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Do optimal thin rods contain homogenized regions ?

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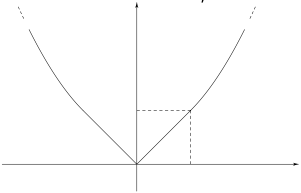
The problem

We consider the problem

$$m(s) := \inf \left\{ \int_D \varphi(\nabla u) \ : \ u \in H_0^1(D)\,,\ \int_D u = s \right\} \ ,$$

where $D \subset \mathbb{R}^2$ is a bounded simply connected domain, s a real parameter and

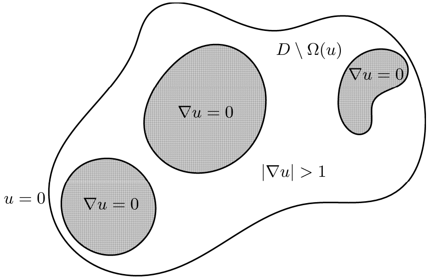
$$\varphi(y) := \begin{cases} \frac{1}{2}\left(1 + |y|^2\right) & |y| \geq 1 \\ |y| & |y| \leq 1 \end{cases}$$



> *Does $m(s)$ admit a solution u such that*
>
> $|\nabla u| \in \{0\} \cup]\,1, +\infty\,[$ *a.e. in D ?*

We call special solution such a minimizer for $m(s)$.

How it looks ?



$\Omega(u) := \{\nabla u = 0\}$ the plateau of u

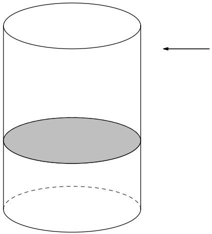
$\Gamma(u) := \partial\Omega(u) \cap D$ the free boundary of u

Outline

1. Mechanical motivation: optimal design of thin torsion rods
   - ▷ GB, Fragalà, Seppecher, *Arch. Rat. Mech. Anal.* (2011).
   - ▷ GB, Fragalà, Lucardesi, Seppecher, *SIAM J. Math. Anal.* (2012).
2. Optimality conditions, existence of a plateau and uniqueness.
3. Free boundary formulation and Cheeger sets
4. Existence results for special solutions
5. Further properties of special solutions and open problems

1. Optimal design of thin torsion rods

Minimize the compliance of an elastic material submitted to torsion, to be placed in a asymptotically thin design region with a prescribed volume fraction.



- shape optimization for the compliance [Allaire, Bonnetier, Cherkaev, Conca, Francfort, Gibiansky, Kohn, Strang, Jouve, Vogelius]
- dimension reduction analysis [Acerbi, Braides, Buttazzo, Ciarlet, Fonseca, Le Dret, Mora, Muller, Murat, Raoult, Percivale, Sili, Tomarelli,Trabucho,Viano]

The compliance of a linear elastic material placed in a subset $\Omega \subset \mathbb{R}^3$ submitted to an external load $F \in H^{-1}(\mathbb{R}^3;\mathbb{R}^3)$, is the opposite of the energy at equilibrium. It is the non negative **shape** functional:

$$C(\Omega) := \sup\left\{ \langle F, u\rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\, dx \ : \ u \in C^\infty(\mathbb{R}^3;\mathbb{R}^3) \right\}.$$

For any given parameter $k > 0$ (Lagrange multiplier), the optimal design problem reads

$$\inf\left\{ C(\Omega) \ : \ \Omega \subset Q, \ + k|\Omega| \right\}$$

- Since j is a quadratic: $C(\Omega) = \frac{1}{2}\langle F, \overline{u}\rangle$, with $\overline{u}$ optimal displacement.
- $j(z) = \frac{\lambda}{2}(\mathrm{tr}(z))^2 + \eta|z|^2$

Assumption on the load

- F is horizontal
- F has a Lebesgue negligible support
- $\langle F, u\rangle = \langle \Sigma, e(u)\rangle, \quad \Sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ with $\Sigma_{33} = 0$

