LAMA Université de Savoie

Sub and supersolutions in shape optimization

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Collaborators

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Generic problem

$$\min_{|\Omega|=m} F(\lambda_1(\Omega), ..., \lambda_k(\Omega)),$$

or (to be simple)

$$\min_{|\Omega|=m} \lambda_k(\Omega),$$

$\Omega \subseteq \mathbb{R}^N$ open, $|\Omega|$ the Lebesgue measure,

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ... \leq \lambda_k(\Omega)$$

the first k eigenvalues of the Laplacian with Dirichlet b.c.
Questions :

- existence of a solution : Ω
- properties of Ω coming from optimality : regularity, symmetry, convexity,...is it the ball ?
- numerical computations

Examples

$$\begin{cases} -\Delta u = & \lambda u \ \text{ in } \ \Omega \\ u = & 0 \ \ \partial\Omega \end{cases}$$

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq \to +\infty$$

Variational definition :

$$\lambda_1(\Omega) = \min_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

$$\lambda_k(\Omega) = \min_{S_k \in H_0^1(\Omega)} \max_{u \in S_k} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

Rayleigh 1877

Faber-Krahn 1920 : the solution of

$$\min_{|\Omega|=m} \lambda_1(\Omega)$$

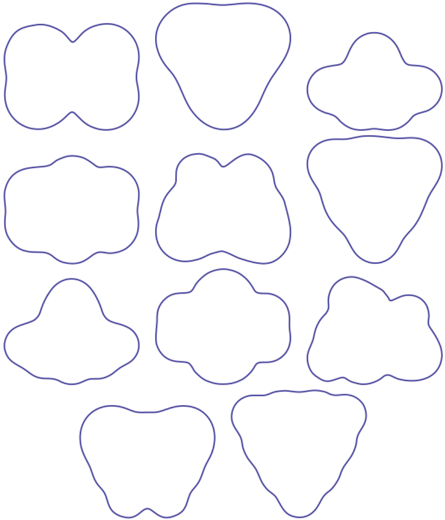
is the ball.

Proof : Schwarz rearrangement of $u \in H_0^1(\Omega), u \geq 0$ on a ball.

Other eigenvalues

- $\min_{|\Omega|=m} \lambda_2(\Omega)$: two balls of volume $\frac{m}{2}$, Faber-Krahn 1923
- $\min \frac{\lambda_1(\Omega)}{\lambda_2(\Omega)}$: ball, Ashbaugh-Benguria 1993
- $\min_{|\Omega|=m} \lambda_3(\Omega)$: conjecture : ball in 2D
- $\min_{|\Omega|=m} \lambda_4(\Omega)$: conjecture : two balls of different measures in 2D
- $\min_{|\Omega|=m} \lambda_{13}(\Omega)$: is not a union of balls, Wolf and Keller 1992
- $\min_{|\Omega|=m} \lambda_5(\Omega)$: is not a union of balls, Oudet 2002

Oudet 2004, **Antunes-Freitas 2012 :** λ_5 **to** λ_{15}



Existence of a solution

Theorem Buttazzo-Dal Maso 1991
Let D be bounded and open. If F is increasing in each variable and l.s.c., the problem

$$\min_{|\Omega|=c, \Omega \subset D} F(\lambda_1(\Omega), ..., \lambda_k(\Omega))$$

has a solution.

Examples :

- $F(\lambda_1, .., \lambda_k) = 5\lambda_1 + \lambda_2\lambda_5$
- $F(\lambda_1, .., \lambda_k) = \lambda_k$
- not admissible $F(\lambda_1, .., \lambda_k) = \lambda_1 - \lambda_2$

Idea of the proof

- $(\Omega_n)_n$ minimizing sequence
- $\Omega_{n_k} \longrightarrow \mu$ in an appropriate sense : γ-convergence
- $\lambda_j(\Omega_{n_k}) \longrightarrow \lambda_j(\mu)$ the spectrum follows the geometry

$$\begin{cases} -\Delta u + \mu u = & \lambda u \text{ in } D \\ u \in & H_0^1(D) \cap L^2(D, \mu) \end{cases}$$

- monon局onicity implies that μ is a true domain

The solution depends on D, and is a priori not smooth (even not open) !

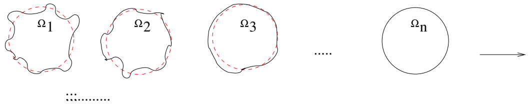
Global minimizers ?

