Compactness estimates for Hamilton-Jacobi equations

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Partial differential equations, optimal design and numerics

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Outline

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1. Hamilton-Jacobi equations
2. Compactness estimates for hyperbolic conservation laws
3. Compactness estimates for Hamilton-Jacobi equations

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2. Compactness estimates for hyperbolic conservation laws
3. Compactness estimates for Hamilton-Jacobi equations

Hamilton-Jacobi equations

$$\begin{cases} u_t(t,x) + H(t,x,\nabla u(t,x)) = 0 & (t,x) \in [0,T] \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

where

- $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a C^2 smooth function such that

$$(a) \quad \lim_{|p|\to\infty} \inf_{(t,x)\in[0,T]\times\mathbb{R}^n} \frac{H(t,x,p)}{|p|} = +\infty$$

$$(b) \quad D_p^2 H(t,x,p) \geqslant \alpha \cdot \mathbb{I}_n, \quad \forall (t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$$

  with $\alpha > 0$

- $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function

play an important role in Dynamic Optimization

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play an important role in Dynamic Optimization

The simplest problem in the calculus of variations

Given a C^2 smooth function $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

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$$(b) \quad D_q^2 L(t,x,q) \geqslant \lambda \cdot \mathbb{I}_n, \quad \forall (t,x,q) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \quad (\lambda > 0)$$

consider the problem of minimizing the functional

$$J(\xi) = \int_0^T L\big(t, \xi(t), \xi'(t)\big)\, dt$$

over all absolutely continuous arcs $\xi : [0, T] \to \mathbb{R}^n$ satisfying

$$\xi(0) = x_0 \quad \text{and} \quad \xi(T) = x_T$$

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The action functional

A particle moving from time 0 to time T between two points $x_0, x_T \in \mathbb{R}^3$ subject to a conservative force

$$F(x) = -\nabla V(x)$$

among all the (admissible) trajectories $\xi(t)$, follows the one that minimizes the action, i.e. the functional

$$J(\xi) = \int_0^T \left[\frac{1}{2} m|\xi'(t)|^2 - V(\xi(t))\right] dt,$$

where m is the mass of the particle and $\frac{1}{2}m|x'(t)|^2$ is its kinetic energy

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Minimal surfaces of revolution

Given $a, b \in \mathbb{R}$ with $a < b$, consider in the space $\mathbb{R}^3$ the circles

$$\begin{cases} y^2 + z^2 = A^2 \\ x = a \end{cases} \qquad \begin{cases} y^2 + z^2 = B^2 \\ x = b \end{cases}$$

For any smooth $\xi : [a, b] \to \mathbb{R}$, with $\xi(x) > 0$, $\xi(a) = A$ and $\xi(b) = B$, consider

- the regular curve $\vec{X}(x) = (x, 0, \xi(x))$ in the xz-plane
- the surface of revolution $\Sigma(\xi)$ generated by the rotation of $\vec{X}$ around the x-axis

Finding the surface of revolution of minimal area amounts to minimizing

$$A(\Sigma(\xi)) = 2\pi \int_a^b \xi(x)\sqrt{1 + \xi'(x)^2}\, dx$$

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Dynamic programming

Replacing the initial point constraint with the initial cost u_0, consider

$$\inf_{\xi(t)=x}\left\{\int_0^t L(s,\xi(s),\xi'(s))\,dt+u_0(\xi(0))\right\}=V(t,x)$$

$$\begin{cases} V_t(t,x)+\underbrace{\sup_{q\in\mathbb{R}^n}\Big\{\langle q,\nabla V(t,x)\rangle - L(t,x,q)\Big\}}_{H\left(t,x,\nabla V(t,x)\right)}=0 & (t,x)\in[0,T]\times\mathbb{R}^n \ a.e. \\ V(0,x)=u_0(x) & x\in\mathbb{R}^n \end{cases}$$

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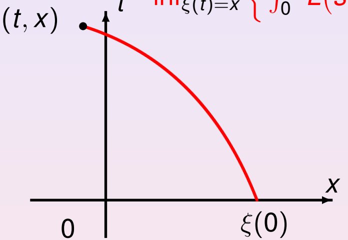

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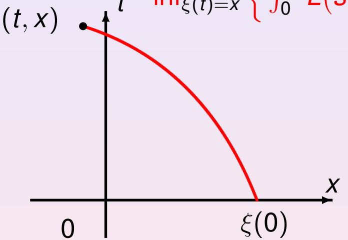