Examples: $\Omega = D \times I \;,\; I = (0,1)$

- $F = (\delta_1 - \delta_0)(x_3)(-\partial_2\psi(x'), \partial_1\psi(x'), 0) \qquad (\psi \in H^1_0(D))$
- $F = \rho(x_3)\tau_{\partial D}(x')\mathcal{H}^1 \llcorner \partial D \qquad (\rho \in L^2_m(I))$

Properties:

- $\langle F, u\rangle = 0 \qquad \forall\, u \in BN(Q) = \Big\{ e_{ij}(u) = 0 \;\forall (i,j) \neq (3,3) \Big\}$
- $\langle F, v\rangle = \langle m_F, c\rangle \quad \forall\, v \in TW(Q) = \Big\{ \big(c(x_3)(-x_2, x_1)\,,\; v_3 \big) \Big\}\;.$
  $m_F := [[x_1 F_2 - x_2 F_1]]$ average momentum

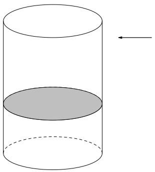
- **Small parameter problem:**

$$Q_\delta = \delta D \times I \quad , \quad F^\delta(x', x_3) = \delta^{-1} F(\delta^{-1} x', x_3)$$

$$\phi^\delta(k) := \inf_{A \subset Q^\delta} \left\{ \mathcal{C}^\delta(A) + k \frac{|A|}{\delta^2} \right\}$$

with $\quad \mathcal{C}^\delta(A) := \sup_{w \in H^1(Q^\delta; \mathbb{R}^3)} \{ \langle F^\delta, w \rangle - \int_A j(e(w)) \}$

$$\begin{cases} \delta \to 0 & \text{infinitesimal cross-section} \\ k = & \text{Lagrange multiplier} \\ (k \to +\infty & \text{for vanishing filling ratio)} \end{cases}$$



- **Reducing on fixed design** $Q = D \times I \ \ (A \subset Q^\delta \rightsquigarrow \omega \subset Q)$

$$C^\delta(\omega) := \sup \left\{ \delta^{-1} \langle F, u \rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u))\, dx \ : \ u \in H^1(Q; \mathbb{R}^3) \right\} .$$

$$e^\delta(u) := \begin{bmatrix} \delta^{-2} e_{\alpha\beta}(u) & \delta^{-1} e_{\alpha 3}(u) \\ \delta^{-1} e_{\alpha 3}(u) & e_{33}(u) \end{bmatrix} .$$

In the limit as $\delta \to 0^+$, displacements tend to belong to Bernouilli Navier $BN(Q)$,
but the load acts only on twist fields in $TW(Q)$.

- **Limit of ϕ_k^δ**

$$\Big\Downarrow \begin{array}{l} \delta \to 0 \\ + \text{ relaxation} \\ \quad \omega^\delta \subset Q \rightsquigarrow \theta \in L^\infty(Q;[0,1]) \end{array}$$

$$\phi(k) = \inf \left\{ \mathcal{C}^{lim}(\theta) + k \int_Q \theta \quad : \quad \theta \in L^\infty(Q;[0,1]) \right\}$$

where

$$\theta = \text{local filling ratio of elastic material}$$

$$C^{lim}(\theta) \;:=\; \sup_{c, v_3} \left\{ \langle m_F, c\rangle_{\mathbb{R}} - \kappa \int_Q \left| c'(x_3)(-x_2, x_1) + \nabla_{x'} v_3 \right|^2 \theta \, dx \right\}$$

Writing $\phi(k) \;=\; \inf_\theta \sup_{c,v} \ldots \;=\; \sup_{c,v} \inf_\theta \ldots$ we eliminate θ and are obtain (after dualizing with respect to pair (c, v)):

- **Dual problem on Q:**

$$\frac{\phi(k)}{2k} = \inf_{L^2(Q;\mathbb{R}^2)} \left\{ \int_Q \varphi(q) \ : \ \mathrm{div}_{x'} q = 0 \, , \ \int_D (x_1 q_2 - x_2 q_1) = -2\, M_F(x_3) \right\}$$

- **Localization on each section**
  - The dual form can be solved for $q(\cdot, x_3)$ section by section
  - The function $q(\cdot, x_3)$ is **divergence free on all** $\mathbb{R}^2$ and if $\mathbb{R}^2 \setminus D$ is connected

$$\exists u \in H_0^1(D) \ : \ q(\cdot, x_3) = (-\partial_2 u, \partial_1 u)$$

q optimal $\iff$ $q(\cdot, x_3) = (-\partial_2 u, \partial_1 u)$ where u optimal for $m(s)$ (with $s = M_F(x_3)$)

Link with *special* solutions?

