Shape optimization for the observability of PDEs

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N-D wave equation

$$\begin{cases} y_{tt} - \Delta y = 0 & (t,x) \in (0,T) \times \Omega \\ y(t,x) = 0 & t \in [0,T],\ x \in \partial\Omega \\ y(0,x) = y^0(x),\ \partial_t y(0,x) = y^1(x) & x \in \Omega. \end{cases} \qquad (1)$$

- $\Omega \subset \mathbb{R}^d$ is bounded
- $T > 0$ fixed

$$\forall (y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$$

$$\exists! y \in \mathcal{C}^0([0,T], H_0^1(\Omega)) \times \mathcal{C}^1([0,T], L^2(\Omega)), \text{ solution of } (1)$$

Observable variable ($\omega \subset \Omega$ of positive measure)

$$z(t,x) = \chi_\omega(x) \partial_t y(t,x)$$

Observability of the N-D wave equation

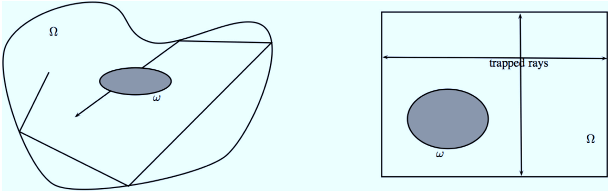
$\hookrightarrow$ Without loss of generality, we consider the wave equation with Dirichlet boundary conditions

Observability inequality

The time T being chosen large enough, how to choose $\omega \subset \Omega$ to ensure that $\forall (y^0, y^1) \in H_0^1(\Omega)(\Omega) \times L^2(\Omega)$

$$C_T \|(y^0, y^1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_0^T \int_\Omega z(t,x)^2 dxdt \ ? \tag{2}$$

- Microlocal Analysis. Bardos, Lebeau and Rauch proved that, roughly in the class of $\mathcal{C}^\infty$ domains, the observability inequality (2) holds iff (ω, T) satisfies the Geometric Control Condition (GCC).



Shape optimization problems

- Observability constant :

$$C_T(\chi_\omega) = \inf_{\substack{y \text{ solution of } (1) \\ (y^0,y^1)\in H_0^1(\Omega)\times L^2(\Omega)}} \frac{\int_0^T \int_\omega y_t(t,x)^2 dxdt}{\|(y^0, y^1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}.$$

A relevant problem when looking for optimal sensors location ?

Fix $L \in (0,1)$. We investigate the problem of maximizing the observability constant $C_T(\chi_\omega)$ over all possible subset $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

Related problems

Optimal design for control/stabilization problems

1. What is the "best domain" for achieving HUM optimal control ?

$$y_{tt} - \Delta y = \chi_\omega u$$

2. What is the "best domain" domain for stabilization (with localized damping) ?

$$y_{tt} - \Delta y = -k\chi_\omega y_t$$

See works by
- P. Hébrard, A. Henrot : theoretical and numerical results in 1D for optimal stabilization (for all initial data).
- A. Münch, P. Pedregal, F. Periago : numerical investigations of the optimal domain (for one fixed initial data). Study of the relaxed problem.
- S. Cox, P. Freitas, F. Fahroo, K. Ito, ... : variational formulations and numerics.
- M.I. Frecker, C.S. Kubrusly, H. Malebranche, S. Kumar, J.H. Seinfeld, ... : numerical investigations (among a finite number of possible initial data).
- K. Morris, S.L. Padula, O. Sigmund, M. Van de Wal, ... : numerical investigations for actuator placements (predefined set of possible candidates), Riccati approaches.
- ...

Modeling of the optimal design problem

Fix $L \in (0,1)$

Optimal design Problem

We investigate the problem of maximizing the quantity $C_T(\chi_\omega)$ over all possible subsets $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

Two difficulties

1. Theoretical difficulties
2. The model is not relevant w.r.t. practical expectation

   The usual observability constant is deterministic and gives an account for the worst case. It is pessimistic.

$\hookrightarrow$ In practice : many experiments, many measures.

$\rightarrow$ Objective : optimize the sensor shape and location **in average**.

$\rightarrow$ randomized observability constant.

A randomized observability constant

Random selection of the initial data

↪ We consider the randomized observability inequality

$$C_{T,\text{rand}}(\chi_\omega)\|(y^0,y^1)\|^2_{H^1_0\times L^2}\leq \mathbb{E}\left(\int_0^T\int_\omega y_t^\nu(t,x)^2\,dxdt\right),$$

for all $y^0(\cdot)\in L^2(\Omega)$ and $y^1(\cdot)\in H^{-1}(\Omega)$, where y^ν denotes the solution of the wave equation with random initial data $y^{0,\nu}$ and $y^{1,\nu}$.

Proposition

For every measurable set $\omega\subset\Omega$,

$$C_{T,\text{rand}}(\chi_\omega)=T\inf_{j\in\mathbb{N}^*}\int_\omega \phi_j(x)^2\,dx.$$

where ϕ_j denotes the j-th eigenfunction of the Laplace-Dirichlet operator on Ω.

There holds $C_{T,\text{rand}}(\chi_\omega)\geq C_T(\chi_\omega)$. There are examples where the inequality is strict.

Optimal observability with respect to the domain

Question

What is the "best possible" observation domain ω of given measure ?

