A result about the estimate of the presssure in a thin domain and its application to elasticity problems

J. Casado-Díaz M. Luna-Laynez F.J. Suárez-Grau

University of Seville

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Motivation

For every $\varepsilon > 0$, we consider the thin domain

$$\Omega_\varepsilon = \omega' \times \varepsilon\omega'' \subset \mathbb{R}^N, \quad \varepsilon > 0,$$

with $\omega' \subset \mathbb{R}^k$, $\omega'' \subset \mathbb{R}^{N-k}$ smooth enough domains ($N \geq 2$, $0 < k < N$), and a solution $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2(\Omega_\varepsilon)$ of the Navier-Stokes problem

$$\begin{cases} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon + (u_\varepsilon \cdot \nabla)u_\varepsilon = f_\varepsilon \text{ in } \Omega_\varepsilon \\ \text{div } u_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ + \text{ boundary conditions} \end{cases}$$

Asymptotic behavior of $(u_\varepsilon, p_\varepsilon)$ as ε tends to zero?

The main result

To estimate the pressure, we often use the well known inequality

$$\|p_\varepsilon - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} p_\varepsilon\, dx\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon}\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N}, \quad (P)$$

for every $p_\varepsilon \in L^2(\Omega_\varepsilon)$, $\varepsilon > 0$.

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for every $p_\varepsilon \in L^2(\Omega_\varepsilon)$, $\varepsilon > 0$.

We improve this inequality by proving the following result

Theorem 1

For every $\varepsilon > 0$ and $p_\varepsilon \in L^2(\Omega_\varepsilon)$ there exist $p_\varepsilon^0 \in H^1(\omega')$ (it does not depend on x'') and $p_\varepsilon^1 \in L^2(\Omega_\varepsilon)$ satisfying

$$p_\varepsilon = p_\varepsilon^0 + p_\varepsilon^1 \quad \textit{in } \Omega_\varepsilon,$$

$$\varepsilon\|\nabla_{x'} p_\varepsilon^0\|_{L^2(\Omega_\varepsilon)^k} + \|p_\varepsilon^1\|_{L^2(\Omega_\varepsilon)} \leq C\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N},$$

with C a positive constant independent of p_ε and ε.

(we write $x \in \mathbb{R}^N$ as $x = (x', x'')$ with $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{N-k}$)

Theorem 1 + Poincaré-Wirtinger's inequality give

Corollary

For every $\varepsilon > 0$ *and* $p_\varepsilon \in L^2(\Omega_\varepsilon)$ *there exist* $\widehat{p}_\varepsilon^0 \in H^1(\omega)$ *(it does not depend on* x''*) and* $\widehat{p}_\varepsilon^1 \in L^2(\Omega_\varepsilon)$ *satisfying*

$$p_\varepsilon - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} p_\varepsilon \, dx = \widehat{p}_\varepsilon^0 + \widehat{p}_\varepsilon^1 \quad \text{in } \Omega_\varepsilon,$$

$$\|\widehat{p}_\varepsilon^0\|_{H^1(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon}\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N}, \quad \|\widehat{p}_\varepsilon^1\|_{L^2(\Omega_\varepsilon)} \leq C\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N},$$

with C *a positive constant independent of* p_ε *and* ε*.*

We decompose p_ε as the sum of a term of order ε^{-1}, which is not only in L^2 but in H^1, plus a term in L^2 of order 1 with respect $\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N}$.

Remark : Let us consider a sequence $p_\varepsilon \in L^2(\Omega_\varepsilon)$ satisfying

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} p_\varepsilon \, dx = 0, \quad \|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N} \leq C, \quad \forall \varepsilon > 0.$$

Then, we have

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon}, \quad \forall \varepsilon > 0.$$

Let v be a smooth enough function and let us define the sequence

$$v_\varepsilon(x) = \varepsilon v\left(x', \frac{x''}{\varepsilon}\right).$$

Observe that $\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon$ and

$$\|\operatorname{div} v_\varepsilon\|_{L^2(\Omega_\varepsilon)} = \|\varepsilon \operatorname{div}_{x'} v + \operatorname{div}_{x''} v\|_{L^2(\Omega_\varepsilon)} \leq C.$$

We can not pass to the limit in $< \nabla p_\varepsilon, v_\varepsilon >$ by using (P),

$$< \nabla p_\varepsilon, v_\varepsilon >= -\int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} v_\varepsilon dx$$

because we would need $\|\operatorname{div} v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon$.

