Controlling Velocities and Domains of the Incompressible Fluid by Means of a Degenerate Distributed Forcing

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Euler/Navier-Stokes (N-S) equation of fluid motion

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u + F(t, x),$$
$$\nabla \cdot u = 0.$$

$u(t, x)$ - field of velocities, $p(t, x)$ - pressure, $F(t, x)$ - controlling forcing term

We use low-dimensional or degenerate control - a linear combination with controlled time-variant coefficients:

$$F(t, x) = \sum_{k \in \mathcal{K}} f_k(t) \phi_k(x),$$

$\mathcal{K}$ is a **fixed** finite set, $\phi_k(x)$ are fixed functions of space variables.

The choice of the class of **degenerate** controls distinguishes the problem setting from other approaches where a "full-dimensional" control is applied on the boundary or on a sub-domain (cf. *J.-M. Coron, A. Fursikov - O. Imanuvilov et al*, 90's)

Exact controllability by means of degenerate control - is not possible for any $\nu \geq 0$.

Adequate controllability setting - approximate controllability, or controllability in finite-dimensional projections.

We will seek for controllability criteria with **fixed sets $\mathcal{K}$ of controlled modes**, which are *independent of the rate of approximation and of the dimension of projections*.

The dimension of needed control is small (≤ 8) in each of the following criteria of approximate controllability.

- sufficient controllability criteria for finite-dimensional Galerkin approximations of the N-S equation 2D and 3D torus (A.Agrachev, AS, 2004, M.Romito, 2004)

- sufficient approximate controllability and controllability in fin.-dim. projections for N-S/Euler equation on:

  flat 2D torus, rectangle, sphere, hemisphere,

  Riemannian surface with generic metric; (A.Agrachev, AS, S.Rodrigues, '04-09)

- extension of the methods on 3D torus (A.Shirikyan, H.Nersessian, '06-'09)

Open questions remaining:

The obtained approximate controllability criteria, are 'basis-dependent'.

The 'controlled modes' $\phi_k(x)$, which form the controlling forcing term $F(t,x) = \sum_{k \in \mathcal{K}} f_k(t)\phi_k(x)$ are to be chosen in relation to **spectral geometry** of the 2D domain (are complex exponentials in the case of torus, spherical harmonics for a sphere/hemisphere etc.)

Desirable: approximate controllability criteria which are **structurally stable** with respect to perturbations of $\phi_k(x)$.

So far controllability of the N-S equation of its (high-dimensional) Galerkin approximations remains unproved for many 2D domains, e.g for a disc D^2.

Geometric control for other classes of non linear (semilinear) equations

Results on approximate controllability obtained for

- defocusing cubic Schroedinger equation (AS, 2012);
- Burgers equation (A.Shirikyan, to appear).

There is work in progress on abstract semilinear equation with a polynomial nonlinearity, driven by a degenerate control.

An interesting problem is an extension of the developed methods on **nonpolynomial** nonlinearities.

Controlling velocity field and the 'domains'

Coming back to the Navier-Stokes equation we extend our problem setting by adding a (Lagrange) equation for the motion of the domains of fluid by virtue of the velocity field.

We consider the flow on 2D torus $\mathbb{T}^2$ with the standard area form.

We seek for a sufficient criteria that two velocity fields - the initial $\tilde{u}(x)$ and the target $\hat{u}(x)$, and two domains - the initial $\tilde{\mathcal{D}} \subset \mathbb{T}^2$ and the target domain $\hat{\mathcal{D}} \subset \mathbb{T}^2$ - of equal volumes can be steered one to another *approximately* by a degenerate forcing acting on the time interval $[0, T]$.

Representation of the Lagrangian part

The domains $\tilde{\mathcal{D}}, \widehat{\mathcal{D}}$ may have several (equal number of) connected components of coinciding volumes:

$$\tilde{\mathcal{D}} = \bigcup_{s=1}^{S} \tilde{\mathcal{D}}_s, \widehat{\mathcal{D}} = \bigcup_{s=1}^{S} \widehat{\mathcal{D}}_s, \ \mathrm{vol}(\widehat{\mathcal{D}}_s) = \mathrm{vol}(\tilde{\mathcal{D}}_s), \ s = 1, \dots, S.$$

Each domain is represented as a regular Lebesgue set of a smooth function: $\mathcal{D}_\chi = \{x \in \mathbb{T}^2 | \ \chi(x) \geq 0\}$.

