Some smoothness results for classical problems in optimal design and applications

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Compliance problem

$$\Omega \subset \mathbb{R}^N, N \geq 2, \text{ bounded, open,}$$
$$\beta > \alpha > 0,\ 0 < \kappa < |\Omega|, \tilde{f} \in H^{-1}(\Omega)$$

$$\max_{|\omega| \leq \kappa} \int_\Omega \left(\alpha\chi_\omega + \beta(1-\chi_\omega)\right)|\nabla u_\omega|^2 dx$$

$$\begin{cases} -\operatorname{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))\nabla u_\omega\big) = \tilde{f} \text{ in } \Omega \\ u_\omega = 0 \text{ on } \partial\Omega \end{cases}$$

Using

$$\int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u_\omega|^2 dx$$

$$= -\left(\int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u_\omega|^2 dx - 2 \prec \tilde{f}, u_\omega \succ\right)$$

$$= -\min_{u\in H_0^1(\Omega)}\left(\int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u|^2 dx - 2 \prec \tilde{f}, u \succ\right).$$

The problem can be stated as

$$\min_{\substack{u\in H_0^1(\Omega)\\ |\omega|\leq\kappa}}\left(\int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u|^2 dx - 2 \prec \tilde{f}, u \succ\right)$$

F. Murat (1972): The problem has not solution in general. A relaxation is needed.

F. Murat, L. Tartar (1985). A relaxation is given by replacing $\alpha\chi_\omega + \beta(1-\chi_\omega)$ by the armonic mean value of α and β with proportions θ and $1-\theta$, with $\theta \in L^\infty(\Omega;[0,1])$, i.e.

$$\min_{\substack{u\in H_0^1(\Omega)\\ \theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa}} \left(\int_\Omega \frac{\alpha\beta|\nabla u|^2}{\beta\theta+\alpha(1-\theta)}dx - 2 \prec \tilde{f}, u \succ\right)$$

$$= \beta \min_{\substack{u\in H_0^1(\Omega)\\ \theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa}} \left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta}dx - 2 \prec f, u \succ\right)$$

or
$$\begin{cases} \displaystyle\max_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa} \int_\Omega \frac{|\nabla u_\theta|^2}{1+c\theta}dx \\ -\mathrm{div}\left(\dfrac{\nabla u_\theta}{1+c\theta}\right) = f \text{ in } \Omega, \quad u_\theta = 0 \text{ on } \partial\Omega \end{cases}$$

$$c = \frac{\beta-\alpha}{\alpha},\ f = \frac{1}{\beta}\tilde{f}$$

Another formulation (F. Murat, L. Tartar (1985)).

Recall: If u_θ is the solution of

$$-\mathrm{div}\frac{\nabla u_\theta}{1+c\theta} = f \text{ in } \Omega, \qquad u_\theta = 0 \text{ on } \partial\Omega.$$

Then, $\sigma_\theta = \frac{\nabla u_\theta}{1+c\theta}$ is the solution of $\min\limits_{\substack{\sigma\in L^2(\Omega)^N \\ -\mathrm{div}\sigma = f \text{ in } \Omega}} \int_\Omega (1+c\theta)|\sigma|^2 dx.$

Thus
$$\min_{\theta\in L^\infty(\Omega;[0,1]), \int_\Omega \theta\, dx\le\kappa} \ \min_{u\in H_0^1(\Omega)} \left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta} dx - 2 \prec f, u \succ \right)$$

$$= -\max_{\theta\in L^\infty(\Omega;[0,1]), \int_\Omega \theta\, dx\le\kappa} \ \min_{\substack{\sigma\in L^2(\Omega)^N \\ -\mathrm{div}\sigma = f \text{ in } \Omega}} \int_\Omega (1+c\theta)|\sigma|^2 dx$$

$$= -\min_{\substack{\sigma\in L^2(\Omega)^N \\ -\mathrm{div}\sigma = f \text{ in } \Omega}} \ \max_{\theta\in L^\infty(\Omega;[0,1]), \int_\Omega \theta\, dx\le\kappa} \int_\Omega (1+c\theta)|\sigma|^2 dx$$

Remark:

The functional $\sigma \mapsto \max_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa} \int_\Omega (1+c\theta)|\sigma|^2 dx$

is strictly convex. So the problem

$$\min_{\substack{\sigma\in L^2(\Omega)^N\\ -\mathrm{div}\sigma=f \text{ in } \Omega}} \max_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa} \int_\Omega (1+c\theta)|\sigma|^2 dx$$

has a unique solution $\hat\sigma$, i.e. although the solution $(\hat\theta,\hat u)$ of

$$\min_{\substack{u\in H_0^1(\Omega)\\ \theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa}} \left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta} dx - 2 \prec f, u \succ\right)$$

can be not unique, $\hat\sigma = \frac{\nabla\hat u}{1+c\hat\theta}$ is unique.

Taking the minimum in θ in

$$\min_{u\in H_0^1(\Omega)} \min_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta\,dx\le\kappa} \left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta}dx - 2 \prec f,u \succ\right),$$

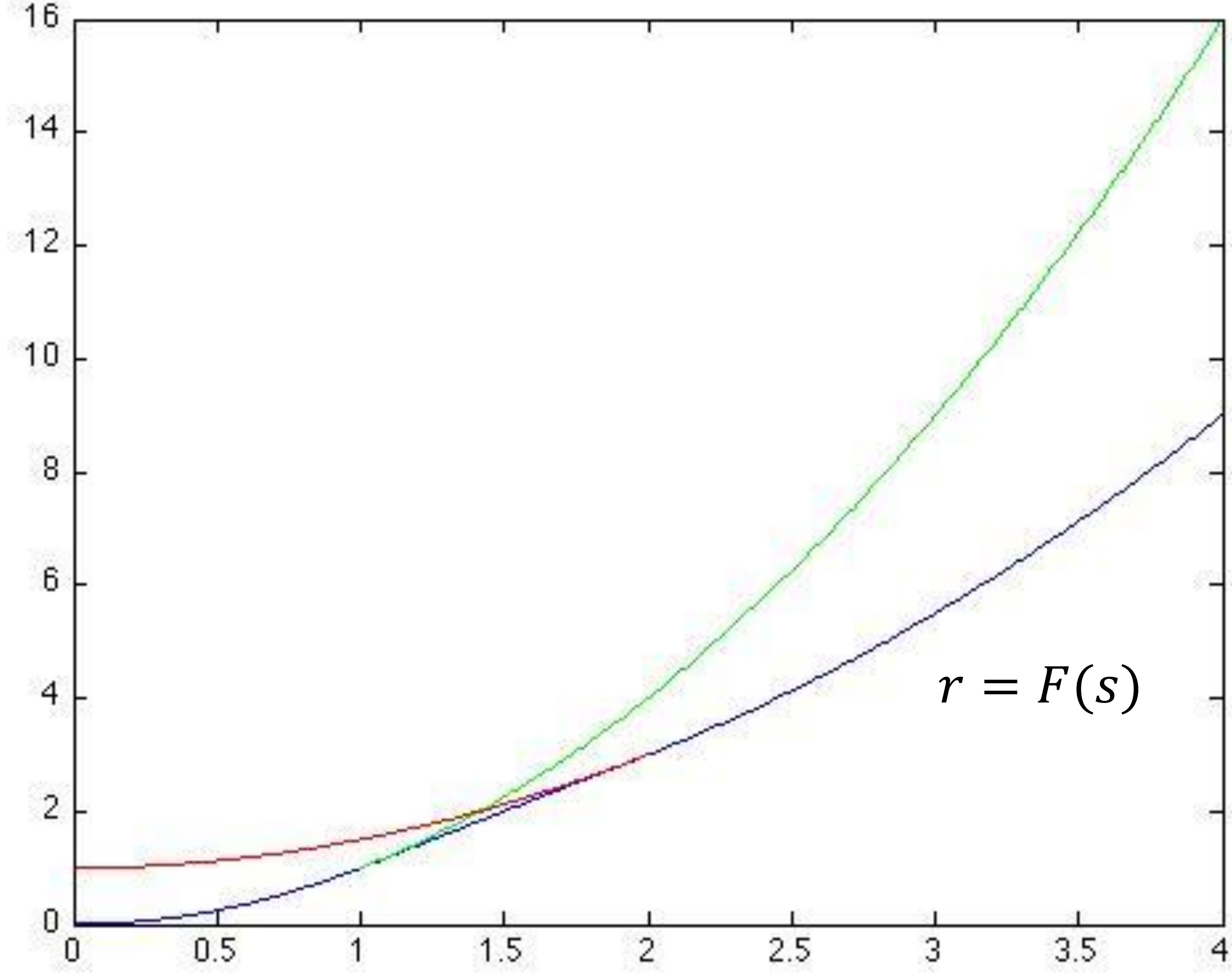
we deduce the existence of $\mu > 0$ such that u is a solution of

$$\min_{u\in H_0^1(\Omega)} \left(\int_\Omega F(|\nabla u|)dx - 2 \prec f,u \succ\right)$$

with $F \in W^{2,\infty}(0,\infty)$ given by

$$2F(s) = \begin{cases} s^2 & \text{if } 0 \le s \le \mu \\ 2\mu s - \mu^2 & \text{if } \mu \le s \le (1+c)\mu \\ \dfrac{s^2}{1+c} + \mu^2 & \text{if } (1+c)\mu \le s. \end{cases}$$

Besides $\theta = 1$ if $|\sigma| < \mu$, $\theta = 0$ if $|\sigma| > \mu$. Thus (θ, u) is unique in $\{|\sigma| \neq \mu\}$



16
14
12
10
8
6
4
2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
$r = F(s)$

Theorem: Assume $\Omega \in C^{2,\gamma}, \gamma \in (0,1]$ ($\Omega \in C^{1,1}$ must be enough)

$$f \in W^{-1,p}(\Omega), \;\; 2 \le p < \infty \Longrightarrow \hat{\sigma} \in L^p(\Omega)^N \Rightarrow u \in W_0^{1,p}(\Omega)$$

$$f \in L^p(\Omega), \;\; N < p \Longrightarrow \hat{\sigma} \in L^\infty(\Omega)^N \Rightarrow u \in W^{1,\infty}(\Omega)$$

$$f \in W^{1,1}(\Omega) \cap L^2(\Omega) \Longrightarrow \begin{cases} \hat{\sigma} \in H^1(\Omega)^N, \;\; P(\hat{\sigma}) = 0 \text{ on } \partial\Omega \\ \partial_i \theta \hat{\sigma}_j - \partial_j \theta \hat{\sigma}_i \in L^2(\Omega), \quad 1 \le i,j \le N \end{cases}$$

P denotes the orthogonal projection on the tangent space.

Chipot, Evans, 1986, Local estimates in $W^{1,\infty}$.

