Exponential Decay: From Semi-Global to Global

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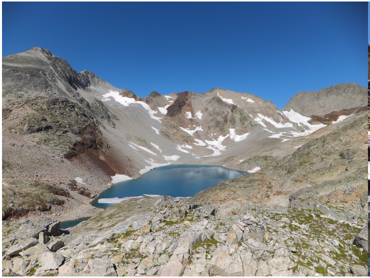


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    - Solution 1: Change the space
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5. Here we consider Solution 2 and present a method that is based upon

   integral inequalities

   to proceed from semi-global to global for stabilized systems.

Example System: The isothermal Euler equations

$$\begin{cases} \rho_t + q_x = 0 \\ \rho_t + (\frac{q^2}{\rho} + a^2 \rho)_x = -\frac{1}{2}\theta \frac{q\,|q|}{\rho} \end{cases}$$

1. Important: The stationary states are **not** constant for $\theta > 0$. Moreover, they become **critical** after a finite length, that is, the velocity

   $$u = \frac{q}{\rho}$$

   approaches the sound speed a and the derivative tends to infinity in a monotone way.

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4. Note that

$$u = -\frac{1}{2}(R_+ + R_-).$$

The System in diagonal form

1. We have

$$R_t + D(R)R_x = S(R)$$

with

$$R = \begin{pmatrix} R_+ \\ R_- \end{pmatrix}, \; D(R) = \begin{pmatrix} a+u & 0 \\ 0 & -a+u \end{pmatrix}, \; S(R) = \begin{pmatrix} \frac{\theta}{2} u^2 \\ \frac{\theta}{2} u^2 \end{pmatrix}.$$

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2. Let $\bar{R}$ denote a stationary state, $D(\bar{R})\bar{R}_x = S(\bar{R})$.
3. We are interested in the difference r to the stationary state that is
$$\boxed{r = R - \bar{R}.}$$
We have
$$r_t + D(\bar{R} + r)r_x = [S(R) - S(\bar{R})] + [D(\bar{R}) - D(\bar{R} + r)]\,\bar{R}_x$$
that is
$$\boxed{r_t + D(\bar{R} + r)r_x = \frac{\theta}{2}\left[\frac{1}{4}(r_+ + r_-)^2 - (r_+ + r_-)\bar{u}\right]\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(r_+ + r_-)\,\bar{R}_x.}$$

The Characteristic Field

1. To define the system in characteristic form, we need the **characteristic curves**

$$\xi_{\pm}^{u}(t,x,t) = x,\ \partial_s \xi_{\pm}^{u}(s,x,t) = u \pm a.$$

$$\boxed{\xi_{\pm}^{u}(t,x,t) = x \pm a(t-t) + \int_t^s u(\tau,\, \xi_{\pm}^{u}(s,x,\tau))\, d\tau \in [0,L]}$$

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2. *Lemma 1* If $u \in C([0,T]\times[0,L])$ is Lipschitz continuous with respect to x (with Lipschitz constant L_u) and there exist a number $u_{\max}$ such that

$$|u(t,x)| \le u_{\max} < a$$

the characteristics are well defined. We have

$$|\xi_\pm^u(s,x,t) - \xi_\pm^v(s,x,t)| \le T\exp(L_u T)\|u-v\|_{C([0,T]\times[0,L])}.$$

Let $t_\pm^u(x,t) \le t$ denote the time where $\xi_\pm(\cdot,x,t)$ hits the boundary of $[0,T]\times[0,L]$. Then

$$|t_\pm^u(x,t) - t_\pm^v(x,t)| \le \frac{1}{a-\bar{u}_{\max}} T\exp(L_u T)\|u-v\|_{C([0,T]\times[0,L])}.$$

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$$r_\pm(t,x) = r_\pm(t_\pm^u(x,t),\ \xi_\pm^u(t_\pm^u(x,t),\ x,t)) + \int_{t_\pm^u(x,t)}^{t} p(r_+ + r_-)(\xi_\pm^u(s,\ x,t))\, ds$$

with $p_\pm(z) = \frac{\theta}{2}[\frac{1}{4}z^2 + (\frac{1}{2}\partial x \bar{R}_\pm - \bar{u})z]$.