The flow of a fluid - a family of volume preserving diffeomorphisms P_t - on $\mathbb{T}^2$ - transforms the domain $\mathcal{D}_\chi$ into the domains $\mathcal{D}_{\chi_t}$, where $\chi_t(x) = \left(\widehat{P}_t^{-1}\chi\right)(x) = \chi\left(P_t^{-1}(x)\right)$.

By direct computation

$$\partial_t \chi_t = -u_t \circ \chi_t = -\nabla \chi_t \cdot u_t.$$

We will seek for approximate controllability in the space of functions χ.

We assume each connected component $\mathcal{D}^j$ to be diffeomorphic to a disc and to be a Lebesgue set $\mathcal{L}^{\chi}_{\geq 0}$ of a smooth function χ_j, with the boundary $\partial\mathcal{D}^j$ being a (connected component of the) regular level set $\chi_j^{-1}(0)$.

Under these assumptions $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$\forall \zeta : \ \|\zeta - \chi\|_{L_2(\mathbb{T}^2)} < \delta \Rightarrow \text{meas}\left(\mathcal{L}^{\chi}_{\geq 0} \triangle \mathcal{L}^{\chi}_{\geq 0}\right) < \varepsilon,$$

so approximating $\chi(x)$ means 'approximation in measure' of its Lebesgue set.

Problem setting

$$\begin{cases} \partial_t \chi_t = -u_t \circ \chi_t = -\nabla \chi_t \cdot u_t. \\ \partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + \sum_{k \in \mathcal{K}} e^{ik \cdot x} v_k(t), \\ \nabla \cdot u = 0, \end{cases} \qquad \textbf{(1)}$$

Definition 1 (approximate controllability) *The system* **(1)** *is time-T approximately controllable, if for any two couples $(\tilde{\chi}, \tilde{u}), (\hat{\chi}, \hat{u})$ and for each $\varepsilon > 0$ there exists a control which steers the system in time T from $(\tilde{\chi}, \tilde{u})$ to the ε-neighborhood of $(\hat{\chi}, \hat{u})$ in the norm of $L_2(\mathbb{T}^2) \times L_2(\mathbb{T}^2)$.* $\square$

Related to this setting is the work by O.Glass and T.Horsin on Lagrangian controllability.

Main result We formulate a sufficient criterion of approximate controllability for the system **(1)**.

Theorem. Choosing the 4-element set of controlled modes $\mathcal{K}_4 = \{(\pm 1, 0), (\pm 1, \pm 1)\} \subset \mathbb{Z}^2$ one achieves time-T approximately controllability of the system (1) for each $T > 0$.

Remark. There is an ample class of sets $\mathcal{K}$ possessing so-called saturating property on $\mathbb{Z}^2$ which suffice for approximate controllability.

Sketch of the proof follows.

Helmholtz form of the N-S equation

It is natural in 2D case to pass to the vorticites $w = \nabla^{\perp} \cdot u$.

Represent the divergence-free velocity field u with $\int_{\mathbb{T}^2} u dx = 0$ as $u = \nabla^{\perp}\Delta^{-1}w = \nabla^{\perp}\psi$, where w is the vorticity and $\psi = \Delta^{-1}w$ is a stream function.

The N-S equation can be given Helmholtz form

$$\partial_t w + \{\Delta^{-1}w, w\} - \nu\Delta w = v(t, x),$$

where $\{\cdot,\cdot\}$ stays for the Poisson bracket, corresponding to a standard symplectic form on the flat torus $\mathbb{T}^2$.

The Lagrangian part can be rewritten as

$$\partial_t \chi = \{\chi, \Delta^{-1}w\}.$$

The control is $v(t,x)=\sum_{k\in\mathcal{K}} v_k(t)e^{ik\cdot x},\ \mathcal{K}\subset\mathbb{Z}^2$.

Thus we get a system

$$\partial_t\chi = \{\chi, \Delta^{-1}w\},$$
$$\partial_t w + \{\Delta^{-1}w, w\} - \nu\Delta w = \sum_{k\in\mathcal{K}} v_k(t)e^{ik\cdot x},$$

which is a particular kind of control-affine infinite-dimensional system

$$\dot{y} = f^0(y) + \sum_{k\in\mathcal{K}} f^k v_k(t)$$

with quadratic polynomial "drift term" $f^0(y)$ and constant ((w,χ)-independent) controlled vector fields f^k.