Sketch of the proof. u satisfies

$$-\mathrm{div}\left(\frac{F'(|\nabla u|)}{|\nabla u|}\nabla u\right) = f \text{ in } \Omega$$

and then

$$-\frac{1}{1+c}\Delta u = f + \mathrm{div}\left(\left(\frac{F'(|\nabla u|)}{|\nabla u|} - \frac{1}{1+c}\right)\nabla u\right) \text{ in } \Omega,$$

Using that $F'(s) = s/(1+c)$ if $s > \mu$, we deduce

$$f \in W^{-1,p}(\Omega),\ \ 2 \le p < \infty \implies u \in W_0^{1,p}(\Omega).$$

Now, deriving formally in

$$-\mathrm{div}\left(\frac{F'(|\nabla u|)}{|\nabla u|}\nabla u\right) = f \text{ in } \Omega$$

with respect to x_i, we deduce

$$-\mathrm{div}\big(M\nabla(\partial_i u)\big) = \partial_i f \text{ in } \Omega$$

with $$M = \begin{cases} I & \text{if } |\nabla u| < \mu \\ \dfrac{\mu}{|\nabla u|}\left(I - \dfrac{\nabla u \otimes \nabla u}{|\nabla u|^2}\right) & \text{if } \mu < |\nabla u| < (1+c)\mu \\ \dfrac{1}{1+c} I & \text{if } (1+c)\mu < |\nabla u|. \end{cases}$$

Assume $f \in H^1(\Omega)$. Multiplying by $\partial_i u \varphi^2$, $\varphi \in C_c^1(\Omega)$, we get

$$\int_\Omega M\nabla(\partial_i u) \cdot \nabla(\partial_i u)\, \varphi^2 dx < \infty.$$

but $\partial_i \hat{\sigma} = M\nabla(\partial_i u)$. Thus

$$\int_\Omega |\partial_i \hat{\sigma}|^2 \varphi^2 dx = \int_\Omega |M\nabla(\partial_i u)|^2 \varphi^2 dx$$

$$\leq \int_\Omega M\nabla(\partial_i u) \cdot \nabla(\partial_i u)\, \varphi^2 dx < \infty.$$

The proof of $\hat{\sigma} \in L^\infty(\Omega)^N$ is based on

$$-\mathrm{div}\big(M\nabla(\partial_i u)\big) = \partial_i f \text{ in } \Omega,$$

with $M = \frac{I}{1+c}$ if $|\nabla u| > \mu$ and Stampacchia's estimates.

Proposition:
If $f \in W^{1,1}(\Omega) \cap L^2(\Omega)$, and there exists an unrelaxed solution ($\theta = \chi_\omega$), then $\mathrm{curl}(\hat{\sigma}) = 0$.

If Ω is simply connected, $\Omega \in C^{2,\gamma}$, then $\hat{\sigma} = \nabla w$, with w the unique solution of

$$\begin{cases} -\Delta w = f \text{ in } \Omega \\ w = 0 \text{ on } \partial\Omega \end{cases}$$

Proof. It is essentially a consequence of

$$\partial_i \theta \hat{\sigma}_j - \partial_j \theta \hat{\sigma}_i \in L^2(\Omega), \quad 1 \leq i, j \leq N$$

Remark: The above conclusions appear in F. Murat, L. Tartar 1985, but assuming the existence of smooth solutions.

Remark: The discontinuity sets of a solution θ must be composed by surface levels of the corresponding function u_θ. Moreover, $\frac{\partial u_\theta}{\partial \nu} =$ constant on these surface levels.

Energy problem

$$\Omega \subset \mathbb{R}^N, N \geq 2, \text{ bounded, open,}$$
$$\beta > \alpha > 0,\ 0 < \kappa < |\Omega|,\ \tilde{f} \in H^{-1}(\Omega)$$

$$\min_{|\omega| \geq \kappa} \int_\Omega (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^2 dx$$

$$\begin{cases} -\mathrm{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))\nabla u_\omega\big) = \tilde{f} \text{ in } \Omega \\ u_\omega = 0 \text{ on } \partial\Omega \end{cases}$$

or equivalently

$$\max_{|\omega| \geq \kappa} \min_{u \in H_0^1(\Omega)} \left(\int_\Omega \big(\alpha\chi_\omega + \beta(1-\chi_\omega)\big)|\nabla u|^2 dx - 2 \prec \tilde{f}, u \succ \right)$$

Relaxed formulation

We now need to consider the arithmetic mean value of α and β with proportions θ and 1- θ.

Denoting $c = \frac{\beta-\alpha}{\beta}$, $f = \frac{1}{\beta}\tilde{f}$. The relaxed problem can be written as

$$\max_{\substack{\theta\in L^\infty(\Omega;[0,1]) \\ \int_\Omega \theta dx\geq\kappa}} \min_{u\in H_0^1(\Omega)} \left(\int_\Omega (1-c\theta)|\nabla u|^2 dx - 2 \prec f, u \succ \right)$$

$$= \min_{u\in H_0^1(\Omega)} \max_{\substack{\theta\in L^\infty(\Omega;[0,1]) \\ \int_\Omega \theta dx\geq\kappa}} \left(\int_\Omega (1-c\theta)|\nabla u|^2 dx - 2 \prec f, u \succ \right)$$

or

$$\min_{\substack{\theta\in L^\infty(\Omega;[0,1]),\sigma\in L^2(\Omega)^N \\ \int_\Omega \theta dx\geq\kappa, -\mathrm{div}\sigma=f}} \int_\Omega \frac{|\sigma|^2}{1-c\theta} dx$$

Now, it is the state function $\hat{u}$ which is unique

Theorem: Assume $\Omega \in C^{2,\gamma}, \gamma \in (0,1]$ ($\Omega \in C^{1,1}$ must be enough)

$$f \in W^{-1,p}(\Omega),\ \ 2 \leq p < \infty \ \Longrightarrow \hat{u} \in W_0^{1,p}(\Omega)$$

$$f \in L^p(\Omega),\ \ N < p \ \Longrightarrow \hat{u} \in W^{1,\infty}(\Omega) \quad (\text{and } C^1(\Omega) \text{ if } N = 2)$$

$$f \in L^2(\Omega) \ \Longrightarrow \begin{cases} \hat{u} \in H^2(\Omega) \\ \nabla\theta \cdot \nabla\hat{u} \in L^2(\Omega). \end{cases}$$

Remark: As for the compliance problem, the function $\hat{u}$ is the solution of a certain non-linear problem. Namely $\hat{u}$ is the limit of $\hat{u}_\varepsilon$ satisfying

$$-\mathrm{div}\big(M_\varepsilon(\nabla\hat{u}_\varepsilon)\nabla\,(\partial_i\hat{u}_\varepsilon)\big) = \partial_i f \quad \text{in} \ \ \Omega,$$

where the matrices $M_\varepsilon(\nabla\hat{u}_\varepsilon)$ are uniformly elliptic but unbounded.

Remark:

If $f \in L^2(\Omega)$, and there exists an unrelaxed solution ($\theta = \chi_\omega$), then from the condition $\nabla\chi_\omega \cdot \nabla\hat{u} \in L^2(\Omega)$ one hopes to deduce $\nabla\chi_\omega \cdot \nabla\hat{u} = 0$. This would imply

$$-(1 - c\chi_\omega)\Delta\hat{u} = -\mathrm{div}\big((1 - c\chi_\omega)\nabla\hat{u}\big) = f \; in \; \Omega$$

and as consequence

$$\text{If } \exists U \subset \Omega \text{ open set with } \omega \cap U \Subset U \Longrightarrow \int_{\omega \cap U} f\, dx = 0$$

$$\text{If } \exists U \subset \Omega \text{ open set with } \omega^c \cap U \Subset U \Longrightarrow \int_{\omega^c \cap U} f\, dx = 0.$$

However the implication

$$\nabla\chi_\omega \cdot \nabla\hat{u} \in L^2(\Omega) \Longrightarrow \nabla\chi_\omega \cdot \nabla\hat{u} = 0,$$

is not clear.

Remark: On the discontinuity surface of a solution θ, we have $\frac{\partial \hat{u}}{\partial \upsilon} = 0$.

Eigenvalue problem

We want to mix two materials α and β in order to minimize the first eigenvalue of the operator

$$-\mathrm{div}(\alpha\chi_\omega + \beta(1-\chi_\omega))$$

Namely, for $0 < \kappa < |\Omega|$, we have the problem

$$(\Lambda_m) \qquad \min_{|\omega|\leq\kappa} \min_{u\in H_0^1(\Omega)} \frac{\int_\Omega (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

Remark: For $A \in L^\infty(\Omega)^N$, elliptic,

$$\lambda_1(A) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega A\nabla u \cdot \nabla u \, dx}{\int_\Omega |u|^2 dx}$$

can be characterized by

$$\frac{1}{\lambda_1(A)} = \max_{\substack{-\mathrm{div}\,(A\nabla u)=f \\ u \in H_0^1(\Omega) \\ \|f\|_{L^2(\Omega)} \le 1}} \int_\Omega A\nabla u \cdot \nabla u \, dx$$

$$= - \min_{\substack{u \in H_0^1(\Omega) \\ \|f\|_{L^2(\Omega)} \le 1}} \left(\int_\Omega A\nabla u \cdot \nabla u \, dx - 2 \int_\Omega f u \, dx \right).$$

Thus, we have the relaxed formulation

$$(\Lambda_m) \quad \min_{\|f\|_{L^2(\Omega)}\leq 1} \; \min_{\substack{u\in H_0^1(\Omega) \\ \int_\Omega \theta dx \leq \kappa}} \left(\int_\Omega \frac{|\nabla u|^2}{1+c\theta}\, dx - 2\int_\Omega f u\, dx \right) \qquad c = \frac{\beta-\alpha}{\alpha}$$

The regularity results for the compliance problem can then be applied.

Theorem: Assume $\Omega \in C^{2,\gamma}, \gamma \in (0,1]$

$$\sigma = \frac{\nabla u}{1+c\theta} \in H^1(\Omega)^N \cap L^\infty(\Omega)^N, \quad \partial_i\theta\sigma_j - \partial_j\theta\sigma_i \in L^2(\Omega), \quad 1 \leq i,j \leq N.$$

Theorem: Assume there exists an unrelaxed solution χ_ω for (Λ_m). Then,

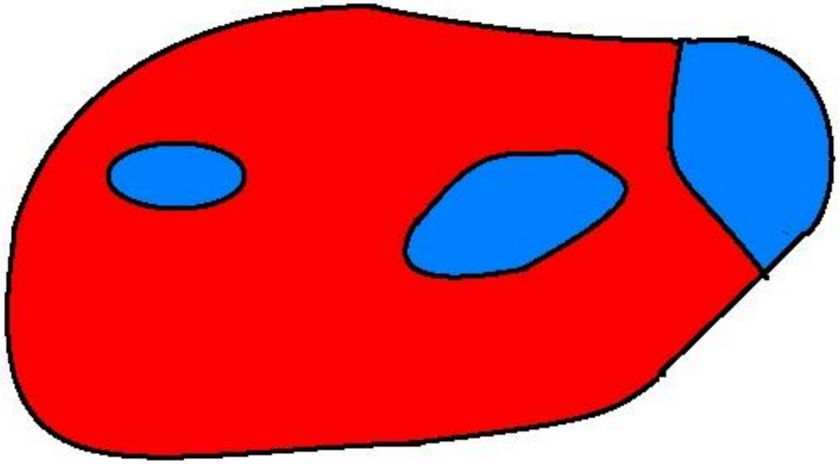
$$\sigma = \left(\alpha\chi_\omega + \beta(1-\chi_\omega)\right)\nabla u \in W^{2,p}(\Omega), \quad \forall p \in [1,\infty), \quad \mathrm{curl}\sigma = 0$$

Moreover, if there exist two open sets $O \Subset U \subset \Omega,\ O \in C^2$, such that $\chi_\omega = r$ in $O, \chi_\omega = 1 - r$ in $U \backslash O$. Then, O is a sphere.

