A 1D model for the recorder

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Benasque's spirit

- Open problem presented in Benasque 2007
  (keep hoping!)
- Greatly inspired by the work of Juan Casado-Díaz, Manuel Luna-Laynez and François Murat
- Involved mathematics applied to a concrete model

Introduction: wind instruments

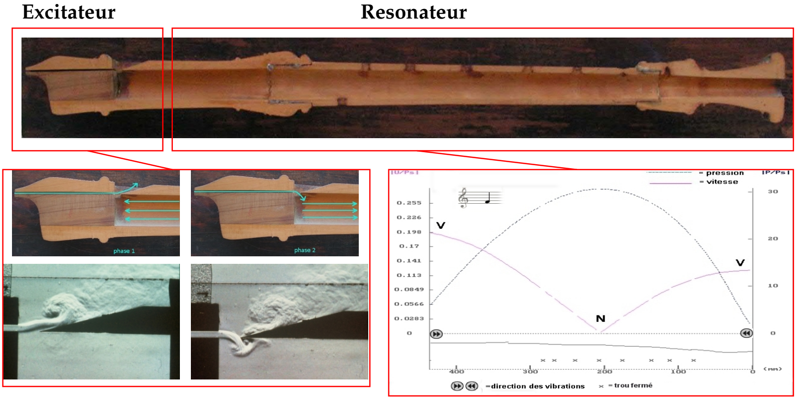
A musical note is the sum of several frequencies:

- A lowest frequency, the fundamental one, which is responsible of the note produced.
- Higher frequencies (or overtones), being multiple of the fundamental one, the harmonics, whose relative amplitudes are responsible of the tone of the note.

A musical instrument is an objet generating sounds, whose overtones are as close as possible of the multiples of the lowest frequency (this is the difference between a bell and a pot).

Introduction: wind instruments

A wind instrument is the combination of an exciter (fipple, reed. . .) and a tube. The exciter creates the oscillation and the eigenfrequencies of the tube selects the produced note.

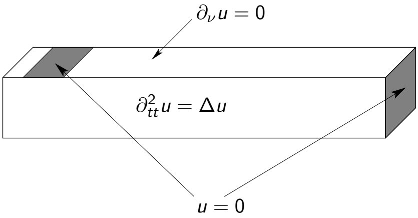


Images coming from: Philippe Bolton *www.flute-a-bec.com*

Introduction: the resonances

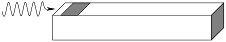
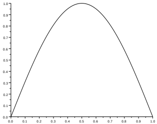
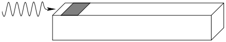
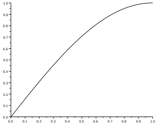
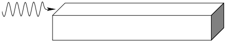
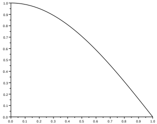
Assume that:

- the pressure u in the tube follows the wave equation $\partial^2_{tt} u = \Delta u$
- at the inner surface of the tube, the pressure satisfies Neumann B.C. $\partial_\nu u = 0$
- at an open part of the tube, the pressure is equal to the exterior pressure $u = 0$.



The resonances are **the square roots of the eigenvalues of the Laplacian operator**.

If the tube is sufficiently thin, then the 3D Laplacian operator is well approximated by the **1D Laplacian operator**.

flute recorder open organ pipe ...			π/L $2\pi/L$ $3\pi/L$ ...
closed organ pipe pan-pipes ...			$\pi/2L$ $3\pi/2L$ $5\pi/2L$ ...
reed instruments (clarinet, oboe...)			$\pi/2L$ $3\pi/2L$ $5\pi/2L$ ...

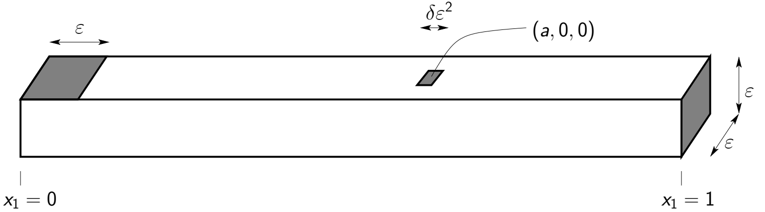
What happens when a hole of the flute is open?

- **1st case: large hole**
  (modern instruments)
  The flute behaves as a tube truncated at the place of the hole. We recover a harmonic sound.