Outline of the approach from *geometric control* viewpoint

Our study of controllability of the equation stems from the method of *iterated Lie extensions*, available in finite-dimensional case.

Lie extension of the control system $\dot{y} = f^0(y) + \sum_{k \in \mathcal{K}} f^k v_k(t)$ is a method of 'enrichment' of the r.-h. side of the control system by extending controlled vector fields *'almost maintaining' controllability properties of the system*.

The additional v.f. are expressed via Lie brackets of f^j. If after a series of extensions one arrives to a system, which is controllable, then controllability of the original system can be concluded.

Examples of extensions

We are in finite-dimensional case

Example 1: extension by a convexification or relaxation
Trajectories of the system

$$\dot{x} = 1, \dot{y} = u, \ u \in \{-1, 1\} \text{ - a nonconvex set}$$

C^0-approximate trajectories of the convexified system

$$\dot{x} = 1, \dot{y} = u, \ u \in [-1, 1] \text{ - convexified set}$$

Example 2: Lie bracket extension Consider a control system on a manifold M

$$\dot{x} = X^1(x)u_1 + X^2(x)u_2,$$

where the v.f. $X^1(x), X^2(x)$ satisfy a **bracket generating condition**. Then the system is globally controllable (Rashevsky-Chow Th.).

Key Lie extension

The following Lie extension will be key (guiding) tool for establishing our approximate controllability result.

Lemma 2 (Lie extension lemma) *Consider control-affine analytic system*

$$\dot{y} = f^0(y) + f^1(y)\widehat{v}_1 + f^2(y)\widehat{v}_2. \quad (2)$$

Let $[f^1, f^2] = 0, \ [f^1, [f^1, f^0]] = 0$. *Then the system*

$$\dot{y} = f^0(x) + f^1(y)\tilde{v}_1 + f^2(y)\tilde{v}_2 + [f^1, [f^2, f^0]](y)v_{12}, \quad (3)$$

is fixed-time Lie extension of (2), meaning that

CLOSURES OF THE TIME-T ATTAINABLE SETS OF (2) and (3) COINCIDE. □

Finite-dimensional analogue: controlling angular velocity and the attitude of a satellite

Controlled attitude motion of a satellite (rigid body) is described by a couple of equations on $SO(3) \times so^*(3)$

$$\begin{aligned} \dot{Q} &= Q\mathcal{J}^{-1}M, \\ \dot{M} &= M \times \mathcal{J}^{-1}M - \nu M + L_1 v_1(t) + L_2 v_2(t), \end{aligned}$$

where $Q \in SO(s)$ describes the attitude, while $M \in so^*(3) \asymp \mathbb{R}^3$ is an angular momentum of the satellite, $\mathcal{J}$ being an inertia tensor, with distinct eigenvalues (principal inertia momenta).

This is an example of *control-affine system*

$$\dot{x} = f^0(x) + f^1 v_1(t) + f^2 v_2(t).$$

In our example the drift v.f. $f^0(x)$ is 2nd degree polynomial and the controlled v.f. $f^1 = L_1, f^2 = L_2$ are constant.

The state (Q, M) is 6-dimensional and there are two controls.

We assume L_1 to be a principal axis of the body, so that $[f^1, [f^1, f^0]]$ vanishes.

Involving the Lie extension lemma we get an **extending control v.f.** $f^{12} = [f^1, [f^2, f^0]]$, multiplied by the extending control $v_{12}(t)$; v.f. f^{12} has vanishing Q-component and constant M-component

$$L_{12} = L_1 \times \mathcal{J}^{-1} L_2 + L_2 \times \mathcal{J}^{-1} L_1.$$

For a generic choice of L_1, L_2 the triple of vectors

$$\begin{pmatrix} 0 \\ L_k \end{pmatrix}$$

is linearly independent.

Computing the Lie brackets $[f^0, f^k], \ k \in 1, 2, 12$ we get another triple of v.f.

$$[f^0, f^k] = \begin{pmatrix} QJ^{-1}L_k \\ * \end{pmatrix}$$

with linearly independent Q-components.

Having 6 controls available makes it easy to conclude controllability (it **is small-time and controls are high-gain**).

