation $T = (T_1, T_2)^\top$ acting on $Y := C([0, t_f]; (L^1(0, L))^2)$

$$T_i(y)(t,x) = \begin{cases} y^*_{i,L/R} + \sum_{j=1}^N \int_{t_1^*}^t \alpha_j(t) g^j(y(\vartheta, s_i(\vartheta; t, x)))\, d\vartheta, & x \in (0, \bar{s}_i) \\ y^*_{i,0} + \sum_{j=1}^N \int_0^t \alpha_j(t) g^j(y(\vartheta, s_i(\vartheta; t, x)))\, d\vartheta, & x \in (\bar{s}_i, L) \end{cases}$$

with appropriate boundary points $y^*_{i,L/R}$, $y^*_{i,0}$, $\bar{s}_i$ and

$$s_i(t; \tau, \sigma) = \sigma \pm a(t - \tau)$$

denoting the characteristic curves. T is a contraction in the norm

$$||y||_\dagger = \sup_{t \in [0, t_f]} e^{-Kt} \left(\int_0^L |y_1(t,x)| dx + \int_0^L |y_2(t,x)| dx \right)$$

for a suitable constant K.

Binary-Relaxation

- We now relax the restriction $\alpha_j(t) \in \{0, 1\}$, $t \in [0, t_f]$ a. e. by

$$\alpha_j(t) \in [0, 1], \ \ t \in [0, t_f] \text{ a. e.}$$

Theorem (Binary-Relaxation)

Let $\omega \in L^\infty(0, t_f; [0, 1]^N)$ be a feasible control of the binary-relaxed problem. Then, for every $\varepsilon > 0$, there exists a feasible control

$$\alpha \colon [0, t_f] \to \{0, 1\}^N \text{ piecewise constant}$$

satisfying

$$\|y(\alpha) - y(\omega)\|_Y \leq C\varepsilon$$

for some constant C (depending only on ω, κ and known data).

For abstract semilinear systems: [H., Sager: Comp. Opt. Appl. 2013]

Proof of the Binary-Relaxation Theorem

Lemma (Sager, Bock, Diehl: Math. Prog. 2011)

Let $\omega \in L^\infty(0, t_f; [0,1]^N)$, $\sum_{i=1}^N \omega_i(t) = 1$ *for a. e.* $t \in [0, t_f]$ *and* $\Delta > 0$. *Then, there exists a function*

$$\alpha\colon [0, t_f] \to \{0,1\}^N \text{ piecewise constant}$$

such that

1. $$\gamma := \max_{i=1,\dots,N} \sup_{t\in[0,t_f]} \left| \int_0^t \alpha_i(\tau) - \omega_i(\tau)\, d\tau \right| \le (N-1)\Delta,$$
2. $$\sum_{i=1}^N \alpha_i(t) = 1 \text{ for all } t \in [0, t_f].$$

Hence, choosing Δ such that $(N-1)\Delta \le \varepsilon$ it suffices to show that

$$\|y(\alpha) - y(\omega)\| \le C\gamma$$

for some C.

Proof of the Binary-Relaxation Theorem (continued)

- Since y is defined by a fixed point of T, it suffices to show

$$(\#) \qquad \int_0^L \Big|[T_i(y(\alpha)) - T_i(y(\omega))](t_f,\cdot)\Big|\,dx \le \tilde{C}\gamma,\ \ i = 1,2.$$

  for some $\tilde{C}$, since then $\|y(\alpha) - y(\omega)\| \le e^{Kt}\tilde{C}$.

- To this end, integration by parts yields

$$\int_0^L \Big|[T_1(y(\alpha)) - T_1(y(\omega))](t,x)\Big|\,dx \le C_1\gamma + \gamma\sum_{j=1}^{N}\Bigg(\int_{t_1^*}^{t}\int_0^L \Big|D_\vartheta g^j(y(\vartheta, s_1(\vartheta;t,x)))\Big|\,dx$$
$$+ \int_0^L |g^j(y(t, s_1(t;t,x)))|dx + \int_0^L |g^j(y(t_1^*(t,x), s_1(t_1^*(t,x);t,x)))|dx\Bigg).$$

  (and a similar estimate for T_2).