- **2nd case: small hole**
  (recorder, baroque flutes)
  Not so simple: fork fingering, half-holes, not harmonic (high frequencies are less modified by the hole)...

Main result

Set $x = (x_1, \tilde{x}) \in \mathbb{R}^3$.
Let $a \in]0,1[$ and $\delta > 0$. Consider the domain Ω_ε as follows.



Gray: Dirichlet B.C., white: Neumann B.C.

Δ_ε is the positive Laplacian operator in Ω_ε with the associated B.C.

Main result

$$\forall u, v \in D(\Delta_\varepsilon) \ , \ \ \langle \Delta_\varepsilon u | v \rangle_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \nabla u \nabla v \ .$$

Let $A : D(A) \longrightarrow L^2(0,1)$ given by $Au = -u''$ and

$D(A) = \{u \in H^2((0,a)\cup(a,1))\cap H_0^1(0,1) \mid u'(a^+) - u'(a^-) = \alpha\delta u(a)\}$

with α depending on the geometry of the hole.

$$\forall u, v \in D(A) \ , \ \ \langle Au | v \rangle_{L^2(0,1)} = \int_0^1 u'(x)v'(x)dx + \alpha\delta u(a)v(a) \ .$$

Main result

Let $0 < \lambda_\varepsilon^1 < \lambda_\varepsilon^2 \leq \lambda_\varepsilon^3 \leq \dots$ be the eigenvalues of Δ_ε.

Let $0 < \lambda^1 < \lambda^2 \leq \lambda^3 \leq \dots$be the eigenvalues of A.

Theorem – R.J. (2011)

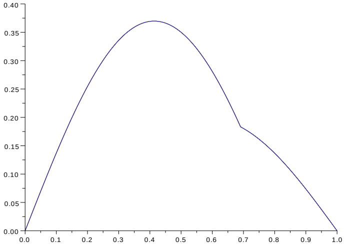
When $\varepsilon \longrightarrow 0$, the spectrum of Δ_ε converges to the one of A, i.e.

$$\forall k \in \mathbb{N}^* \ , \quad \lambda_\varepsilon^k \xrightarrow[\varepsilon \longrightarrow 0]{} \lambda^k \ .$$

Main idea: use the techniques of J. Casado-Díaz, M. Luna-Laynez and F. Murat dealing with domains with several orders of thickness.

Discussion

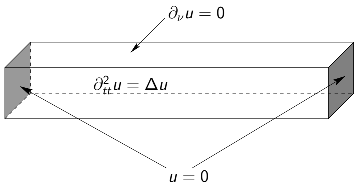
μ^2 is an eigenvalue of A iff $\alpha\delta = \dfrac{-\mu \sin \mu}{\sin(\mu a)\sin(\mu(1-a))}$.



- It is the expected profile of pressure.
- One can adjust a or $\delta \Rightarrow$ creation of fork fingerings, half-holes. . .
- The overtones are not really harmonic.

Sketch of the proof (simplified case)

First consider the tube without open holes.



Min-Max principle

$$\lambda_\varepsilon^k = \min_{E^k \text{ vect. space of dim } k \text{ of } H_0^1(\Omega_\varepsilon)} \max_{u \in E^k} \frac{\int |\nabla u|^2}{\int |u|^2} .$$

- Lower-semicontinuity of the spectrum: Min-Max principle and embedding of the eigenfunctions of A in $H_0^1(\Omega_\varepsilon)$.
- Upper-semicontinuity of the spectrum: Min-Max principle and weak convergence of the $k-$th eigenfunction of Δ_ε to an eigenfunction of A.

Proof of the lower-semicontinuity of the spectrum:

- Let $\varphi^k = \sin(k\pi\cdot)$ be the eigenfunctions of $\partial^2_{x_1x_1}$. We embed φ^k in $D(\Delta_\varepsilon)$ by setting $\varphi^k_\varepsilon(x) = \sin(k\pi x_1)$, we get

$$\langle\varphi^j_\varepsilon|\varphi^k_\varepsilon\rangle_{L^2(\Omega_\varepsilon)} = \varepsilon^2\langle\varphi^j|\varphi^k\rangle_{L^2(0,1)}$$

$$\langle\nabla\varphi^j_\varepsilon|\nabla\varphi^k_\varepsilon\rangle_{L^2(\Omega_\varepsilon)} = \varepsilon^2\langle\partial_{x_1}\varphi^j|\partial_{x_1}\varphi^k\rangle_{L^2(]0,1[)}$$