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Weak solutions to Hamilton-Jacobi equations

$$\begin{cases} u_t(t,x) + H(t,x,\nabla u(t,x)) = 0 & (t,x) \in [0,T] \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases} \qquad (HJ)$$

- has no global smooth solution due to crossing of characteristics
- may have infintely many Lipschitz solutions satisfying (HJ) a.e.
  - Dacorogna and Marcellini (1999)
- has a unique viscosity solution
  - Crandall and Lions (1983), Crandall, Evans and Lions (1984)
  - Bardi and Capuzzo Dolcetta (1997), Fleming and Soner (1993)
- the viscosity solution is the unique semiconcave u satisfying (HJ) a.e.
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Viscosity solutions

A function $u \in \mathcal{C}([0,T] \times \mathbb{R}^n)$ is a viscosity solution of

$$u_t + H(t, x, \nabla u) = 0 \qquad \text{in }]; [0, T] \times \mathbb{R}^n$$

if for every $(t, x) \in (0, T) \times \mathbb{R}^n$ and every $\phi \in \mathcal{C}^1((0, T) \times \mathbb{R}^n)$

- $u - \phi$ has a local maximum at $(t, x) \Rightarrow \phi_t(t, x) + H(t, x, \nabla\phi(t, x)) \leqslant 0$
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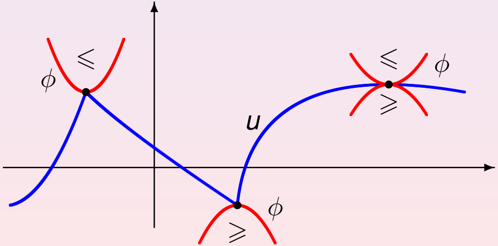

Semiconcave functions

Definition

We say that $u : \mathbb{R}^N \to \mathbb{R}$ is (linearly) semiconcave *if there exists a constant $K > 0$ (a* semiconcavity constant *for u) such that*

$$\lambda u(x) + (1-\lambda)u(y) - u\big(\lambda x + (1-\lambda)y\big) \leqslant K\lambda(1-\lambda)\frac{|y-x|^2}{2}$$

for all $x, y \in \mathbb{R}^N$ and all $\lambda \in [0,1]$

- u is semiconcave with semiconcavity K if any only if the function

$$\tilde{u}(x) = u(x) - \frac{K}{2}|x|^2$$

  is concave

- v is semiconvex with semiconvexity constant K if $-v$ is semiconcave with semiconcavity constant K

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For more on semiconcave functions see

- Control theory

  Hrustalev (1978), C – Soner (1987), C – Frankowska (1991)

  Fleming – McEneaney (2000), Rifford (2000, 2002)

- Nonsmooth and variational analysis

  Rockafellar (1982)

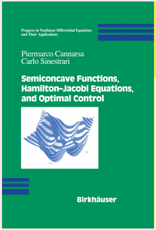
  Colombo – Marigonda (2006), Colombo – Nguyen (2010)

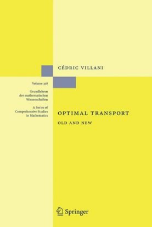
- Differential geometry Perelman (1995), Petrunin (2007)

- Monographs

  C – Sinestrari (Birkhäuser, 2004)

  Villani (Springer, 2009)





From Hamilton-Jacobi equations to conservation laws

When $n = 1$ the Hamilton-Jacobi equation

$$\begin{cases} u_t(t,x) + H\big(t,x,u_x(t,x)\big) = 0 & (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R} \end{cases}$$

can be reduced to the conservation law

$$\begin{cases} v_t(t,x) + H\big(t,x,v(t,x)\big)_x = 0 & (t,x) \in [0,T] \times \mathbb{R} \\ v(0,x) = u_0'(x) & x \in \mathbb{R} \end{cases}$$

taking $v(t,x) = u_x(t,z)$

Scalar conservation laws

u is an entropy solution of

$$u_t + f(u)_x = 0 \quad \text{in} \quad [0, +\infty) \times \mathbb{R}$$

where $f : \mathbb{R} \to \mathbb{R}$ is is (uniformly) strictly convex

$$f''(u) \geqslant c > 0 \quad \forall u \in \mathbb{R}$$

if

- u distributional solution

$$\iint [u\varphi_t + f(u)\varphi_x]\, dxdt = 0 \quad \forall \varphi \in C_c^1([0, +\infty) \times \mathbb{R}) \qquad (D)$$

- Lax stability condition

$$u(t, x-) \geq u(t, x+) \quad \text{for a.e } t > 0, \quad \forall x \in \mathbb{R}$$

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Compactness of the semigroup $(S_t)_{t\geq 0}$

$S_t : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ associates to every initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ the unique entropy solution $u(t) = S_t(u_0)$ of

$$\begin{cases} u_t + f(u)_x = 0 & [0,+\infty) \times \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R} \end{cases}$$

Theorem (Lax, 1954)

The map $S_t : L^1(\mathbb{R}) \longrightarrow L^1_{\mathrm{loc}}(\mathbb{R})$ is compact for every $t > 0$

A question (by P. Lax) :

is it possible to give a quantitive estimate of the compactness of S_t ?