In the PDE case the respective Lie algebraic and differential geometric tools are not available

We will employ fast-oscillating controls, whose use underlies Lie extensions method in finite dimension.

Specially designed resonances between such controls result in a motion which provides (*approximates*) motion in *extending* direction of a Lie bracket.

Structure of the control

The control we are going to construct is a concatenation of two controls.

First control steers in time T the initial domain $\tilde{D}$ to a neighborhood of the target domain $\hat{D}$.

At the same time it keeps the terminal vorticity $w(T)$ in a bounded domain.

The **second control** steers the field of velocities to a neighborhood of the target field, while "not moving much" the domain.

Controlling the domain

The construction goes in 3 steps.

1. Moving the domain by ANY hamiltonian dynamics.

2. Moving the domain (approximately) by virtue of N-S equation with HIGH-DIMENSIONAL forcing.

3. Moving the domain (approximately) by virtue of N-S equation with LOW-DIMENSIONAL (degenerate) forcing.

1. Volume-preserving diffeotopy. (Anosov-Katok-)Krygin's Theorem

Our construction starts with the result on existence of an area preserving dynamics, which transforms the domain $\tilde{D}$ into $\hat{D}$ exactly.

Theorem 3 (A-K-K, 1970,1971) *For two sets* $\tilde{D}^1, \dots, \tilde{D}^\alpha$, $\hat{D}^1, \dots, \hat{D}^\alpha$ *of disjoint domains diffeomorfic to a closed discs with* $Vol(\tilde{D}^j) = Vol(\hat{D}^j)$, *and* $T > 0$, *there exists a volume-preserving diffeotopy* $P_t, (P_0 = Id)$, *such that* $P_T(\tilde{D}^j) = \hat{D}^j$.

In 2D case

Corollary 4 *The above mentioned* volume-preserving diffeotopy *is a flow generated by a time-variant Hamiltonian vector field (with a continuous in time hamiltonian* $\tau \to h_\tau$*).*

2. High-dimensional control for the volume element transfer

We start with constructing a control, supported on a small interval $[0, N^{-1}]$, $N >> 1$

$$v(t,x) = N\partial_t V(t,x),$$

where $V(t,x) = \sum_\ell v_\ell(t) e^{i\ell\cdot x}$ is a (high-order) trigonometric polynomial. The control is **high-gain**.

Substituting it into the r.-h. side of our system we get

$$\partial_t \chi = \{\chi, \Delta^{-1} w\},$$
$$\partial_t w + \{\Delta^{-1} w, w\} - \nu \Delta w = N\partial_t V(t,x),$$

and proceed with the time-variant substitution

$$w(t) = w^*(t) + NV(t,x),^*$$

*We impose the condition $V|_{t=0} = V|_{t=T} = 0$ to preserve boundary data.

Rescaling the time to $Nt = \tau \in [0,1]$ one arrives to the equations

$$\begin{aligned} \partial_\tau \chi &= \{\chi, \Delta^{-1}V(\tau)\} + N^{-1}\{\phi, \Delta^{-1}w^*\}, \\ \partial_\tau w^* &= N\{\Delta^{-1}V(\tau), V(\tau)\} + \\ +\{N^{-1}\Delta^{-1}w^* &+ \Delta^{-1}V(\tau), w^*\} + \{\Delta^{-1}w^*, V(\tau)\} - \nu\Delta(N^{-1}w^* + V(\tau)) \end{aligned}$$

We wish to:

steer the χ-component to a neighborhood of the target $\hat{\chi}$

AND

maintain the terminal value $w(1)$ of the w-component in a bounded domain.

SEEMINGLY we get a contradiction

Fast-oscillating control and relaxation metric

The construction involves **fast-oscillating functions***.

For finite-dimensional *time-variant* ODE $\partial_\tau y = Y_\tau(y)$ it is known (since the work of L.C.Young on 'generalized curves') that the trajectories depend continuously on the C^k-norms of the primitives $\int_0^\tau Y_\theta d\theta$ of the r.-h. side.

This continuity underlies in particular 'theory of relaxed controls; therefore we describe the convergence of the primitives via relaxation seminorms $\|\int_0^\cdot Y_\theta d\theta\|_{C^k}$.

If Y_τ is fast-oscillating in time τ, e.g. $Y_\tau = \tilde{Y}(y)\sin N\tau, \ N >> 1$, then the evolution of $y(t)$ in time 1 will be $O(N^{-1})$.