In fact, it holds (up to negligible subset)

$$\{0 < |\nabla u| < 1\} \;\subset\; \{0 < \theta < 1\} \;\subset\; \{0 < |\nabla u| \leq 1\}$$

Special solutions for $m(s)$ $\iff$ Classical solution for $\phi(k)$

(no homogenization)

2. Existence, optimality conditions and uniqueness

$$m(s) := \inf \left\{ \int_D \varphi(\nabla u) \ : \ u \in H^1_0(D)\,,\ \int_D u = s \right\} \ ,$$

Proposition

The map $s \mapsto m(s)$ *is convex even and* $\displaystyle \lim_{|s|\to\infty} \frac{m(s)}{s^2} = \tau_D > 0$ *where (Saint-Venant torsional rigidity)*

$$\tau_D := \frac{1}{2} \inf \left\{ \int_D |\nabla u|^2 \ : \ u \in H^1_0(D)\,,\ \int_D u = 1 \right\} \ ,$$

For every $s \in \mathbb{R}$*, the minimum* $m(s)$ *is achieved. Moreover*

If $m(s)$ *admits a special solution, then there is no other solution.*

Dual problem

The Fenchel conjugate of m reads

$$m^*(\lambda) = \min_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D \varphi^*(\sigma) \ : \ -\mathrm{div}\sigma = \lambda \right\} ,$$

where $$\varphi^*(\xi) = \frac{1}{2} \left(|\xi|^2 - 1\right)_+^2$$

Proposition (optimality conditions)

Let $s, \lambda \in \mathbb{R}$, $u \in H_0^1(D)$, and $\sigma \in L^2(D;\mathbb{R}^2)$. There holds the following equivalence

$$(i) \begin{cases} u \text{ solution to } m(s) \\ \sigma \text{ solution to } \ m^*(\lambda) \\ \lambda \in \partial m(s)\,. \end{cases} \iff (ii) \begin{cases} \int_D u = s \\ -\mathrm{div}\sigma = \lambda \\ \sigma \in \partial\varphi(\nabla u) \text{ a.e. in } D\,. \end{cases}$$

Remark: at every $s \neq 0$, $m(s)$ is differentiable and $m'(s) > 0$.

Argument for uniqueness

Take $\lambda \in \partial m(s)$ and a particular solution $\overline{\sigma}$ for $m^*(\lambda)$. Let

$$Q_s \ := \ \{|\overline{\sigma}| > 1\}$$

Then any solution u for $m(s)$ satisfies $\nabla u = \overline{\sigma}$ on $D \setminus Q_s$.
($\partial\varphi$ satisfies $\partial\varphi(\xi) = \xi$ if $|\xi| > 1$, and $\partial\varphi(0) = \overline{B(0,1)}$.)

$\square$

Existence of a plateau

Proposition

For every $s > 0$, any solution u to $m(s)$ is Lipschitz continuous and the maximal set $\{u = \max u\}$ has positive measure

Proof: Let $\lambda \in \partial m(s)$. Then, for every $v \in H^1_0(D)$:

$$\int_D \varphi(\nabla u) - \lambda \int_D u \ \le \ \int_D \varphi(\nabla v) - \lambda \int_D v \ .$$

Take $t > 0$ and $v = \min\{u, t\}$. As $\varphi(z) \geq |z|$, we get

$$\int_{u>t} |\nabla u| \ \le \ \lambda \int_{u>t} u \ .$$

By coarea formula and isoperimetric inequality, the non increasing $\alpha(s) = |\{u > s\}|$ satisfies

$$\int_t^{\infty} \sqrt{\alpha(s)}\, ds \ \le C \ \int_t^{\infty} \alpha(s)\, ds \quad , \quad C = \frac{\lambda}{2\sqrt{\pi}} \, .$$