A new "Second Problem" (energy concentration criterion)

We investigate the problem of maximizing

$$\frac{C_{T,\text{rand}}(\chi_\omega)}{T} = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\, dx.$$

over all possible subset $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

Solving of the optimal design problem

Relaxation procedure

Second problem

$$\sup_{\substack{\omega\subset\Omega \\ |\omega|=L|\Omega|}} J(\chi_\omega) := \sup_{\substack{\omega\subset\Omega \\ |\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2 dx$$

- Admissible set for this problem :

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of measure } L|\Omega|\}.$$

- Closure of this set for the weak-star topology of L^∞ :

$$\overline{\mathcal{U}}_L = \left\{ a \in L^\infty(\Omega;[0,1]) \mid \int_\Omega a(x)dx = L|\Omega| \right\}.$$

Relaxed problem

$$\sup_{a\in\overline{\mathcal{U}}_L} J(a) := \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2 dx$$

Solving of the optimal design problem

What we know about it

(L^∞-weak Quantum Ergodicity) Assumption

- The sequence $(\phi_j^2)_{j\in\mathbb{N}^*}$ is uniformly bounded in L^∞ norm
- There exists a subsequence such that $\phi_j^2 \rightharpoonup \frac{1}{|\Omega|}$ vaguely as $j \to +\infty$

We have

$$\sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2 dx = L \quad \text{(reached with } a = L\text{)}$$

Remarks.

- L^∞-WQE holds true in any flat torus
- if Ω is a convex ergodic billiard with $W^{2,\infty}$ boundary then $\phi_j^2 \rightharpoonup \dfrac{1}{|\Omega|}$ vaguely for a subset of indices of density 1.

Gérard-Leichtnam (Duke Math. 1993), Zelditch-Zworski (CMP 1996), Burq-Zworski (SIAM Rev. 2005), see also Shnirelman, Colin de Verdière,...

Solving of the optimal design problem

Theorem

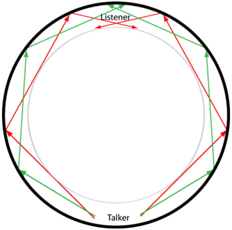
Under L^∞-WQE, there is no gap, that is :

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega \chi_\omega(x)\phi_j(x)^2\,dx = \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2\,dx = L.$$

$\rightarrow$ the maximal value of the time-asymptotic / randomized observability constant is L.

Remark

L^∞-WQE is not a sharp assumption :
the result also holds also true in the Euclidean disk, for which however the eigenfunctions are not uniformly bounded in L^∞ (whispering galleries phenomenon).



Conjecture

For generic domains Ω and generic values of L, the supremum is not reached and hence there does not exist any optimal set.

A truncated criterion : $J_N(a) = \inf_{1\leq j\leq N} \int_\Omega \chi_\omega(x)\phi_j(x)^2\,dx$

A truncated shape optimization problem

$$\boxed{\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{1\leq j\leq N} \int_\Omega \chi_\omega(x)\phi_j(x)^2\,dx}$$

Theorem

Let $L \in (0,1)$. The shape optimization problem above has a unique solution ω_N^*.

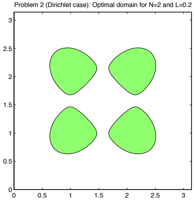
- A Γ-convergence result :

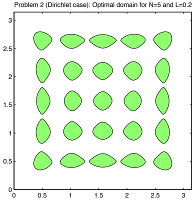
$$\lim_{N\to+\infty} \sup_{\chi_\omega \in \mathcal{U}_L} J_N(\chi_\omega) = \sup_{a\in\overline{\mathcal{U}}_L} J(a)$$

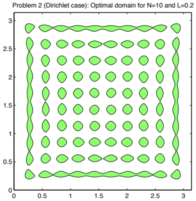
- Convergence of $(\chi_{\omega_N^*})_{N\in\mathbb{N}^*}$ to a minimizer of the optimal design problem.

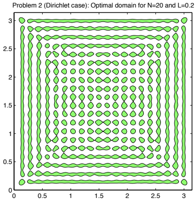
Several numerical simulations : $\Omega = [0, \pi]^2$

For 4, 25, 100 and 500 eigenmodes and $L = 0.2$









Conclusion of this talk

- Intimate relations between domain optimization and quantum chaos (quantum ergodicity properties).
- Ongoing works :
  - optimal design for the heat equation.
  - optimal design for boundary observability. (with P. Jounieaux, Paris 6) Ω being assumed bounded connected and its boundary $\mathcal{C}^2$, maximize

$$\inf_{j\in\mathbb{N}^*} \frac{1}{\lambda_j(\Omega)} \int_\Sigma \left| \frac{\partial \phi_j}{\partial n} \right|^2 dx$$

    over all possible subsets $\Sigma \subset \partial\Omega$ of given Hausdorff measure.
  - new strategies to avoid spillover phenomena when solving optimal design problems.
  - discretization issues. Do the numerical designs converge to the continuous optimal design as the mesh size tends to 0 ?

Y. Privat, E. Trélat, E. Zuazua, *Optimal observation of the one-dimensional wave equation*, to appear in J. Fourier Analysis Appl.

Y. Privat, E. Trélat, E. Zuazua, *Optimal location of controllers for the one-dimensional wave equation*, to appear in Ann. Inst. H. Poincaré.

Y. Privat, E. Trélat, E. Zuazua, *Optimal observability of wave and Schrödinger equations in ergodic domains*, Preprint (2012).

Thank you for your attention