However if we use Theorem 1

$$< \nabla p_\varepsilon, v_\varepsilon >= \int_{\Omega_\varepsilon} \nabla p_\varepsilon^0 v_\varepsilon dx - \int_{\Omega_\varepsilon} p_\varepsilon^1 \operatorname{div} v_\varepsilon dx$$

we would need $\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon$, $\|\operatorname{div} v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C$.

Remark : We can deal with more general thin domains Ω_ε. For example, we can consider thin domains with rough boundary as

$$\Omega_\varepsilon = \left\{ (x_1, x_2, x_3) \in \omega \times \mathbb{R} : \; -\delta_\varepsilon \Psi \left(\frac{x_1}{r_\varepsilon}, \frac{x_2}{r_\varepsilon} \right) < x_3 < \varepsilon \right\}.$$

In a recent paper we have studied the asymptotic behavior of

$$\begin{cases} -\mu \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon \;\; \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0 \;\; \text{in } \Omega_\varepsilon, \\ u_\varepsilon \nu = 0, \quad T\left(\mu \dfrac{\partial u_\varepsilon}{\partial \nu} + \dfrac{\gamma}{\varepsilon} u_\varepsilon \right) = 0 \;\; \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0 \;\; \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \end{cases}$$

where $\qquad T\xi = \xi - (\xi\nu)\nu, \quad \forall \xi \in \mathbb{R}^3, \;\; \text{a.e. on } \partial\Omega_\varepsilon,$

$$\Gamma_\varepsilon = \left\{ x \in \mathbb{R}^3 : \; x' \in \omega, \, x_3 = -\delta_\varepsilon \Psi \left(\frac{x'}{r_\varepsilon} \right) \right\}$$

when $\delta_\varepsilon \ll r_\varepsilon \ll \varepsilon$. The asymptotic behavior depends on the value

$$\lambda = \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{r_\varepsilon^{3/2}} \varepsilon^{1/2}.$$

- If $\lambda = \infty$, then the fluid behaves as if we also imposed an adherence condition on Γ_ε.
- If $\lambda \in (0, +\infty)$, then the roughness is not strong enough to give the adherence condition in the limit but it is enough to obtain a new friction term in the limit.
- If $\lambda = 0$ the roughness is so weak that the fluid behaves as if the rough wall was plane.

D. Bresch, D. Bucur, E. Feireisl, E. Fernández-Cara, W. Jager, A. Mikelic, N. Nečsová, J. Simon ...

Korn's inequality

It is well known that from inequality

$$\|p_\varepsilon - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} p_\varepsilon \, dx\|_{L^2(\Omega_\varepsilon)} \leq \frac{C}{\varepsilon}\|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N}, \quad (P)$$

$\forall p_\varepsilon \in L^2(\Omega_\varepsilon)$, $\varepsilon > 0$,
we can prove Korn's inequality in Ω_ε.

Analogously, from Theorem 1 we can deduce the following result

Theorem 2

For every $\varepsilon > 0$ *and* $u_\varepsilon \in H^1(\Omega_\varepsilon)^N$ *there exist*