Proof.

It is a consequence of $\begin{cases} -\Delta u = \lambda_1 u \quad \text{in } O \\ u = \text{constant on } \partial O, \ \ \dfrac{\partial u}{\partial \nu} = \text{constant on } \partial O. \end{cases}$

and Serrin's theorem.



It would be only possible if the interior blue zones were circles

Counterexample: $\Omega = \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^{N-1}$, $\alpha = 1, \beta = 2$. For $\varepsilon > 0$ small enough the solutions θ of

$$\min \left\{ \frac{\int_\Omega \frac{|\nabla u|^2}{1+\theta} dx}{\int_\Omega |u|^2 dx} : u \in H_0^1(\Omega), \theta \in L^\infty(\Omega, [0,1]), \int_\Omega \theta dx \leq |\Omega| - \varepsilon \right\}$$

is not a characteristic

Proof. If $(\chi_{\omega_\varepsilon}, u_\varepsilon)$ were a solution then $u_\varepsilon \approx \cos(2x_1) \prod_{j=2}^N \cos(x_j)$.

$\exists$ a smooth connected component O_ε of $\Omega \backslash \omega_\varepsilon$,

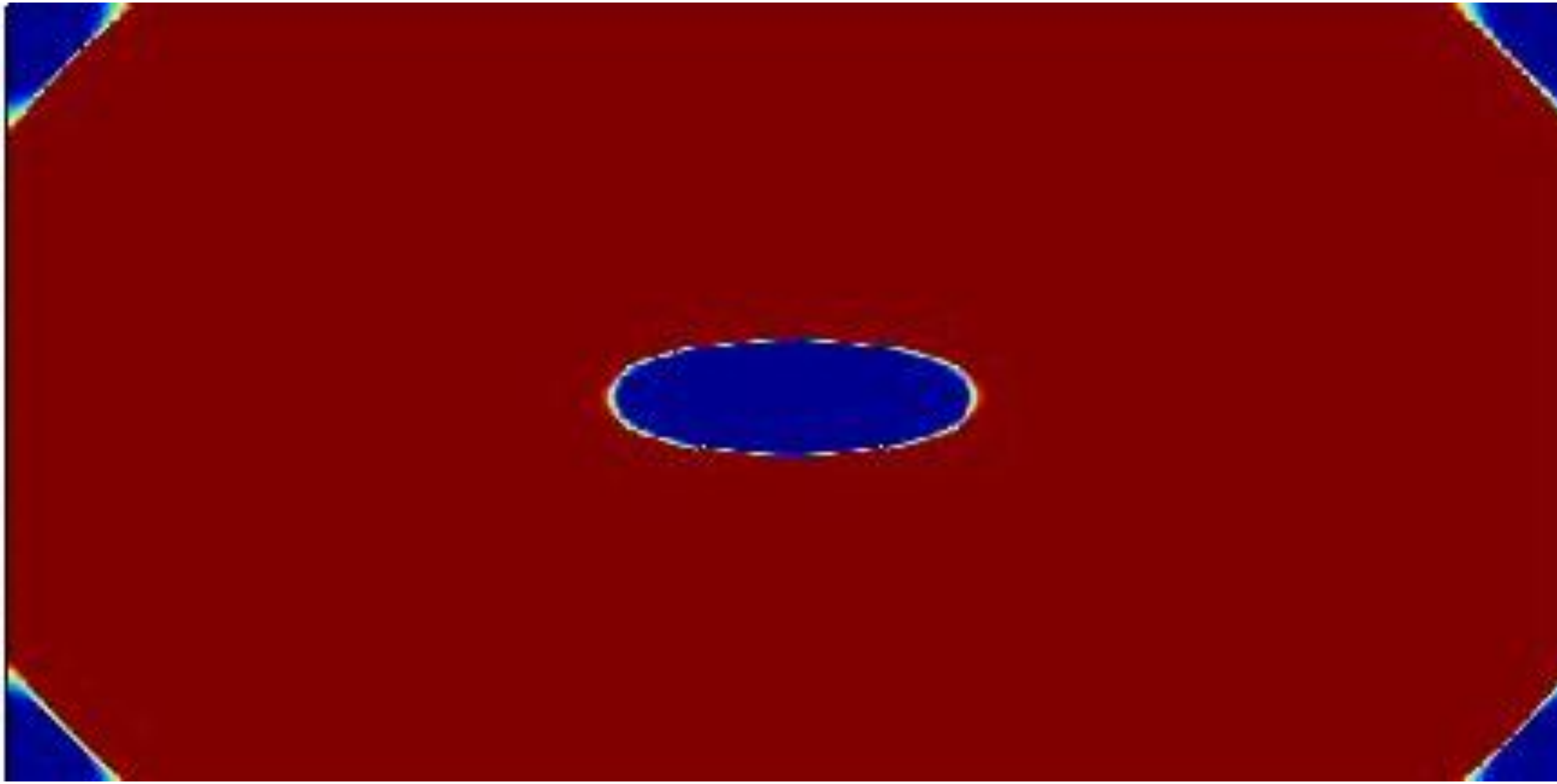
$$O_\varepsilon \approx \left\{ \frac{x_1^2}{8} + \sum_{i=2}^N \frac{x_i^2}{2} = 1 - c_\varepsilon \right\}, \quad c_\varepsilon \searrow 0$$

Numerical experiments.

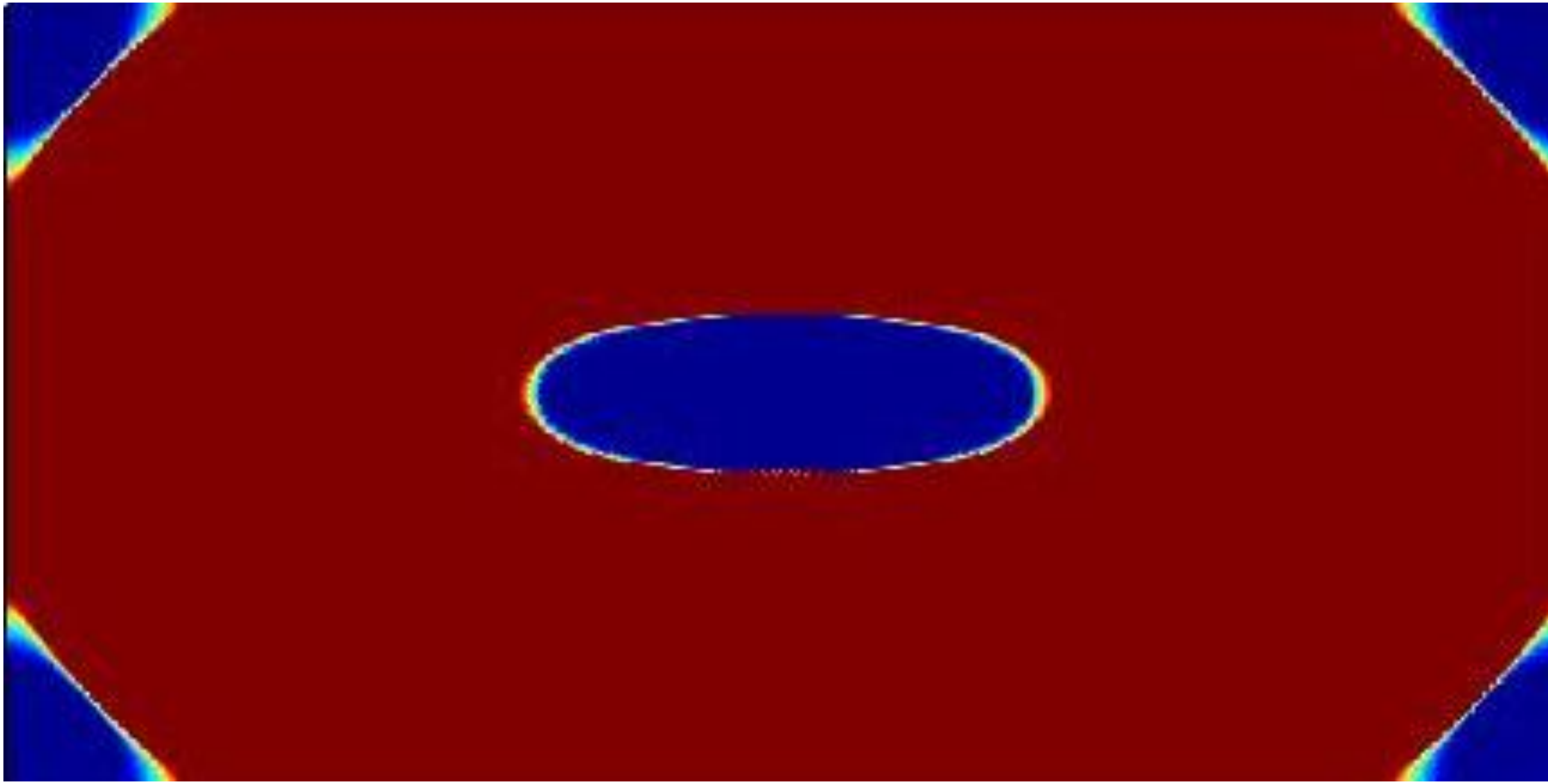
Problem $\Omega = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $|\Omega| \approx 4{,}935$, $\alpha = 1, \beta = 2$

$$\min \frac{\int_\Omega \frac{|\nabla u|^2}{1+\theta} dx}{\int_\Omega |u|^2 dx}$$