- We apply Min-Max principle

$$\begin{aligned}\lambda^k_\varepsilon &= \min_{E^k \text{ vect. space of dim } k \text{ of } H^1_0(\Omega_\varepsilon)} \ \max_{u\in E^k} \frac{\int|\nabla u|^2}{\int|u|^2}\\ &\leq \max_{u\in vect(\varphi^j_\varepsilon)_{j\leq k}} \frac{\int|\nabla u|^2}{\int|u|^2}\\ &\leq \lambda^k\end{aligned}$$

Sketch of the proof (simplified case)

Proof of the upper-semicontinuity of the spectrum:

- Up to extracting, λ_ε^k converges to λ_0^k.
- We move to $\Omega = (0,1)^3$ by setting $v_\varepsilon^k(x) = \varphi_\varepsilon^k(x_1, \varepsilon\tilde{x})$. We have
$$\int_\Omega |\partial_{x_1} v_\varepsilon^k|^2 + \frac{1}{\varepsilon^2}|\partial_{\tilde{x}} v_\varepsilon^k|^2 = \lambda_\varepsilon^k \ .$$
- Up to extracting, v_ε^k converges to v^k weakly in $H^1(0,1)$ and strongly in $H^{3/4}(0,1)$.

Sketch of the proof (simplified case)

Let ψ be a 1D test function. We embed ψ in Ω_ε by setting $\psi_\varepsilon(x) = \psi(x_1)$. We get

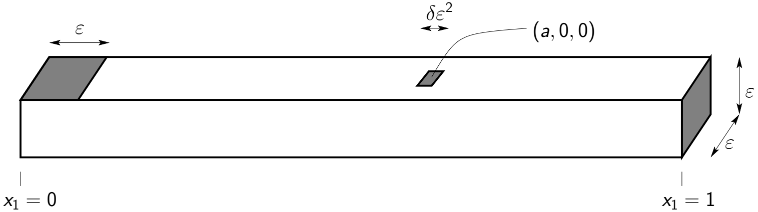
$$\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} \nabla\varphi_\varepsilon^k \nabla\psi_\varepsilon = \frac{1}{\varepsilon^2}\lambda_\varepsilon^k \int_{\Omega_\varepsilon} \varphi_\varepsilon^k \psi_\varepsilon$$

$$\xrightarrow[\varepsilon \longrightarrow 0]{} \int_0^1 \partial_{x_1} v^k \partial_{x_1}\psi = \lambda_0^k \int_0^1 v^k \psi \ .$$

Thus, v^k is a eigenfunction of $\partial^2_{x_1 x_1}$ for the eigenvalue λ_0^k and $\lambda_0^k \geq \lambda^k$ because the v^k are orthogonal.

Sketch of the proof

Consider the original domain with an open hole

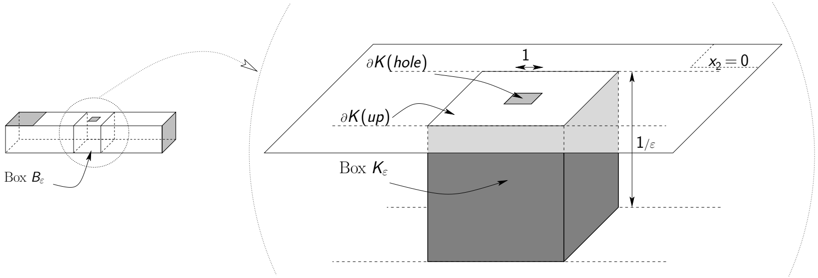


We cannot use the canonical embedding 1D$\rightarrow$3D: one needs to study what happens around the hole.

Method: one introduces a zoom close to the hole and the corresponding functional spaces following
[J. Casado-Díaz, M. Luna-Laynez and F. Murat].

Zoom around the hole

Let K be the half-space $\{x, x_2 < 0\}$ with the boundary condition $u = 0$ on $\partial K(hole)$ and $\partial_\nu u = 0$ on $\partial K(up)$.



$$\dot{H}^1(K) = \{v \in H^1_{loc}(K)\ ,\quad \nabla v \in L^2(K) \text{ and } u_{|\partial K(hole)} = 0\}$$

$$\dot{H}^1_0(K) = \text{ closure of } \mathcal{C}^\infty_0(K) \text{ in } \dot{H}^1(K)$$

We endow both spaces of the scalar product $\langle u|v\rangle = \int \nabla u \nabla v$.