- $\widehat{a}_\varepsilon \in \mathbb{R}^N$, $\widehat{B}_\varepsilon \in \mathbb{R}^{N\times N}$ *skew-symmetric*
- $\widehat{u}''_\varepsilon \in H^2(\omega')^{N-k}$, $\widehat{w}_\varepsilon \in H^1(\Omega_\varepsilon)^N$, $\widehat{C}_\varepsilon \in H^1(\omega')^{(N-k)\times(N-k)}$ *skew-symmetric*

satisfying

$$u_\varepsilon(x) = \widehat{a}_\varepsilon + \widehat{B}_\varepsilon x + \begin{pmatrix} -D_{x'}\widehat{u}''_\varepsilon(x')^t \dfrac{x''}{\varepsilon} \\ \dfrac{1}{\varepsilon}\widehat{u}''_\varepsilon(x') + \widehat{C}_\varepsilon(x')\dfrac{x''}{\varepsilon} \end{pmatrix} + \widehat{w}_\varepsilon(x), \qquad (1)$$

$$\|\widehat{u}''_\varepsilon\|_{H^2(\Omega_\varepsilon)^{N-k}} \leq C\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)^{N\times N}},$$

$$\|\widehat{C}_\varepsilon\|_{H^1(\Omega_\varepsilon)^{(N-k)\times(N-k)}} \leq C\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)^{N\times N}},$$

$$\|\widehat{w}_\varepsilon\|_{W^{1,q}(\Omega_\varepsilon)^N} \leq C\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)^{N\times N}}$$

A simple application: a thin beam in $\mathbb{R}^3$

We consider

$$\Omega_\varepsilon = (0,1) \times \varepsilon\omega''$$

and we denote

$$\Gamma_\varepsilon = \{0,1\} \times \varepsilon S.$$

In Ω^ε we consider the elasticity problem

$$\begin{cases} -divAe(u_\varepsilon) = F_\varepsilon \quad \text{in } \Omega_\varepsilon \\ A^\varepsilon e(u_\varepsilon)\nu = 0 \quad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon \\ u_\varepsilon = 0 \quad \text{on } \Gamma^\varepsilon \end{cases} \qquad (E)$$

where $F_\varepsilon \in L^2(\Omega_\varepsilon)$ and $A \in \mathcal{L}(\mathbb{R}^{3\times 3}_s)$ satisfies

$$A\xi : \xi \geq m|\xi|^2, \quad \forall \xi \in \mathbb{R}^{3\times 3}_s$$

F. Murat, A. Sili (1999)
For $U_\varepsilon \in H^1((0,1)\times\omega'')^3$ defined by

$$U_{\varepsilon,1}(y_1,y_2,y_3) = u_{\varepsilon,1}(y_1,\varepsilon y_2,\varepsilon y_3),$$
$$U_{\varepsilon,2}(y_1,y_2,y_3) = \varepsilon u_{\varepsilon,2}(y_1,\varepsilon y_2,\varepsilon y_3)$$
$$U_{\varepsilon,3}(y_1,y_2,y_3) = \varepsilon u_{\varepsilon,3}(y_1,\varepsilon y_2,\varepsilon y_3)$$

there exist a Bernouilli-Navier displacement U, a rotation V, and a displacement orthogonal to the rigid displacements W, satisfying a system of PDE (limit problem) and such that

$$U_\varepsilon(y) \sim U(y) + \varepsilon V(y) + \varepsilon^2 W(y)$$

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$$U_{\varepsilon,3}(y_1,y_2,y_3) = \varepsilon u_{\varepsilon,3}(y_1,\varepsilon y_2,\varepsilon y_3)$$

or equivalently, there exist $\zeta_1 \in H^1(0,1)$, $\zeta_2,\zeta_3 \in H^2(0,1)$, $c \in H^1(0,1)$, $v_1, w_2, w_3 \in L^2(0,1;H^1(\omega''))$ such that

$$\begin{cases} U_{\varepsilon,1}(y) \sim \zeta_1(y_1) - \dfrac{d\zeta_2}{dy_1}(y_1)y_2 - \dfrac{d\zeta_3}{dy_1}(y_1)y_3 + \varepsilon v_1(y), \\ U_{\varepsilon,2}(y) \sim \zeta_2(y_1) + \varepsilon c(y_1)y_3 + \varepsilon^2 w_2(y), \\ U_{\varepsilon,3}(y) \sim \zeta_3(y_1) - \varepsilon c(y_1)y_2 + \varepsilon^2 w_3(y), \end{cases}$$