Compactness of the semigroup $(S_t)_{t\geq 0}$

$S_t : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ associates to every initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ the unique entropy solution $u(t) = S_t(u_0)$ of

$$\begin{cases} u_t + f(u)_x = 0 & [0, +\infty) \times \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R} \end{cases}$$

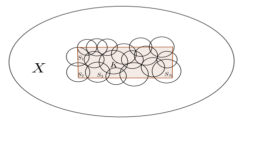
Theorem (Lax, 1954)

The map $S_t : L^1(\mathbb{R}) \longrightarrow L^1_{\mathrm{loc}}(\mathbb{R})$ is compact for every $t > 0$

A question (by P. Lax) :
is it possible to give a quantitive estimate of the compactness of S_t ?

Kolmogorov ε-entropy

Let (X, d) be a metric space and K a totally bounded subset of X



For any $\varepsilon > 0$, let $N_\varepsilon(K)$ be the minimal number of sets in a cover of K by subsets of X having diameter no larger than 2ε

Definition

The ε-entropy of K is defined as

$$\mathcal{H}_\varepsilon(K \mid X) = \log_2 N_\varepsilon(K)$$

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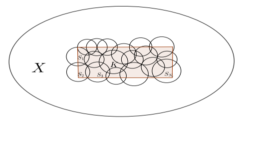

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Applications

one relies on Kolmogorov's ε-entropy to:

- provide estimates on the accuracy and resolution of numerical methods
- analyze computational complexity of conservation laws (derive number of needed operations to compute solutions with an error $< \varepsilon$)

Upper estimate

Given $L, m, M > 0$, define

$$\mathcal{C}_{[L,m,M]} = \left\{ u_0 \in L^1(\mathbb{R}) \;:\; \mathrm{spt}(u_0) \subset [-L, L]\,,\; \|u_0\|_{L^1} \leq m\,,\; \|u_0\|_{L^\infty} \leq M \right\}$$

Our goal: to give an upper bound for

$$\mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{[L,m,M]}) \Big|\, L^1(\mathbb{R})\Big)$$

Theorem (De Lellis and Golse, 2005)

For any $\varepsilon > 0$ and $T > 0$, one has

$$\mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{[L,m,M]}) \Big|\, L^1(\mathbb{R})\Big) \leqslant \frac{C_T}{\varepsilon}$$

for some constant $C_T > 0$

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For any $T > 0$ and for $\varepsilon > 0$ sufficiently small, one has

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By the upper and lower bounds, we conclude

$$\mathcal{H}_\varepsilon \Big(S_T(\mathcal{C}_{[L,m,M]}) \Big| \ L^1(\mathbb{R}) \Big) \approx \varepsilon^{-1}$$

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Back to Hamilton-Jacobi equations

Consider the Hamilton-Jacobi equation ($n \geqslant 1$)

$$u_t(t,x) + H(\nabla u(t,x)) = 0 \qquad (t,x) \in [0,+\infty) \times \mathbb{R}^n$$

with $H \in C^2(\mathbb{R}^n)$ satisfying

(H1) superlinearity: $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty$

(H2) uniform convexity: $D^2 H(p) \geqslant \alpha \cdot \mathbb{I}_N, \quad \forall p \in \mathbb{R}^n$

where $\alpha > 0$ and $\mathbb{I}_n$ is the identity $n \times n$ matrix

Legendre transform of H

$$H^*(q) = \max_{p\in\mathbb{R}^n} \{\langle p,q\rangle - H(p)\} \qquad (q \in \mathbb{R}^n)$$

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Hopf-Lax semigroup

For any $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$ the Cauchy problem

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$$u(t,x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot H^* \left(\frac{x-y}{t} \right) + u_0(y) \right\}, \quad \forall (t,x) \in]0,+\infty[\times \mathbb{R}^n$$

Our goal: to obtain upper and lower compactness estimates for

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Semiconcavity of the Hopf-Lax semigroup