- The estimate (#) then follows from a subtle regularity result for semilinear hyperbolic systems: $y_1(t,\cdot), y_2(t,\cdot)$ is piecewise $W^{1,1}$ [Oberguggenberger: 1986].

Limit of the solution for $\kappa \to 0$?

Conjecture

Suppose that $TV(\rho_L)$, $TV(\rho_R)$ and $TV(\rho_0)$ are sufficiently small. Then, the L^1-solutions (as a fixed-point of T) converge to the weak entropy solutions of the LWR model with boundary conditions

$$\overline{F}(\rho(t,0),\rho_L(t),j(t)) = 0, \quad \underline{F}(\rho(t,L),\rho_R(t),j(t)) = 0$$

where

$$\overline{F}(\rho_1,\rho_2,j) = \sup_{\rho\in I(\rho_1,\rho_2)} (\mathrm{sg}(\rho_1-\rho_2)(f^j(\rho_1)-f^j(\rho)),$$

$$\underline{F}(\rho_1,\rho_2,j) = \inf_{\rho\in I(\rho_1,\rho_2)} (\mathrm{sg}(\rho_1-\rho_2)(f^j(\rho_1)-f^j(\rho))$$

and $I(\rho_1,\rho_2) = [\inf(\rho_1,\rho_2), \sup(\rho_1,\rho_2)]$.

- For L^1_{loc}-solutions on $\mathbb{R}$ without boundary conditions, the convergence was proved in [Bianchini: 2001].
- Can show a uniformity of C w.r.t. κ?

Other Open Questions

- For $\varepsilon \to 0$, we obtain convergence of the optimal state and the optimal cost. Do the controls $\omega(\varepsilon)$ converge? In which topology?
- Does it also work with other cost functions, e. g., total travel time?
- How to include switching costs?
- Can we obtain similar results for non-linear hyperbolic systems, e. g., the Euler gas equations?
- Does binary-relaxation work with boundary control?

Conclusions

- Considered switching control for the flux function of a conservation law, exemplary for LWR-traffic.
- Have a binary-relaxation result based on a hyperbolic relaxation.
- All proofs are constructive providing a numerical method.
- Have an implementation based on IMEX-schemes (using adjoint based derivatives to compute ω and SUR for α)
- Have similar results for semilinear parabolic equations with distributed mixed-integer control on reflexive spaces.
- Many open questions left for switching systems with PDEs.

Selected Publications

Optimal Switching Control of PDEs

- Hante, Leugering, Seidman: Modeling and Analysis of Modal Switching in Networked Transport Systems. *Appl. Math. and Opt.*, 2009.
- Hante, Leugering: Optimal Boundary Control of Convention-Reaction Transport Systems with Binary Control Functions. In *Hybrid Systems: Computation and Control*, Springer LNCS, 2009.
- Hante, Leugering, Seidman: An augmented BV setting for switching feedback control. *J. Syst. Sci. Compl.*, 2010.
- Hante: *Hybrid Dynamics Comprising Modes Governed by Partial Differential Equations: Modeling, Analysis and Control for Semilinear Hyperbolic Systems in One Space Dimension*. University Erlangen-Nuremberg, Dissertation, 2010.
- Hante, Sager: Relaxation Methods for Mixed-Integer Optimal Control of Partial Differential Equations. *Comp. Opt. Appl.*, 2013.

Stability Analysis of Switching PDEs

- Hante, Sigalotti: Converse Lyapunov Theorems for Switched Systems in Banach and Hilbert Spaces. *SIAM J. Cont. Opt.*, 2011.
- Hante, Sigalotti, Tucsnak: On conditions for asymptotic stability of dissipative infinite-dimensional systems with intermittent damping. *J. Diff. Eq.*, 2012.
- Amin, Hante, Bayen: Exponential Stability of Switched Linear Hyperbolic Initial-Boundary Value Problems. *IEEE TAC*, 2012.