In the original variables ($x_1 = y_1$, $x_2 = \varepsilon y_2$, $x_3 = \varepsilon y_3$) this reads as

$$\begin{cases} u_{\varepsilon,1}(x) \sim \zeta_1(x_1) - \dfrac{d\zeta_2}{dy_1}(x_1)\dfrac{x_2}{\varepsilon} - \dfrac{d\zeta_3}{dy_1}(x_1)\dfrac{x_3}{\varepsilon} + \varepsilon v_1(x_1, \dfrac{x_2}{\varepsilon}, \dfrac{x_3}{\varepsilon}), \\ u_{\varepsilon,2}(x) \sim \dfrac{1}{\varepsilon}\zeta_2(x_1) + c(x_1)\dfrac{x_3}{\varepsilon} + \varepsilon w_2(x_1, \dfrac{x_2}{\varepsilon}, \dfrac{x_3}{\varepsilon}), \\ u_{\varepsilon,3}(x) \sim \dfrac{1}{\varepsilon}\zeta_3(x_1) - c(x_1)\dfrac{x_2}{\varepsilon} + \varepsilon w_3(x_1, \dfrac{x_2}{\varepsilon}, \dfrac{x_3}{\varepsilon}), \end{cases}$$

The main difficulty to prove Murat, Sili's result is that we only have a good bound for the symmetric part of the derivative of u_ε:

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} |e(u^\varepsilon)|^2 dx \leq C, \quad \forall \varepsilon > 0.$$

From this estimate for $e(u_\varepsilon)$ and Theorem 2, we deduce there exist $\widehat{a}_\varepsilon \in \mathbb{R}^3$, $\widehat{B}_\varepsilon \in \mathbb{R}^{3\times 3}$ skew-symmetric, $(\widehat{u}_{\varepsilon,2}, \widehat{u}_{\varepsilon,3}) \in H^2(0,1)^2$, $\widehat{c}_\varepsilon \in H^1(0,1)$, $\widehat{w}_\varepsilon \in H^1(\Omega_\varepsilon)^3$ such that

$$u_\varepsilon(x) = \hat{a}_\varepsilon + \hat{B}_\varepsilon x + \begin{pmatrix} -\dfrac{d\widehat{u}_{\varepsilon,2}}{dx_1}(x_1)\dfrac{x_2}{\varepsilon} - \dfrac{d\widehat{u}_{\varepsilon,3}}{dx_1}(x_1)\dfrac{x_3}{\varepsilon} + \hat{w}_{\varepsilon,1}(x) \\ \dfrac{1}{\varepsilon}\widehat{u}_{\varepsilon,2}(x_1) + \widehat{c}_\varepsilon(x_1)\dfrac{x_3}{\varepsilon} + \hat{w}_{\varepsilon,2}(x) \\ \dfrac{1}{\varepsilon}\widehat{u}_{\varepsilon,3}(x_1) - \widehat{c}_\varepsilon(x_1)\dfrac{x_2}{\varepsilon} + \hat{w}_{\varepsilon,3}(x) \end{pmatrix}$$

and

$$\|\widehat{u}_{\varepsilon,2}\|_{H^2(0,1)} + \|\widehat{u}_{\varepsilon,3}\|_{H^2(0,1)} \leq C$$

$$\|\widehat{c}_\varepsilon\|_{H^1(0,1)} \leq C$$

$$\|\widehat{w}_\varepsilon\|_{H^1(\Omega_\varepsilon)^3} \leq C\varepsilon \Longleftrightarrow \frac{1}{|\Omega_\varepsilon|}\left(\int_{\Omega_\varepsilon} |\widehat{w}_\varepsilon|^2 dx + \int_{\Omega_\varepsilon} |D\widehat{w}_\varepsilon|^2 dx\right) \leq C$$