Zoom around the hole

Let $\chi \in \mathcal{C}^\infty(\overline{K})$ satisfying the boundary conditions and $\chi \equiv 1$ outside a bounded ball.

Theorem

$\dot{H}^1(K)$ and $\dot{H}^1_0(K)$ are Hilbert spaces and

$$\dot{H}^1(K) = \dot{H}^1_0(K) \oplus \mathbb{R}\chi \ .$$

Moreover, $u \in \dot{H}^1(K)$ belongs to $\dot{H}^1_0(K)$ if and only if $u \in L^6(K)$. Finally, $u \in \dot{H}^1(K)$ splits in $u = \dot{u} + \overline{u}\chi$ where

$$\dot{u} \in \dot{H}^1_0(K) \qquad \overline{u} = \lim_{\varepsilon \to 0} \frac{1}{K_\varepsilon} \int_{K_\varepsilon} u(x) dx \ .$$

Zoom around the hole

We define ζ as the unique solution in $\dot{H}^1(K)$ of

$$\begin{cases} \Delta\zeta = 0 \\ \overline{\zeta} = 1 \\ +B.C. \end{cases}$$

Let $\alpha = \int_K |\nabla\zeta|^2$.
ζ is the orthogonal projection of χ on the orthogonal space of $\dot{H}^1_0(K)$

$$\langle u|v\rangle_{\dot{H}^1(K)} = \int \nabla\dot{u}\nabla\dot{v} \; + \; \alpha\,\overline{u}\,\overline{v}$$

Proposition

There exists a sequence (ζ_ε) converging to ζ in $\dot{H}^1(K)$ such that $\zeta_\varepsilon \equiv 1$ outside K_ε.

We set $\tilde{\zeta}_\varepsilon = \zeta_\varepsilon(\cdot/\varepsilon^2)$ which is supported in B_ε.

Sketch of the proof of the lower-semicontinuity of the spectrum:

- Let (φ^k) be the eigenfunctions of A. We embed φ^k in $D(\Delta_\varepsilon)$ to φ_ε^k such that

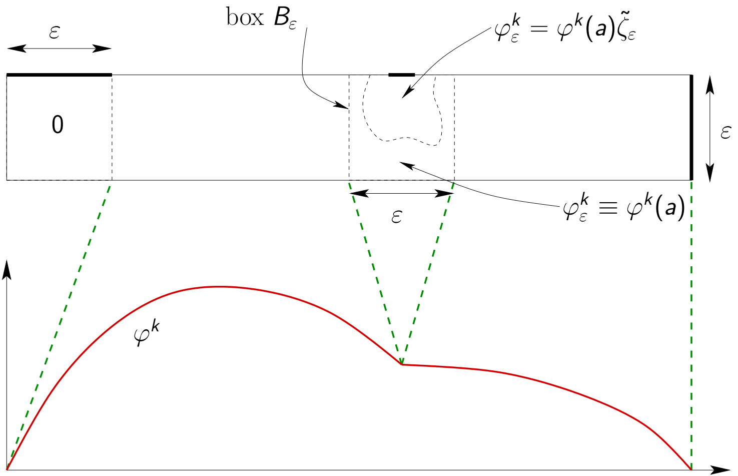
$$\langle \varphi_\varepsilon^j | \varphi_\varepsilon^k \rangle_{L^2(\Omega_\varepsilon)} = \varepsilon^2 \langle \varphi^j | \varphi^k \rangle_{L^2(0,1)} + o(\varepsilon^2)$$

$$\langle \Delta_\varepsilon \varphi_\varepsilon^j | \varphi_\varepsilon^k \rangle_{L^2(\Omega_\varepsilon)} = \varepsilon^2 \langle A\varphi^j | \varphi^k \rangle_{L^2(0,1)} + o(\varepsilon^2)$$

- We apply Min-Max principle

$$\begin{aligned}
\lambda_\varepsilon^k &= \min_{E^k \text{ vect. space of dim } k \text{ of } H_0^1(\Omega_\varepsilon)} \ \max_{u \in E^k} \frac{\int |\nabla u|^2}{\int |u|^2} \\
&\leq \max_{u \in vect(\varphi_\varepsilon^j)_{j \leq k}} \frac{\int |\nabla u|^2}{\int |u|^2} \\
&\leq \lambda^k + o(\varepsilon)
\end{aligned}$$