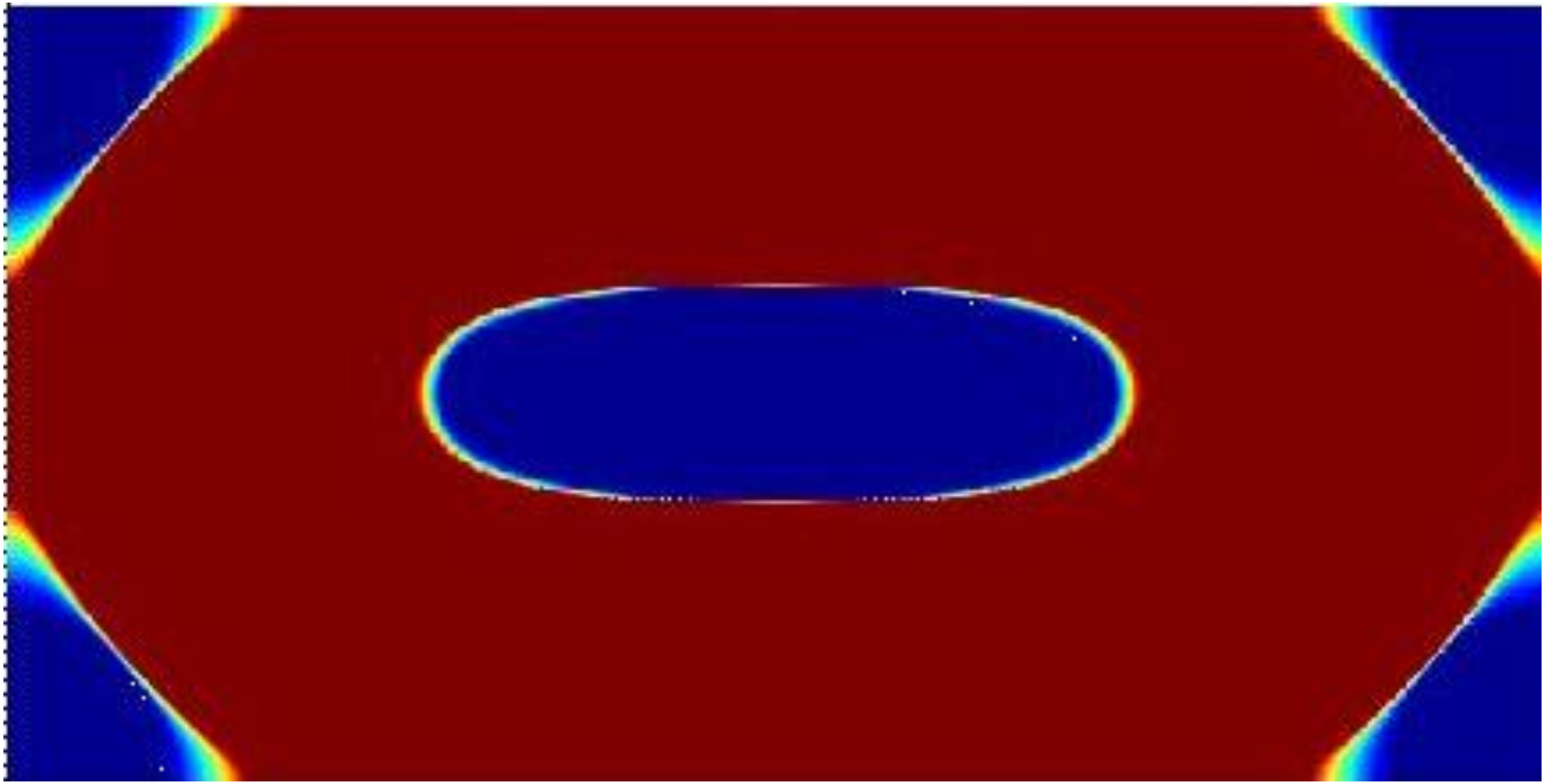
$$u \in H_0^1(\Omega), \int_\Omega \theta dx \leq \kappa$$