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4. For example: With continuous $(r_+,\ r_-)$ that are Lipschitz with constant 1, provided that $\theta \leq \exp(-2T)/2$, the stationary state is sufficiently small in $C^1(0,L)$ and the initial data is sufficiently small in $C(0,L)$ with small Lipschitz constant.

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   $$h(t) = \|r_+\|_{C[0,L]} + \|r_-\|_{C[0,L]}.$$
   For $t \in [0, 2T]$ we have the *a priori bound* $h(t) \leq M$ with $M \leq 1$ depending on the (sufficiently small) initial data. Let $D = [0, 2T] \times [0, L]$. Then
   $$M = h(0) \exp\left(2T\theta\left(\frac{1}{4} + \|\partial_x \bar{R}_+\|_{C(D)} + \|\partial_x \bar{R}_-\|_{C(D)} + \|\bar{u}\|_{C(D)}\right)\right).$$

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3. For $t \in [0, T]$ we have
   $$h(T+t) \leq \theta\left(M + \|\partial_x \bar{R}_+\|_{C(D)} + \|\partial_x \bar{R}_-\|_{C(D)} + \|\bar{u}\|_{C(D)}\right) \int_t^{t+T} h(s)\, ds.$$

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$$h(T+t) \quad \leq \quad \theta\left(M + \|\partial_x \bar{R}_+\|_{C(D)} + \|\partial_x \bar{R}_-\|_{C(D)} + \|\bar{u}\|_{C(D)}\right) \int_t^{t+T} h(s)\, ds.$$
4. Let $\lambda = \theta(M + \|\partial_x \bar{R}_+\|_{C(D)} + \|\partial_x \bar{R}_-\|_{C(D)} + \|\bar{u}\|_{C(D)})$. Then
$$h(T+t) \leq \lambda \int_t^{t+T} h(s)\, ds \leq \lambda T M.$$
   By controlling the initial data and the stationary state, we can make the factor (λT) arbitrarily small.

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3. After Step 3, if we are sure that the Lipschitz constant at time $2T$ is small enough, we can continue the solution to $[2T, 4T]$ and proceed inductively.

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1. For $k \geq 2$, we get inductively

$$\begin{aligned} h(kT + t) &\leq \lambda \int_{(k-1)T+t}^{kT+t} h(s)\, ds \\ &= \lambda \int_t^{T+t} h((k-1)T + s)\, ds \\ &\leq \lambda \int_0^T (\lambda T)^{k-1} M\, ds \\ &\leq (\lambda T)^k M. \end{aligned}$$

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2. In this way we get the global existence and at the same time the exponential decay of $h(t)$ with the rate

$$\mu = \frac{|\ln(\lambda T)|}{T}.$$

The Exponential Decay

Define $\bar{U} = \|\partial_x \bar{R}_+\|_{C(D)} + \|\partial_x \bar{R}_-\|_{C(D)} + \|\bar{u}\|_{C(D)}$. Then we have the decay rate

$$\mu = -\frac{1}{T}\ln\left(T\theta\left[h(0)\exp\left(2T\theta\left(\frac{1}{4}+\bar{U}\right)\right)+\bar{U}\right]\right).$$

We have

$$\boxed{h(kT) \leq \exp(-\mu(kT))M = \exp\left(2T\theta\left(\tfrac{1}{4}+\bar{U}\right)\right)h(0)\exp(-\mu(kT)).}$$

The decay rate μ can be made arbitrarily large by choosing $h(0)$ and $\bar{U}$ sufficiently small.

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Conclusion

- LYAPUNOV-**functions** are an excellent tool to show exponential stability provided that the corresponding **local solutions** are available.
- However, also other approaches are possible that use **semiglobal** solutions This is useful if the system decay has a **stepwise** rather than a **continuous** character.
- In engineering practice, we often have nonlinear dynamics on networks:

There are lots of open questions!
Find a better feedback law!

Thank you for your attention!

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