Given $K, L, M > 0$, define

$$\begin{aligned}\mathcal{C}_{[L,M]} &= \left\{u \in \mathrm{Lip}(\mathbb{R}^n) \ : \ \mathrm{spt}(u) \subset [-L, L]^n, \ \|\nabla u\|_{L^\infty(\mathbb{R}^N)} \leqslant M\right\} \\ \mathcal{SC}_{[K,L,M]} &= \left\{u \in \mathcal{C}_{[L,M]} \ : \ u \ \text{semiconcave with constant} \ K\right\}\end{aligned}$$

Proposition

For any $L, M, T > 0$ and every $u \in \mathcal{C}_{[L,M]}$

1. *$S_T(u)$ is semiconcave with constant $\frac{1}{\alpha T}$*
2. $\|\nabla S_T(u)\|_{L^\infty(\mathbb{R}^n)} \leqslant M$
3. *$spt(S_T(u) + T \cdot H(0)) \subset [-L_T, L_T]^n$ where $L_T = L + T \cdot \sup_{|p| \leq M} |DH(p)|$*

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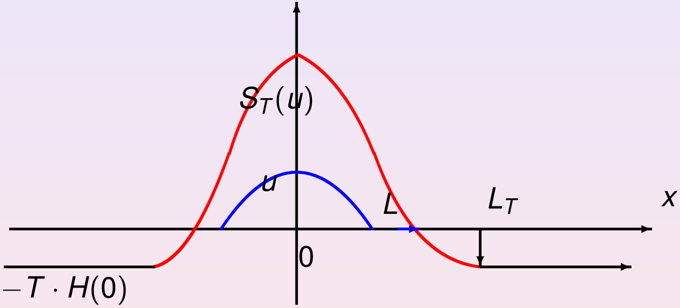
flow associated with H-L semigroup

$$\boxed{S_T(\mathcal{C}_{[L,M]}) + T \cdot H(0) \subset \mathcal{SC}_{[\frac{1}{\alpha T}, L_T, M]}}$$



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For any $L, M, T > 0$ *there exist constant* $\varepsilon_0 = \varepsilon_0(L, M, T) > 0$ *and* $C = C(L, M, T) > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$,

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Main steps of the proof

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$$S_T\big(\mathcal{C}_{[L,M]}\big) + T \cdot H(0) \subset \mathcal{SC}_{[\frac{1}{\alpha T}, L_T, M]}$$

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Let $M > 0$ be fixed
Then, for all $T > 0$ there exist constants $\Gamma_T > 0$ and $\Lambda_T \geqslant 0$ such that

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Main ideas of the proof of the lower estimate

1. Controllability type result: introduce a parameterized class $\mathcal{U}$ of smooth function and show that any element of such a class can be attained, at any given time $T > 0$, by the Hopf-Lax flow $S_T(u)$ for a suitable $u \in \mathcal{C}_{[L,M]}$
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Reachability of semiconcave functions

Theorem

Given $K, L, M > 0$, let $T > 0$ be such that

$$K\,T \le \frac{1}{2\alpha_M} \quad \textit{where} \quad \alpha_M = \sup_{|p|\leqslant M} \|D^2 H(p)\|$$

Then

$$\mathcal{SC}_{[K,L,M]} - T\cdot H(0) \subset S_T(\mathcal{C}_{[L_T,M]})$$

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Reachability of semiconcave functions

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Given $K, L, M > 0$, let $T > 0$ be such that

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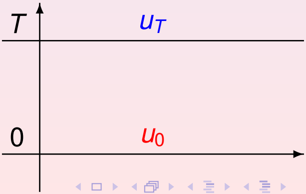

Backward construction

Solve the equation backwards: set $\boxed{v(t,x) = S_t(v_0)(x)}$ with

$$v_0(x) = -u_T(-x)$$

and define

$$u(t,x) = -v(T-t,-x) \qquad (t,x) \in [0,T] \times \mathbb{R}^n$$

Then

- $u(T,\cdot) = u_T$
- $u_0 \doteq u(0,\cdot) \in \mathcal{C}_{[L_T,M]}$ by the properties of S_T
- $u_t(t,x) + H(\nabla u(t,x)) = 0$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}^n$

Therefore,

$$u \text{ viscosity solution} \quad \Longrightarrow \quad u_T = S_T(u_0)$$

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Lower bound for $\mathcal{H}_\varepsilon\left(\mathcal{SC}_{[K,L,M]} \mid W^{1,1}(\mathbb{R}^n)\right)$