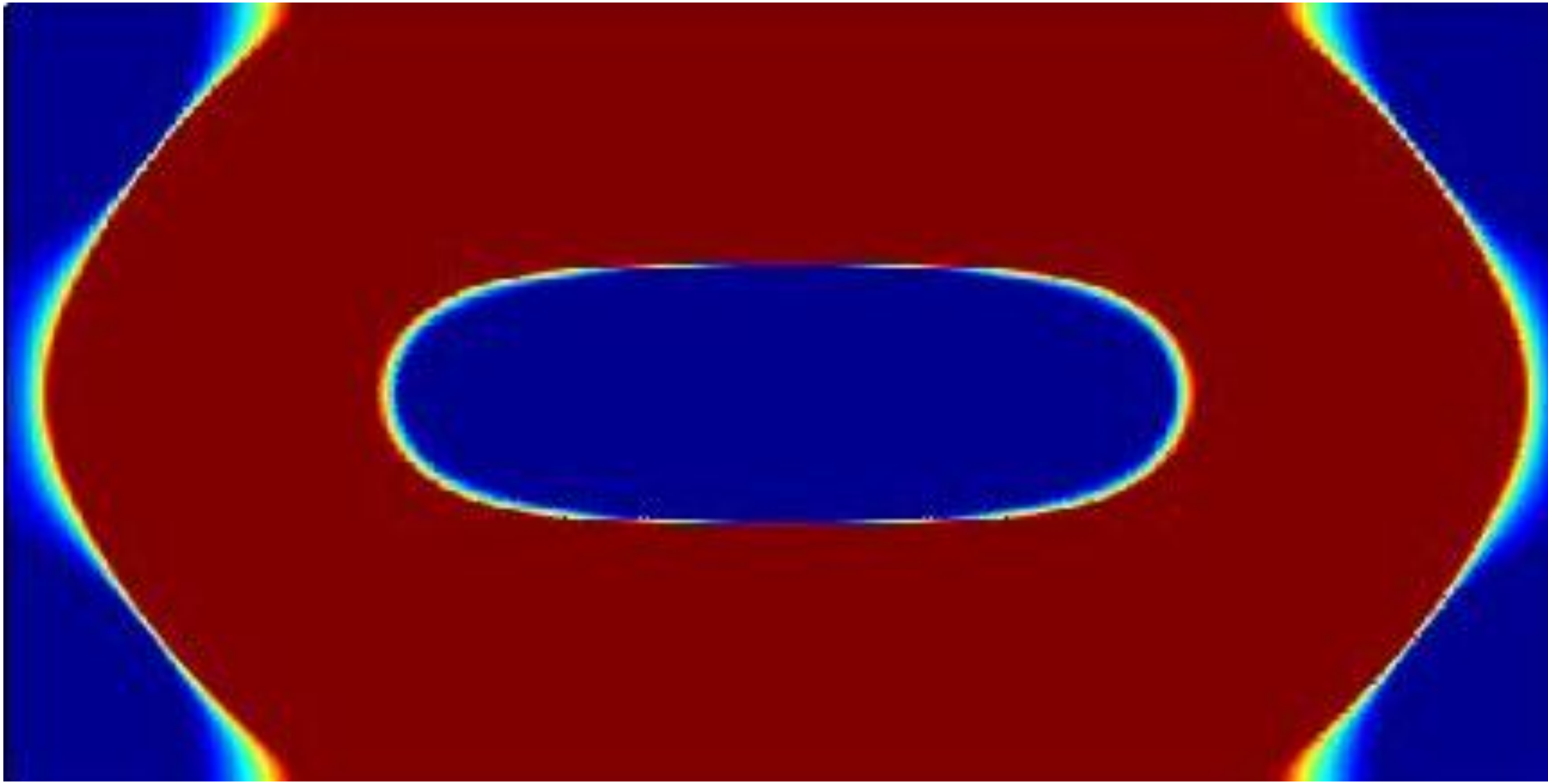
$\alpha = 1, \beta = 2, \kappa = 4.685$



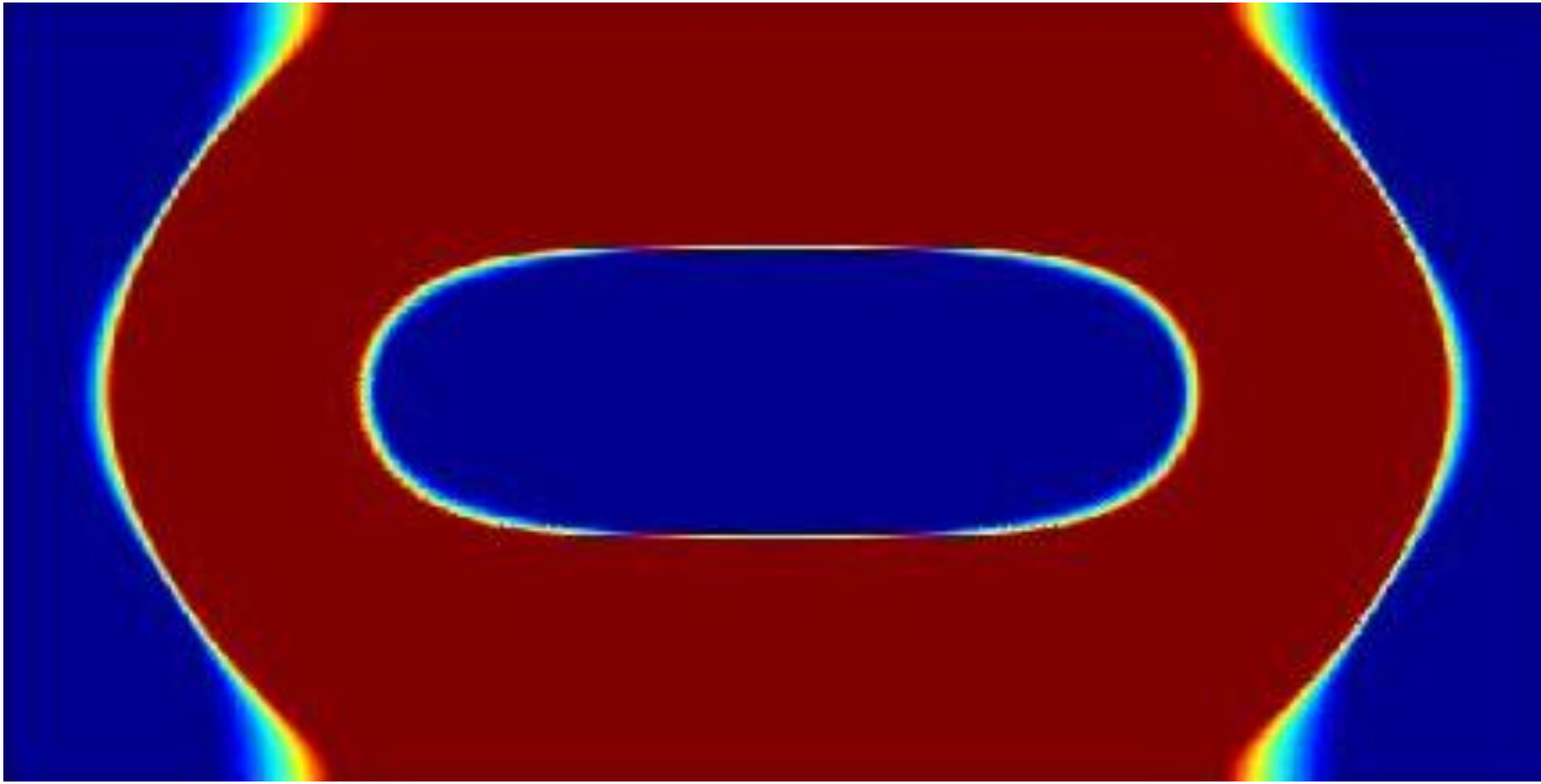
$\alpha = 1, \beta = 2, \kappa = 4.435$



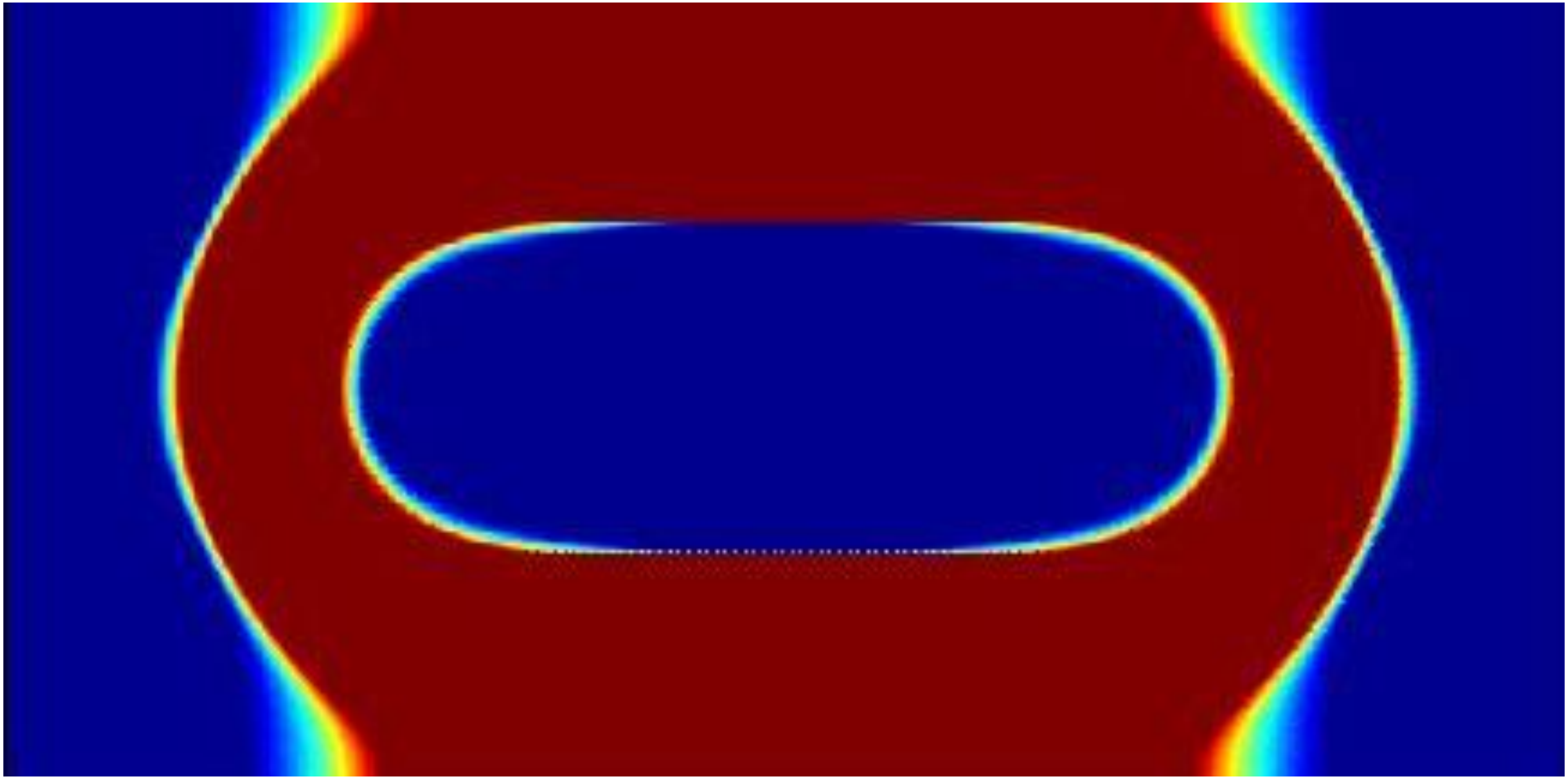
$\alpha = 1, \beta = 2, \kappa = 3.935$



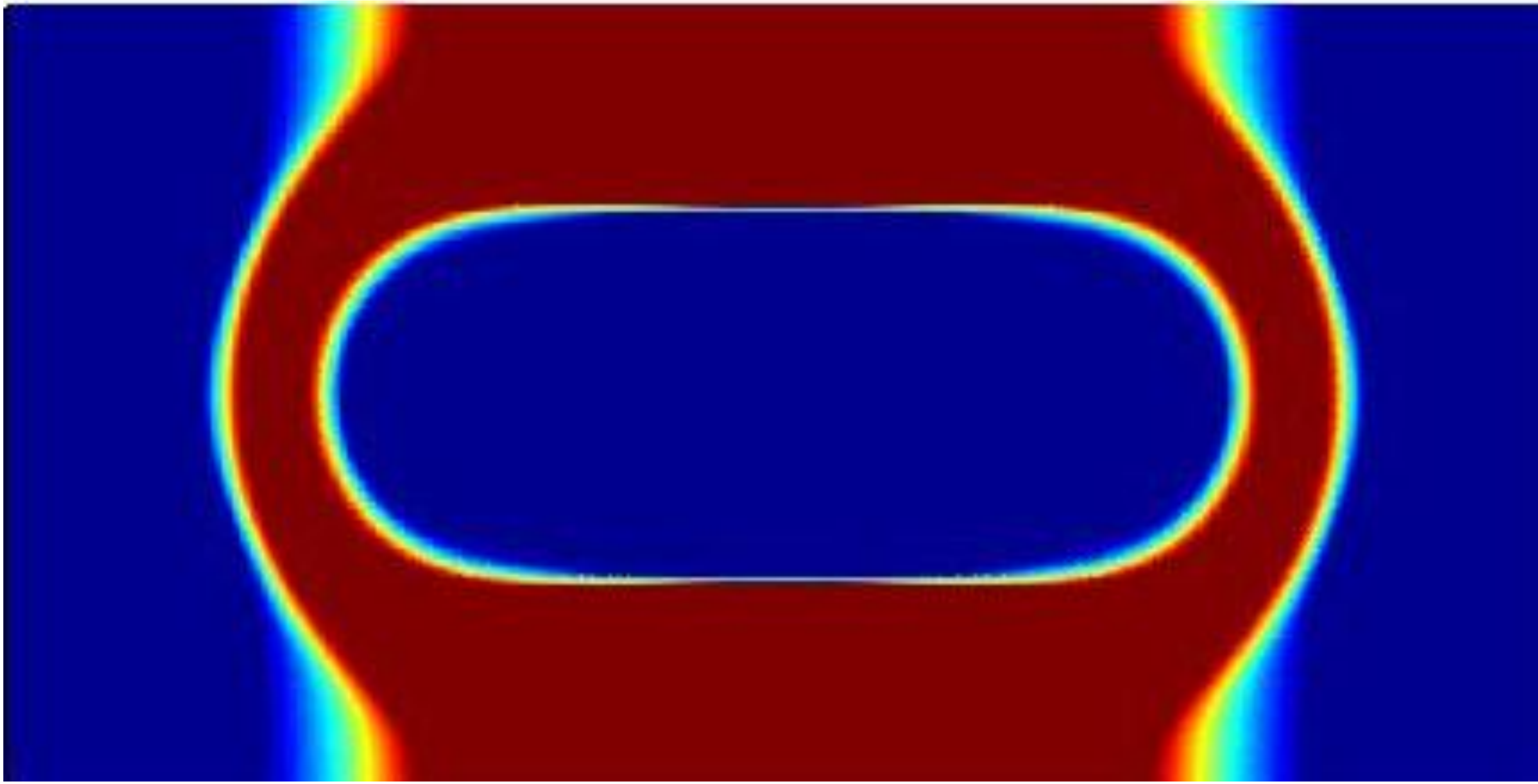
$\alpha = 1, \beta = 2, \kappa = 3.435$



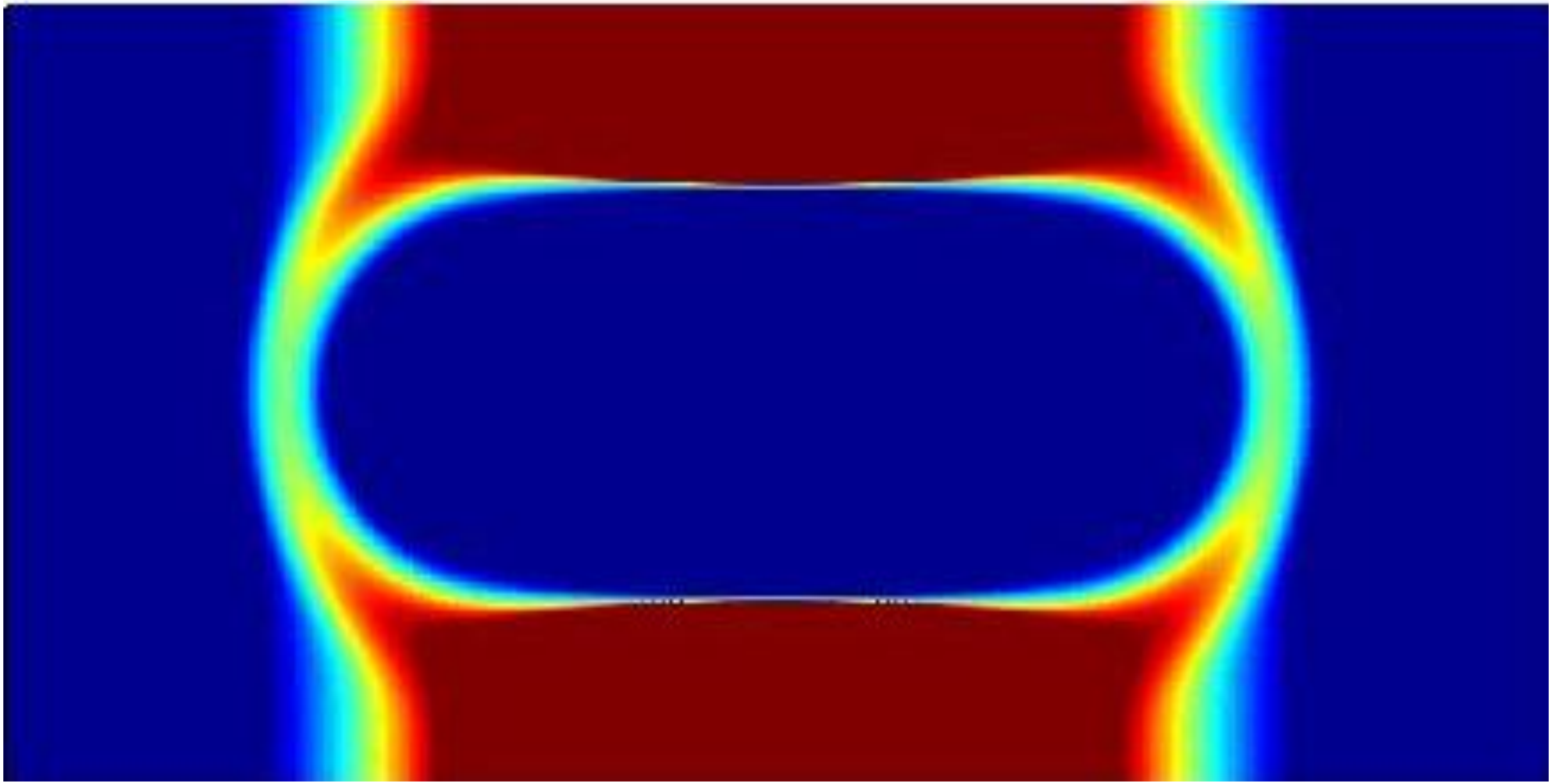
$\alpha = 1, \beta = 2, \kappa = 2.935$