- replace $D = \mathbb{R}^N$: key point for optimality conditions

Global minimization in $\mathbb{R}^N$

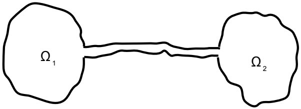
Concentration-compactness principle for functions P.L. Lions : 3 possibilities

- compacntess :



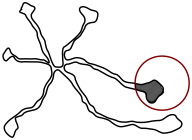
by translation one can concentrate the mass

- dichotomy :



"two distancing pieces"

- vanishing :



nowhere mass concentration

Theorem B. 1998

Denote $R_\Omega := (-\Delta)^{-1} : L^2(\mathbb{R}^N) \to H_0^1(\Omega) \subset L^2(\mathbb{R}^N)$.

Let $\{\Omega_n\}_n$ (quasi) open, $|\Omega_n| = c$.

- compactness : $\exists y_k \in \mathbb{R}^N$, $\exists \mu$ t.q.

$$R_{\Omega_{n_k}+y_k} \longrightarrow R_\mu, \text{ in } \mathcal{L}(L^2(\mathbb{R}^N))$$

- dichotomy : $\exists \Omega_k^1, \Omega_k^2$, t.q.

$$d(\Omega_k^1, \Omega_k^2) \to +\infty$$

$$\Omega_k^1 \cup \Omega_k^2 \subseteq \Omega_{n_k}$$

$$\liminf_{n\to\infty} |\Omega_k^i| > 0$$

$$\|R_{\Omega_{n_k}} - R_{\Omega_k^1} - R_{\Omega_k^2}\|_{\mathcal{L}(L^2(\mathbb{R}^N))} \to 0$$

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Application to λ_3

B. Henrot 1998

$$\min_{|\Omega|=m} \lambda_3(\Omega)$$

- if compactness, we construct μ and by monotonicity $\Longrightarrow$ existence of a solution
- if dichotomy, we replace the minimizing sequence by $\Omega_k^1 \cup \Omega_k^2$! The optimum consists on three balls...

Mazzoleni and Pratelli JMPA 2013

Different approach for existence in $\mathbb{R}^N$.

Theorem (Independent on the optimization problem)

For every smooth open set there exists a bounded open set of the same measure, with controlled diameter and lower or equal first k-eigenvalues.

Consequence : the existence result of Buttazzo-Dal Maso is global in $\mathbb{R}^N$! Existence of one bounded minimizer.

Free boundary approach

Alt and Caffarelli 1981 : the capacity problem
K given

$$\min\{\int |\nabla u|^2 dx + |\{u > 0\}| : u \in H_0^1(box), u = 1 \text{ on } K\}.$$

The free boundary $\partial\{u > 0\}$ = the shape

- exists
- is smooth
- satisfies density estimates

Tool : local perturbations in balls.

Energy and λ_1 : Pierre-Briançon-Hayouni-Lamboley-Landais 1996-2008

Free boundary approach : shape subsolutions

For $f \equiv 1$, the torsion energy

$$E(\Omega) = \min_{u \in H_0^1(\Omega)} \frac{1}{2} \int |\nabla u|^2 dx - \int u dx.$$

Definition

The set Ω is a shape subsolution for the energy problem $\exists\, c > 0$ such that

$$\forall \tilde{\Omega} \subseteq \Omega \quad E(\Omega) + c|\Omega| \leq E(\tilde{\Omega}) + c|\tilde{\Omega}|. \tag{1}$$

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For $f \equiv 1$, the torsion energy

$$E(\Omega) = \min_{u \in H_0^1(\Omega)} \frac{1}{2} \int |\nabla u|^2 dx - \int u dx = -\frac{torsion(\Omega)}{2}.$$

Definition

The set Ω is a shape subsolution for the energy problem $\exists\, c > 0$ such that

$$\forall \tilde{\Omega} \subseteq \Omega \quad torsion(\Omega) - torsion(\tilde{\Omega}) \geq c(|\Omega| - |\tilde{\Omega}|). \qquad (2)$$

Free boundary approach : shape subsolutions

Theorem (B. ARMA 2012)

If Ω is a subsolution then Ω is bounded and has finite perimeter.

- Only inner perturbations alowed !
- If $\int_{B_{2r}(x)} u dx$ is small enough, then $u \equiv 0$ on $B_r(x) \Longrightarrow$ inner density, boundedness and control of the diameter.
- Control on $\int_0^\varepsilon |\nabla u| dx \Longrightarrow$ finite perimeter.

Free boundary approach : shape subsolutions

Theorem

For every $k \in \mathbb{N}, m > 0$, there exists $c > 0$ such that every solution of

$$\min_{|\Omega|=m} \lambda_k(\Omega)$$

is a subsolution of the energy problem $E(\Omega) + c|\Omega|$.