Proposition

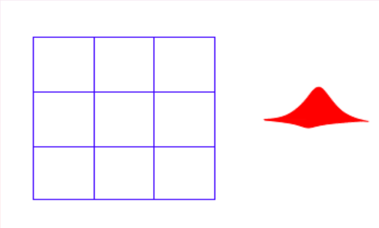
Given $K, L, M > 0$, for any $\varepsilon > 0$

$$\mathcal{H}_\varepsilon\left(\mathcal{SC}_{[K,L,M]} \mid W^{1,1}(\mathbb{R}^n)\right) \geqslant \frac{\Gamma(K,L,M)}{\varepsilon^n}$$

Sketch of the proof ($n = 2$):

Given $N \geqslant 1$ integer, divide $[-L, L]^2$ into N^2 squares of side $\frac{2L}{N}$

$$[-L, L]^2 = \bigcup_{i,j=1,\dots,N} \square_{ij}$$



Construct bump functions $b_{ij} : \square_{ij} \to \mathbb{R}$ such that

- $\|\nabla b_{ij}\|_{\mathbb{L}^\infty} \leqslant \frac{KL}{12N}$, $\|b_{ij}\|_{\mathbb{W}^{1,1}} \leqslant \frac{C}{N^3}$
- ∇b_{ij} Lipschitz with constant K

The class $\mathcal{U}_N$ of smooth functions

Let

$$\Delta_N = \left\{ \delta = (\delta_{ij})_{i,j=1}^N \; : \; \delta_{ij} \in \{-1, 1\} \right\}$$

Consider the class of smooth functions

$$\mathcal{U}_N = \left\{ u_\delta = \sum_{i,j=1}^N \delta_{ij} \cdot b_{ij} \; : \; \delta \in \Delta_N \right\}$$

Then $\#(\mathcal{U}_N) = 2^{N^2}$. Also, one can show that

- $\mathcal{U}_N \subset \mathcal{SC}_{[K,L,M]}$
- $\|u_{\delta'} - u_\delta\|_{W^{1,1}(\mathbb{R}^2)} \leqslant \varepsilon$ if $\#\{(i,j) \; : \; \delta'_{ij} \neq \delta_{ij}\} \leqslant C_{K,L} N^{n+1} \varepsilon$

Choosing $N \approx \frac{1}{\varepsilon}$, by a combinatorial argument one can show that

$$\#\left\{ \delta' \in \Delta_N \; : \; \|u_{\delta'} - u_\delta\|_{W^{1,1}(\mathbb{R}^2)} \leqslant \varepsilon \right\} \leqslant 2^{N^2} e^{-N^2/8} = e^{-N^2/8} \#(\mathcal{U}_N)$$

which yields

$$\mathcal{H}_\varepsilon\left(\mathcal{U}_N \mid W^{1,1}(\mathbb{R}^2)\right) \geqslant \frac{\Gamma}{\varepsilon^2}$$

with $\Gamma = \Gamma(K, L, M) > 0$. Therefore,

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End of the proof of the lower estimate

want to show

Let $M > 0$ be fixed. Then, $\forall T > 0$ there exist constants $\Gamma_T > 0$ and $\Lambda_T \geqslant 0$ such that

$$\mathcal{H}_\epsilon\Big(S_T(\mathcal{C}_{[L,M]}) + T \cdot H(0) \mid W^{1,1}(\mathbb{R}^n)\Big) \geqslant \frac{\Gamma_T}{\varepsilon^n} \qquad \forall L > \Lambda_T\,,\ \forall \varepsilon > 0$$

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Concluding remarks

- combining the upper and lower estimates (near $\varepsilon = 0$)

$$\mathcal{H}_\epsilon\Big(S_T(\mathcal{C}_{[L,M]}) + T \cdot H(0) \mid W^{1,1}(\mathbb{R}^n)\Big) \approx \varepsilon^{-n}$$

- compactness estimates can be extended to

$$u_t(t,x) + H(t,x,\nabla u(t,x)) = 0 \qquad (t,x) \in [0,T] \times \mathbb{R}^n$$

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$$\mathcal{H}_\epsilon\Big(S_T(\mathcal{C}_{[L,M]}) + T \cdot H(0) \mid W^{1,1}(\mathbb{R}^n)\Big) \approx \varepsilon^{-n}$$

- compactness estimates can be extended to

$$u_t(t,x) + H\big(t, x, \nabla u(t,x)\big) = 0 \qquad (t,x) \in [0,T] \times \mathbb{R}^n$$

  (no Hopf-Lax formula available)

- reachability example of a controllability result for Hamilton-Jacobi equations

Thank you for your attention and thanks to

for the hospitality