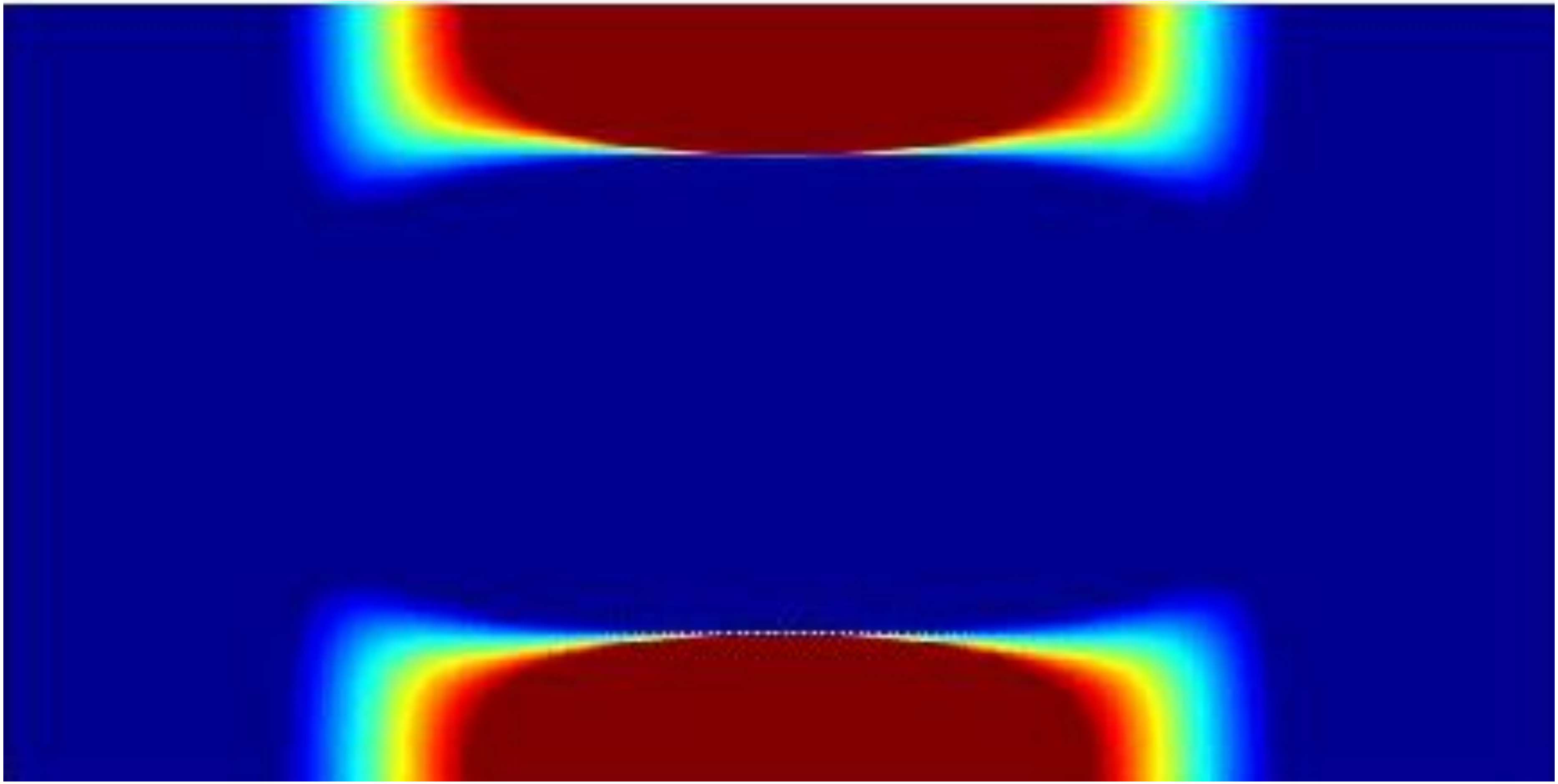
$\alpha = 1, \beta = 2, \kappa = 2.435$



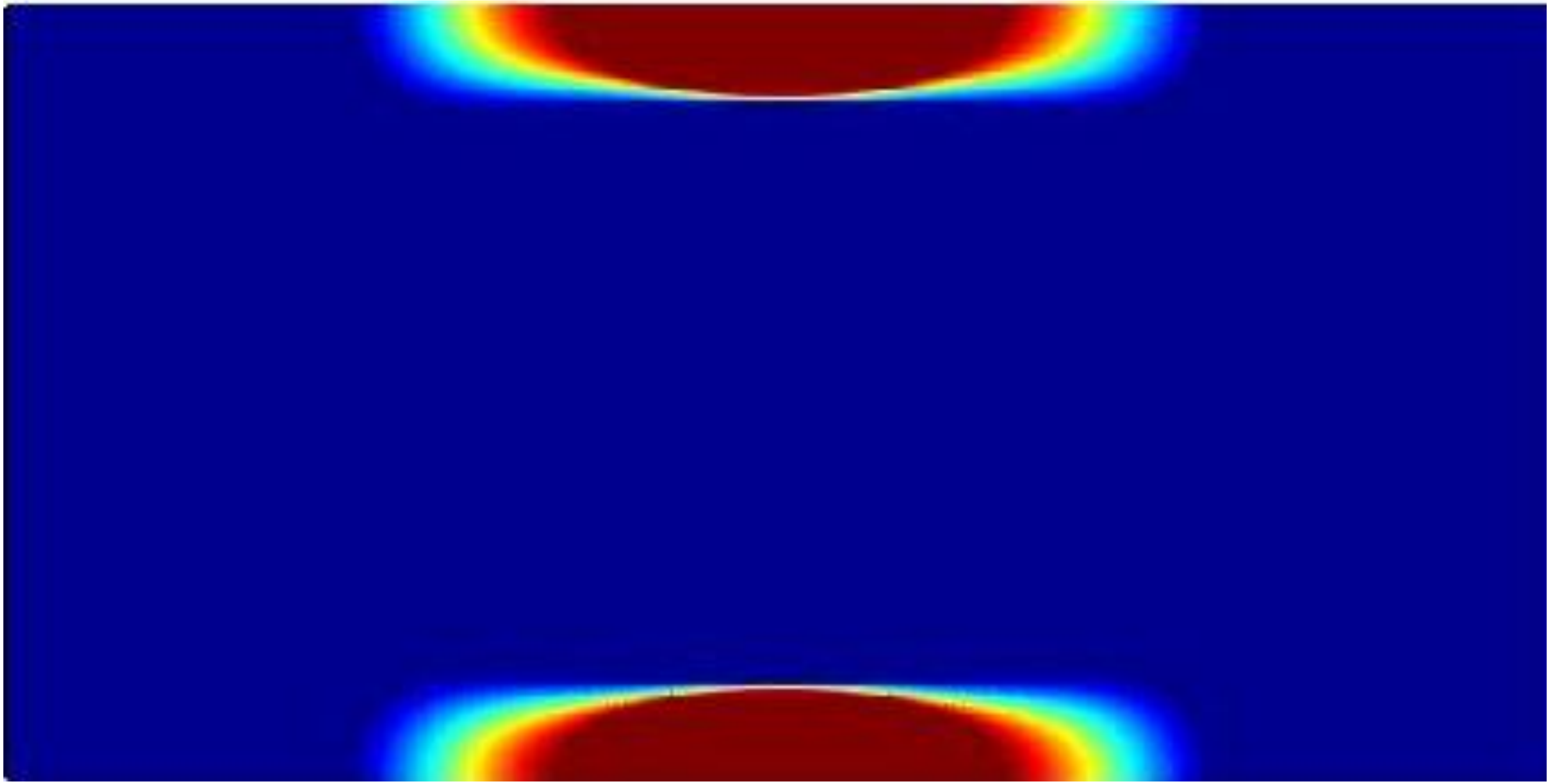
$\alpha = 1, \beta = 2, \kappa = 1.935$



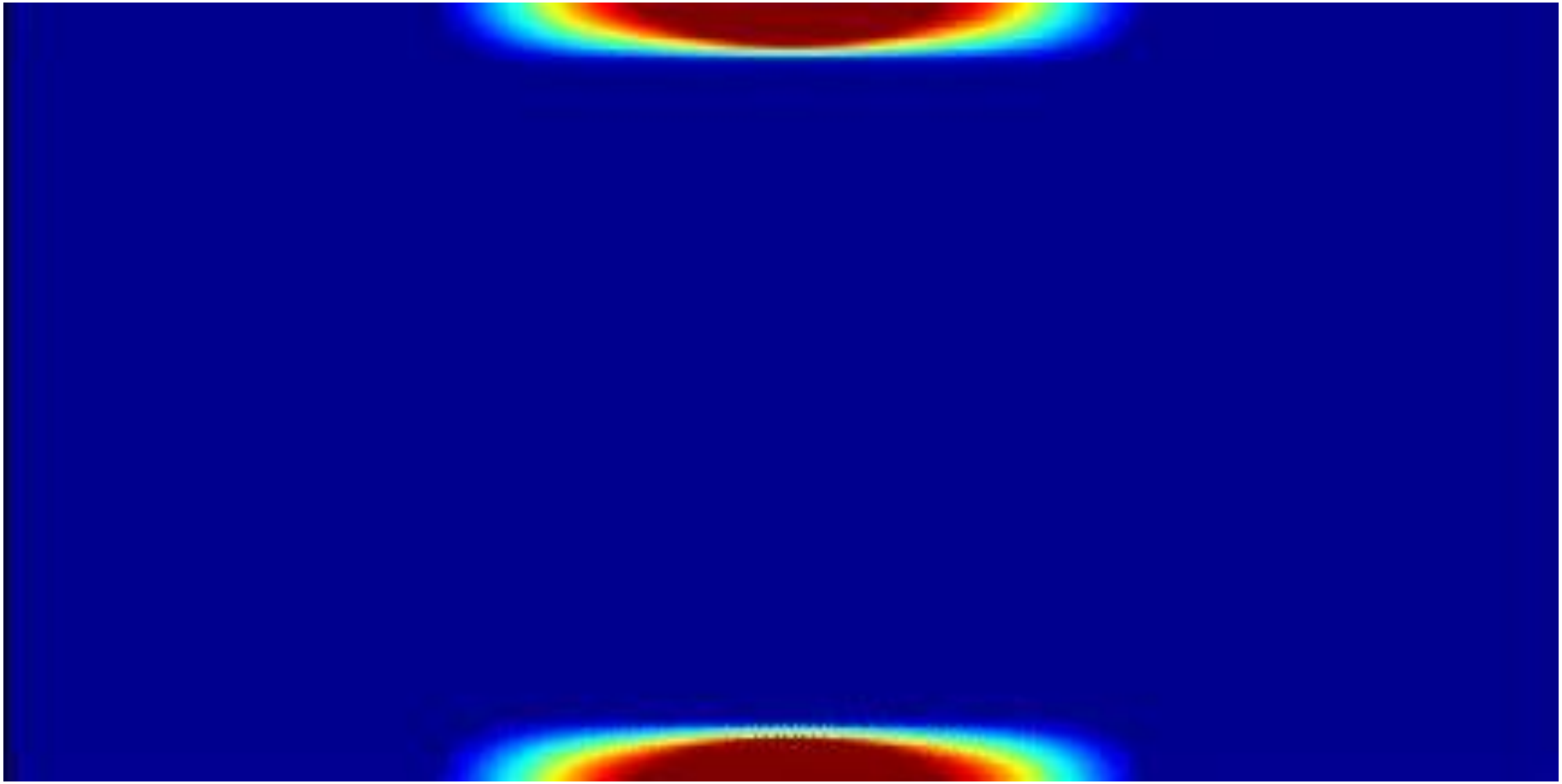
$\alpha = 1, \beta = 2, \kappa = 1.435$



$\alpha = 1, \beta = 2, \kappa = 0.935$



$\alpha = 1, \beta = 2, \kappa = 0.435$

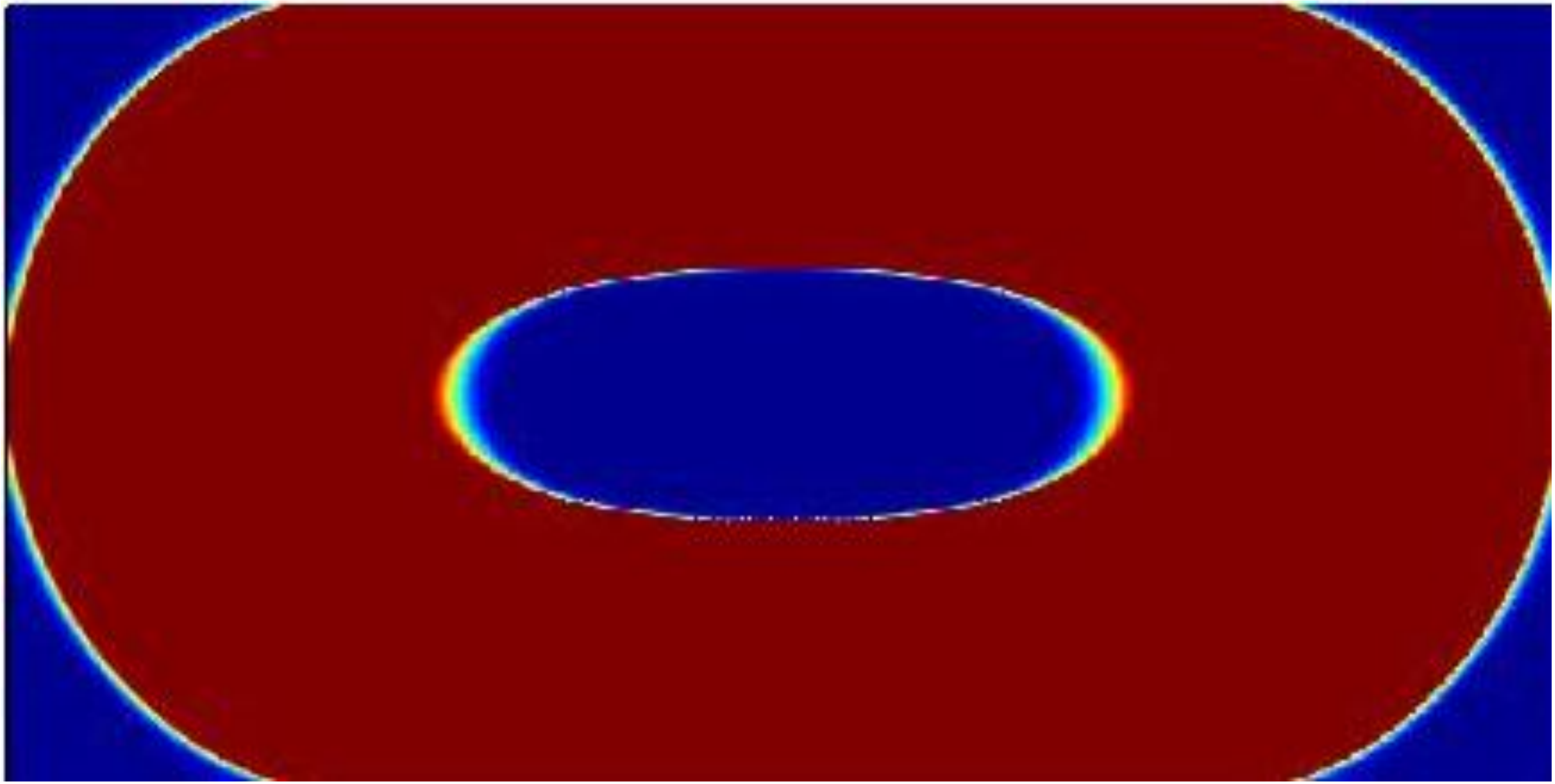