Lower-semicontinuity



$$\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon^k|^2 \simeq \int_0^1 |\partial_x \varphi^k|^2 + \delta|\varphi^k(a)|^2 \int_K |\nabla \zeta|^2$$

Sketch of the proof of the upper-semicontinuity of the spectrum:

- Up to extracting, λ_ε^k converges to λ_0^k.
- We move to $\Omega = (0,1)^3$ by setting $v_\varepsilon^k(x) = \varphi_\varepsilon^k(x_1, \varepsilon\tilde{x})$. We have
$$\int_\Omega |\partial_{x_1} v_\varepsilon^k|^2 + \frac{1}{\varepsilon^2}|\partial_{\tilde{x}} v_\varepsilon^k|^2 = \lambda_\varepsilon^k \ .$$
- Up to extracting, v_ε^k converges to v^k weakly in $H^1(0,1)$ and strongly in $H^{3/4}(0,1)$.
- We prove that v^k is an eigenfunction of A for the eigenvalue λ_0^k.

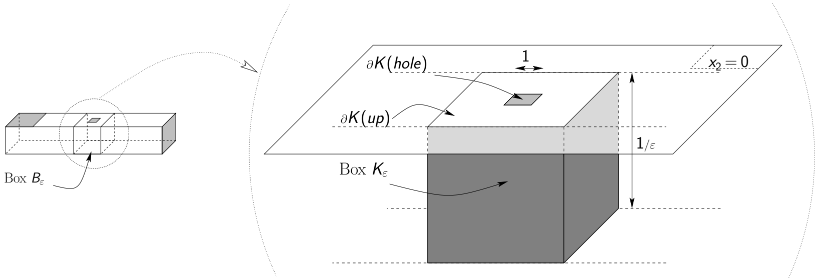
Upper-semicontinuity

Let $\psi \in H_0^1(]0,1[)$ be a test function. We embed ψ in Ω_ε as above and we show that

$$\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} \nabla\varphi_\varepsilon^k \nabla\psi_\varepsilon = \frac{1}{\varepsilon^2}\lambda_\varepsilon^k \int_{\Omega_\varepsilon} \varphi_\varepsilon^k \psi_\varepsilon$$

$$\xrightarrow[\varepsilon\longrightarrow 0]{} \int_0^1 \partial_{x_1} v^k \partial_{x_1}\psi + \alpha\delta v^k(a)\psi(a) = \lambda_0^k \int_0^1 v^k\psi \ .$$

Crucial point: what happens in the box B_ε?



Upper-semicontinuity

Why

$$\frac{1}{\varepsilon^2}\int_{B_\varepsilon} \nabla\varphi_\varepsilon^k \nabla\psi_\varepsilon = \int_{K_\varepsilon} \nabla w_\varepsilon \nabla(\psi(a)\zeta_\varepsilon)$$

converges to $v^k(a)\psi(a)\int_K |\nabla\zeta|^2$?

- By construction $\psi(a)\zeta_\varepsilon \to \psi(a)\zeta$ in $\dot{H}^1(K)$.
- We can assume that w_ε converges weakly to w_0 in $\dot{H}^1(K)$
- If $\phi \in \mathcal{C}_0^\infty(K)$, then $\int_K \nabla w_0 \nabla\phi = 0$
- The mean value of w_0 is given by

$$\overline{w}_0 = \lim_{\varepsilon_n \to 0} \frac{1}{|K_{\varepsilon_n}|}\int_{K_{\varepsilon_n}} w_{\varepsilon_n} = v^k(a)\ .$$

- Thus $w_\varepsilon \rightharpoonup w_0 = v^k(a)\zeta$ in $\dot{H}^1(K)$.

Further discussions

- Several holes: several discontinuities of the derivative.
- Cylindrical flutes: no changes for the first order.
- Conical flutes: the $1D$ Laplacian operator is replaced by a Laplacian operator with a different metric $\frac{1}{S(x)}\partial_x(S(x)\partial_x \cdot)$ where $S(x)$ is the sectional volume.
- External radiation: one has to compute the second order.
- Interaction with the exciter: ???

Thank you for your attention