*we mean fast-oscillating functions with zero average

Fast-oscillating controls and continuity results for (perturbative) N-S equation

There is no general theory of relaxed controls available for general non-linear PDE *

Continuity results for the 'perturbative N-S equation' (the 2nd equation) have been proved recently (A.Agrachev, AS, S.Rodrigues for different 2D domains and boundary conditions; A.Shirikyan for $\mathbb{T}^3$)

*Several results for differential inclusions in infinite-dimensional spaces and for semilinear evolution PDE's

Going back to our system

$$\begin{aligned}\partial_\tau \chi &= \{\chi, \Delta^{-1}V(\tau)\} + O(N^{-1}),\\ \partial_\tau w^* &= N\{\Delta^{-1}V(\tau), V(\tau)\} + O(1),\end{aligned}$$

we search for $V(\tau)$ with fast-oscillating and non oscillating parts, so that

$\{\Delta^{-1}V(\tau), V(\tau)\}$ is fast oscillating with the relaxation seminorms $O(N^{-1})$.

$\Delta^{-1}V(\tau)$ approximates the Hamiltonian vector field $\vec{h}_\tau$, coming from Krygin's theorem the relaxation seminorms.

From low- to high-dimensional control by Lie extension.

Passage from high-dimensional to low-dimensional control in Euler equation goes by (inverse) induction with an induction step being a Lie extension which is a key tool for establishing controllability.

Lemma 5 (Lie extension lemma) *Consider control-affine analytic system*

$$\dot{y} = f^0(y) + f^1(y)\widehat{v}_1 + f^2(y)\widehat{v}_2. \quad (2)$$

Let $[f^1, f^2] = 0, \ [f^1, [f^1, f^0]] = 0$. *Then the system*

$$\dot{y} = f^0(x) + f^1(y)\tilde{v}_1 + f^2(y)\tilde{v}_2 + [f^1, [f^2, f^0]](y)v_{12}, \quad (3)$$

is fixed-time Lie extension of (2), meaning that

CLOSURES OF THE TIME-T ATTAINABLE SETS OF (2) and (3) COINCIDE. $\square$

In our case the drift $f^0(w) = -\{\Delta^{-1}w, w\} + \nu\Delta w$, while the controlled vector fields $f^k = e^{ik\cdot x}, f^\ell = e^{i\ell\cdot x}$ are constant (w.r.t. w).

The Lie bracket $[f^1,[f^1,f^0]] = \beta\{\Delta^{-1}e^{ik\cdot x}, e^{ik\cdot x}\}$ **vanishes**, while

the Lie bracket $[f^1,[f^2,f^0]] = \gamma e^{i(k+\ell)\cdot x}$, with γ **non-vanishing**, whenever $k\wedge\ell \neq 0, |k| \neq |\ell|$.

Extension for the controlled N-S equation via resonances

Lie brackets are unavailable for the nonlinear PDE setting BUT... we use the proof of Lie Extension Lemma, based on 'design of resonances' of fast oscillations.

In (VERY) short, the end-points $w(1)$ of the trajectories of the N-S equation guided by the control

$$V^e(t,x) = \tilde{v}_k(t)e^{ik\cdot x} + \tilde{v}_\ell(t)e^{i\ell\cdot x} + \tilde{v}_{k+\ell}(t)e^{i(k+\ell)\cdot x},$$

can be approximated by the end-points $w^r(1)$ of the trajectories of the same equation guided by the 'reduced' control

$$V(t,x) = \hat{v}_k(t)e^{ik\cdot x} + \hat{v}_\ell(t)e^{i\ell\cdot x}.$$

Saturation of the set of controlled modes

This result allows us to add to the set $\mathcal{K}$ of original controlled modes the extended controlled modes $k+\ell$, where $k,\ell \in \mathcal{K}, \ |k| \neq |\ell|, \ k \wedge \ell \neq 0$.

A simple linear algebra on $\mathbb{Z}^2$ shows that iterated application of the extension $\{k, \ell\} \to \{k, \ell, k+\ell\}$ to the 4-element set $\mathcal{K} = \{(\pm 1, 0), (\pm 1, \pm 1)\}$ spans the whole $\mathbb{Z}^2$ wherefrom we can conclude approximate controllability by means of controls applied to the modes from $\mathcal{K}$.

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THANK YOU!