$\alpha = 1, \beta = 2, \kappa = 0.46$

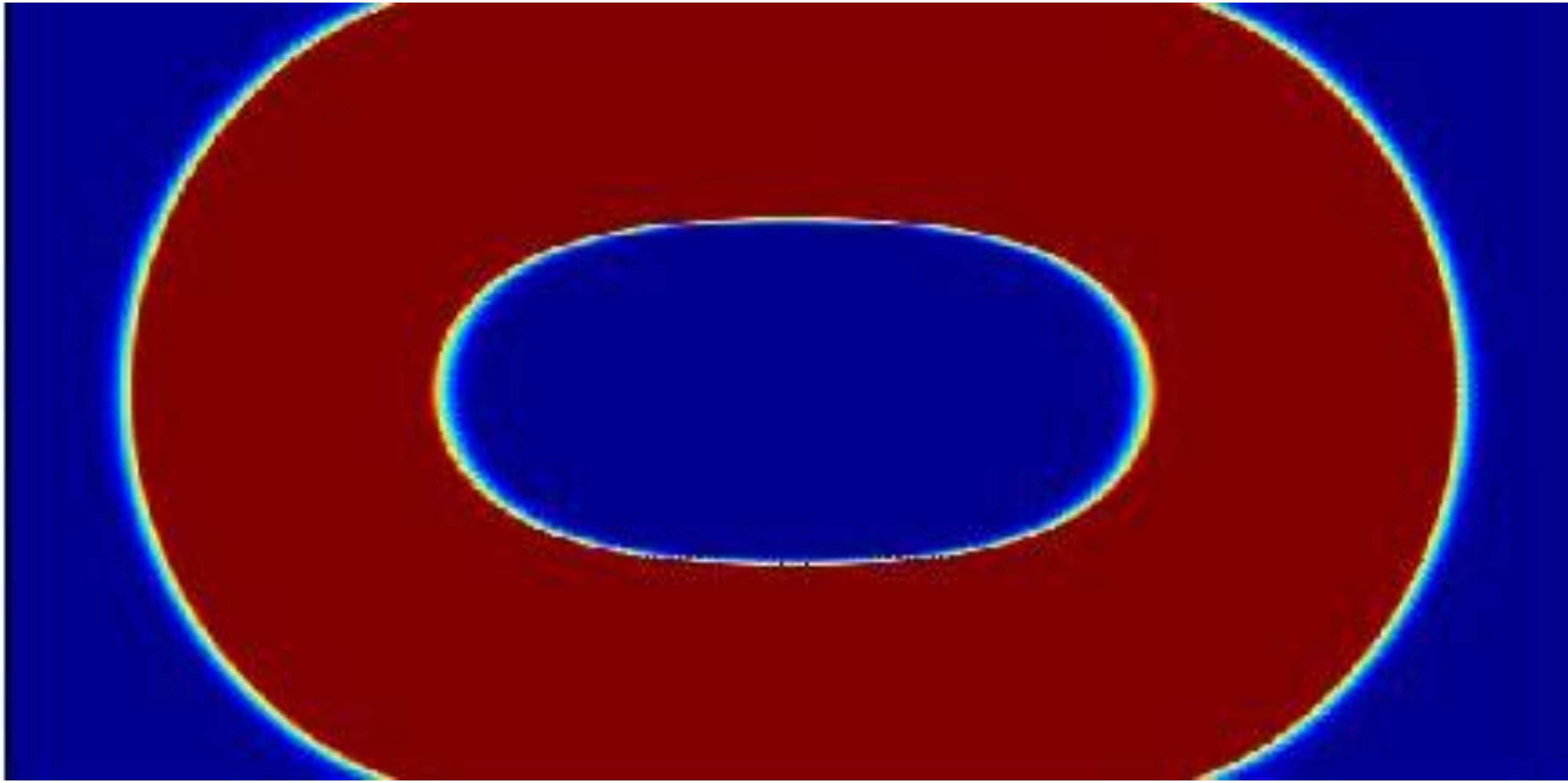
Problem $\Omega = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \ |\Omega| \approx 4{,}935, \ \alpha = 1, \beta = 20$

$$\min \frac{\int_\Omega \frac{|\nabla u|^2}{1 + 19\theta} dx}{\int_\Omega |u|^2 dx}$$

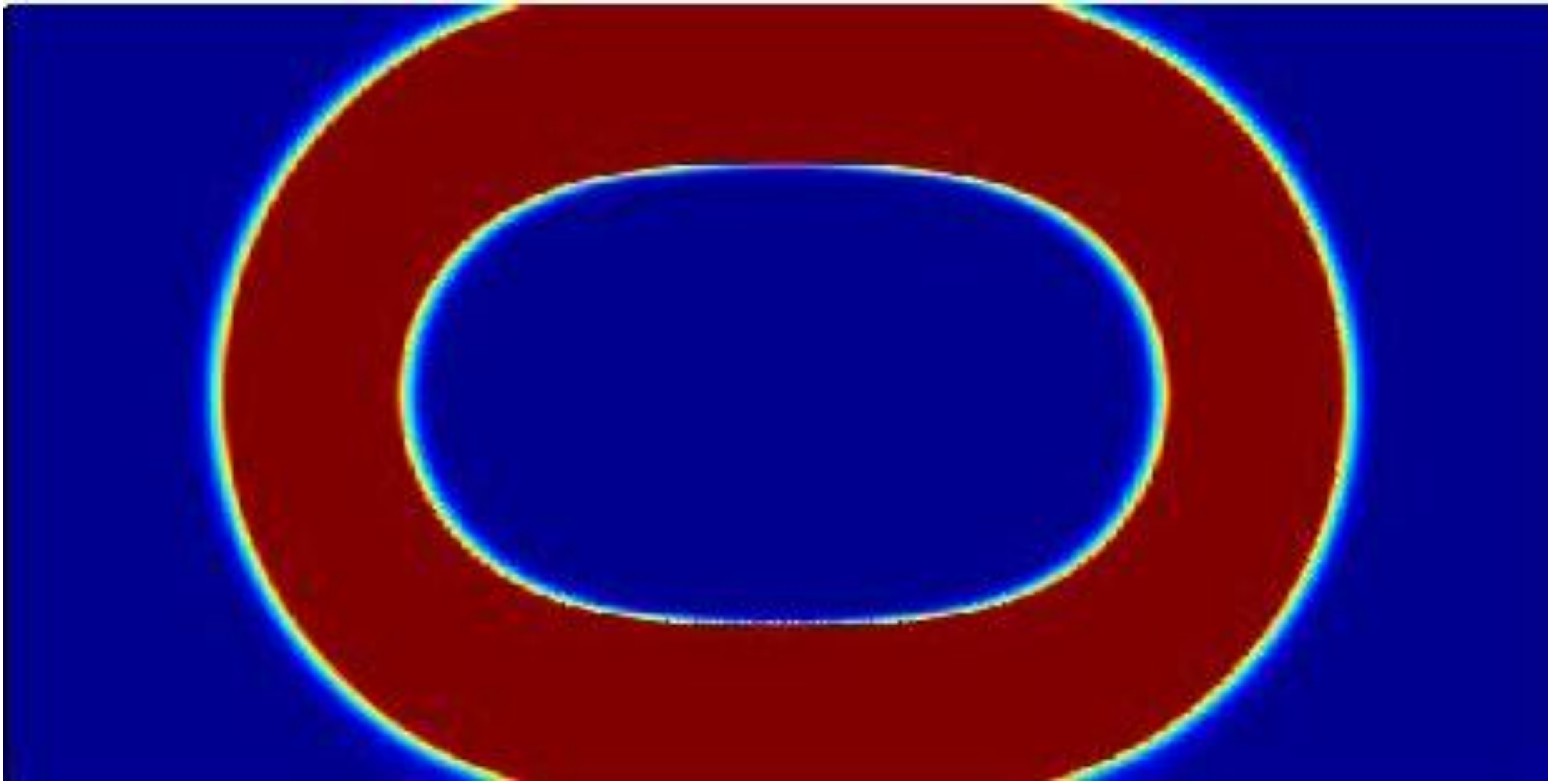
$$u \in H_0^1(\Omega), \int_\Omega \theta dx \leq \kappa$$