$\Longrightarrow$ existence of minimizers for λ_k + boundedness, finite perimeter, inner density and control of the diameter of every mini minimizer

The variation of the torsion controls the variation of the eigenvalue, with a constant depending on the larger domain :

$$c(|\Omega| - |\tilde{\Omega}|) \le \lambda_k(\tilde{\Omega}) - \lambda_k(\Omega) \le C_\Omega(torsion(\Omega) - torsion(\tilde{\Omega})).$$

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Shape supersolutions : regularity of minimizers

Supersolution= optimal for outer geometric perturbations !

Theorem (Alt-Caffarelli, ... Pierre)

Let $f \in L^\infty(\Omega)$ *and the function* $u \in \tilde{H}^1_0(\Omega)$ *which satisfies the following conditions :*

(a) $-\Delta u = f$ *in* $[\tilde{H}^1_0(\Omega)]'$;

(b) *there are constants* $r_0 \leq 1$ *and* C_b *such that for every* $x \in R^d$, *every* $0 < r \leq r_0$ *and every* $\varphi \in H^1_0(B_r(x))$ *we have*

$$|\langle \Delta u + f, \varphi \rangle| \leq C_b \|\nabla \varphi\|_{L^2} |B_r|^{1/2}. \qquad (3)$$

Then u *is Lipschitz continuous on* $\mathbb{R}^d$*, with controlled constant.*

If λ_k were simple $\Longrightarrow$ the result applies !

Shape supersolutions : openness, regularity of minimizers

Conjecture : for every minimizer of λ_k, the k-th eigenvalue is multiple and equals λ_{k-1}.
Numerical evidence by Oudet 2004, Antunes-Freitas 2012, Kao-Osting 2012.

Numerical observation by Kao and Osting : assume Ω_k^* is optimal for λ_k. There exists $\varepsilon > 0$ such that Ω_k^* is also optimal for

$$(1-\varepsilon)\lambda_k(\Omega) + \varepsilon\lambda_{k-1}(\Omega).$$

Ideas :

- work with supersolutions of $(1-\varepsilon)\lambda_k(\Omega) + \varepsilon\lambda_{k-1}(\Omega)$
- if Ω is supersolution for λ_k then it is also supersolution (with different constant) for λ_{k-1}

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Theorem (B., Mazzoleni, Pratelli, Velichkov 2013)

For every bounded shape supersolution Ω^* of the problem

$$\min\left\{\lambda_k(\Omega):\ \Omega\subset\mathbb{R}^d,\ |\Omega|=c\right\},$$

there exists a Lipschitz k-th eigenfunction.

Example : under the constraint $|\Omega| = m$, any minimizer Ω of

$$\lambda_1(\Omega)+...+\lambda_k(\Omega)$$

is an open set and all eigenfunctions up to k are Lipschitz.

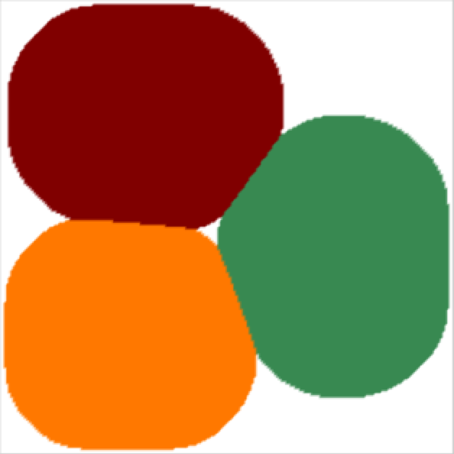
Multiphase shape optimization (B. Velichkov 2013)

$$\min\{\sum_{i=1}^N \lambda_{k_i}(\Omega_i) + c|\bigcup_{i=1}^N \Omega_i| : \Omega_i \cap \Omega_j = \emptyset,\ \Omega_i \subseteq D\}$$

- general existence of quasi-open sets B. Buttazzo, Henrot 1998
- each cell is a sub solution : finite perimeter, density estimates
- no triple junction points : multiphase monotonicity formula Caffarelli, Jerison and Kenig

$$\prod_{i=1}^3 \left(\frac{1}{r^{2+\epsilon}} \int_{B_r} \frac{|\nabla u_i|^2}{|x|^{d-2}}\, dx \right) \le C_d \left(1 + \sum_{i=1}^3 \int_{B_1} \frac{|\nabla u_i|^2}{|x|^{d-2}}\, dx \right).$$

Numerical results (Bogosel 2013)



3 cells

Numerical results (Bogosel 2013)

3 cells

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Thank you for your attention !