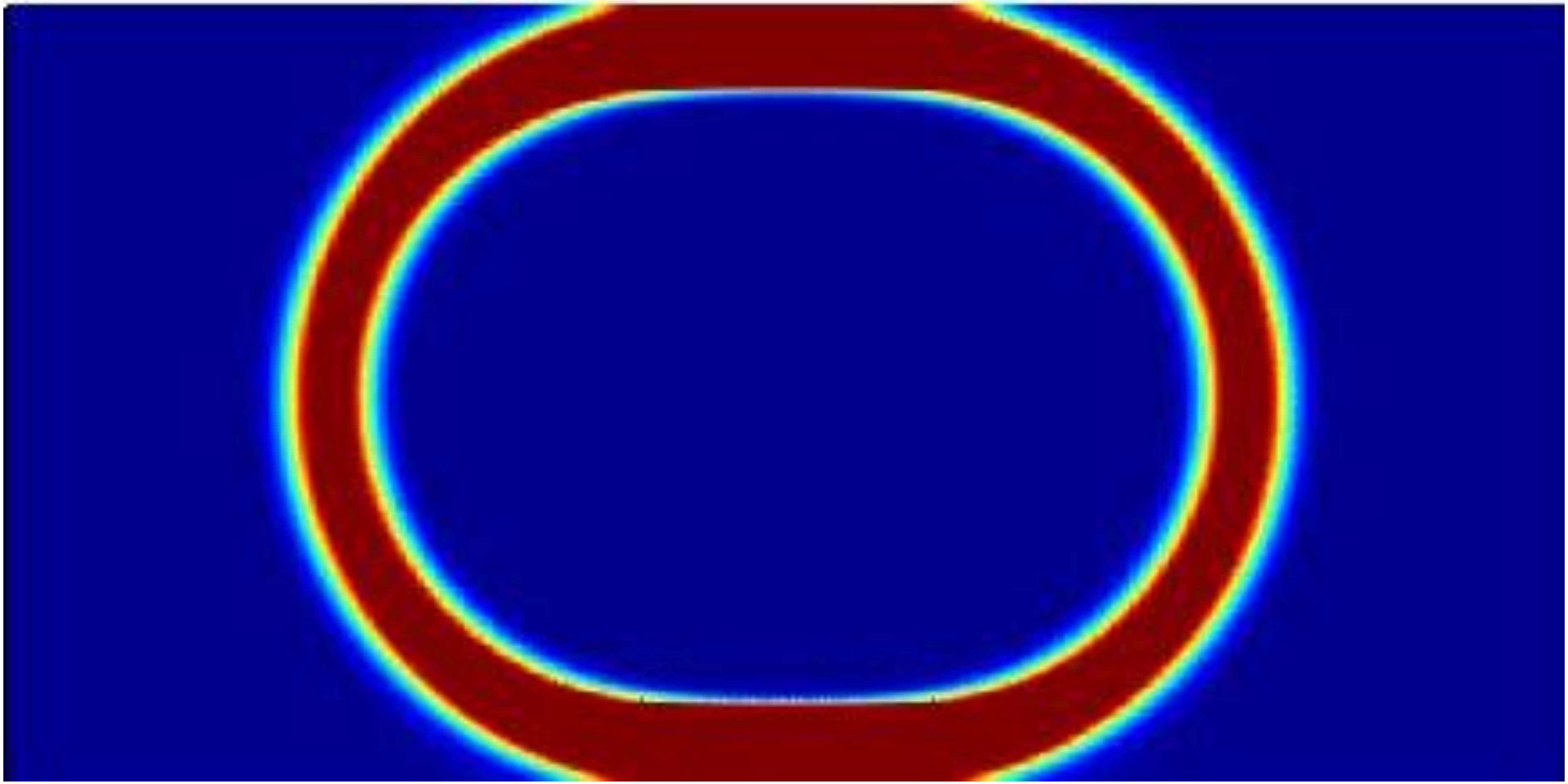
$\alpha = 1, \beta = 20, \kappa = 3.935$



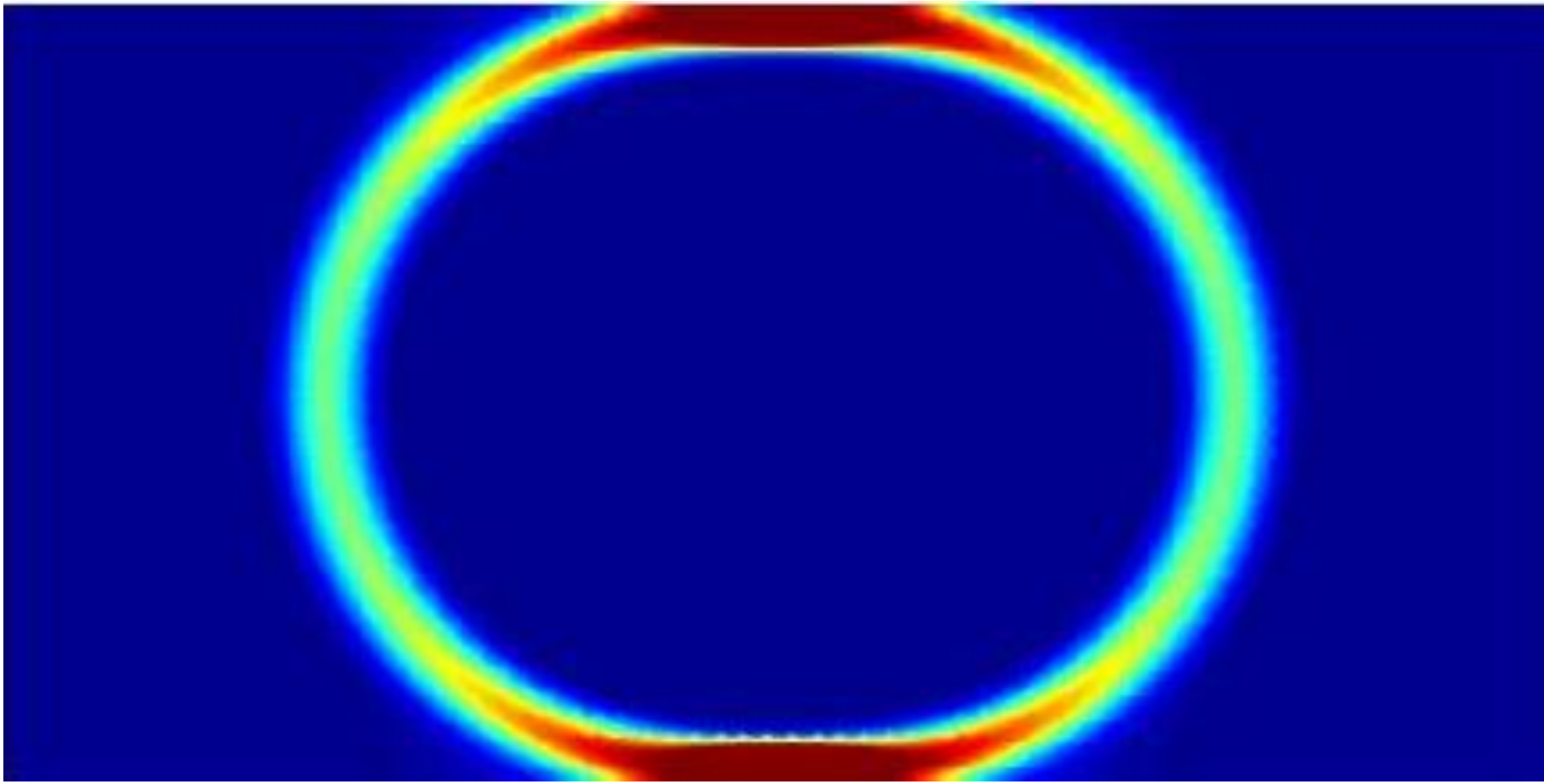
$\alpha = 1, \beta = 20, \kappa = 2.935$



$\alpha = 1, \beta = 20, \kappa = 1.935$



$\alpha = 1, \beta = 20, \kappa = 0.935$



$\alpha = 1, \beta = 20, \kappa = 0.435$

Remark. Similar results can be obtained for the probems ($p > 1$)

$$\max_{|\omega|\le\kappa} \int_\Omega (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^p dx$$

$$\begin{cases} -\mathrm{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^{p-2}\nabla u_\omega\big) = \tilde{f} \text{ in } \Omega \\ u_\omega = 0 \text{ on } \partial\Omega \end{cases}$$

and

$$\min_{|\omega|\ge\kappa} \int_\Omega (\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^p dx$$

$$\begin{cases} -\mathrm{div}\big((\alpha\chi_\omega + \beta(1-\chi_\omega))|\nabla u_\omega|^{p-2}\nabla u_\omega\big) = \tilde{f} \text{ in } \Omega \\ u_\omega = 0 \text{ on } \partial\Omega, \end{cases}$$

which admit the relaxed formulations

$$\min_{\theta\in L^\infty(\Omega;[0,1]),\int_\Omega \theta dx\leq\kappa} \min_{u\in H_0^1(\Omega)} \left(\int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}} dx - p \prec f, u \succ\right)$$

with $\quad c = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}} - 1, \quad f = \frac{1}{\beta}\tilde{f}$

and

$$\max_{\substack{\theta\in L^\infty(\Omega;[0,1]) \\ \int_\Omega \theta dx\geq\kappa}} \min_{u\in H_0^1(\Omega)} \left(\int_\Omega (1-c\theta)|\nabla u|^p dx - p \prec f, u \succ\right)$$

with $\quad c = \frac{\beta-\alpha}{\beta}, \quad f = \frac{1}{\beta}\tilde{f}$