Thus $\exists t^* \ : \ \alpha(t) = 0$ for $t \geq t^*$ and $\alpha(t) \geq \frac{1}{C^2}$ for $t < t^*$ $\square$

Free boundary problem

In view of previous optimality conditions, looking for a special solution amounts to find

- a function $u \in H_0^1(D)$ with

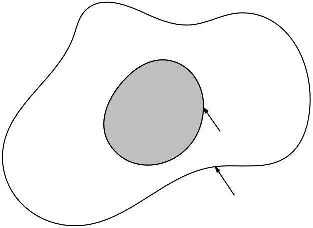
$$\begin{cases} u = const. & \text{in a subset } \Omega \subset D \\ |\nabla u| > 1 & \text{in } D \setminus \Omega \end{cases}$$

- a vector field $\sigma \in L^2(D; \mathbb{R}^2)$ with

$$\begin{cases} -\mathrm{div}\sigma = \lambda & \text{in } D \\ \sigma = \nabla u & \text{in } D \setminus \Omega \quad (\Rightarrow \; -\triangle u = \lambda \text{ in } D \setminus \Omega) \\ \|\sigma\|_\infty \leq 1 & \text{in } \Omega \end{cases}$$

($\partial\varphi$ satisfies $\partial\varphi(\xi) = \xi$ if $|\xi| > 1$, and $\partial\varphi(0) = \overline{B(0,1)}$.)

This leads to a free boundary value problem: find a subset $\Omega = \Omega(u) \subset D$ such that



$$\begin{cases} -\triangle u = \lambda\,,\ |\nabla u| > 1 & \text{in } D \setminus \Omega(u) \\ |\nabla u| = 1 & \text{on } \partial\Omega(u) \\ u \text{ constant on each connected component of} & \Omega(u) \end{cases}$$

BUT needs more in order to construct σ and specify constant λ

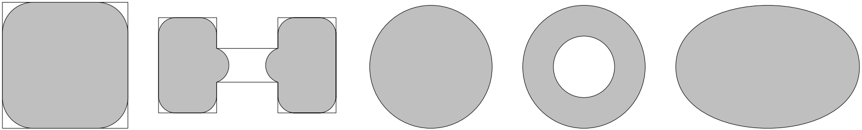
$\Rightarrow$ geometrical condition on set $\Omega(u)$

3- Free boundary problem and Cheeger sets

Let E be a bounded domain of $\mathbb{R}^2$. The *Cheeger constant* of E is defined as

$$h_E := \inf_{\substack{A \subset \overline{E} \\ Per(A) < +\infty}} \frac{|\partial A|}{|A|} = \inf_{\substack{v \in BV_0(E) \\ \int_E v = 1}} \int_E |\nabla v|$$

A minimizer for h_E is called a *Cheeger set* of E. It exists (sub-levels of any v_{opt}), but in general is not unique. However
If E is convex, then: $\exists!$ Cheeger set C_E and $v_{opt} = 1_{C_E}$.



Cheeger constant

Proposition

The subdifferential of m at the origin is $\partial m(0) = [-h_D, h_D]$. , *i.e.*

$$\lim_{s \to 0^+} \frac{m(s)}{s} \; = \; h_D$$

Remark: the behaviour of $m(s)$ near $s = 0$ is related with the limit $k \longrightarrow +\infty$ in the original torsion problem.

Proof:

$$\begin{aligned} m'_+(0) &= \lim_{s \to 0^+} \frac{m(s)}{s} \\ &= \lim_{s \to 0^+} \frac{1}{s} \inf_{\substack{v \in H_0^1(D) \\ \int_D v = 1}} \int_D \varphi(s \nabla v) = \inf_{\substack{v \in H_0^1(D) \\ \int_D v = 1}} \int_D |\nabla v| \end{aligned}$$

where in last line we switch symbols inf and $\int$ $\square$

Calibrable sets

Let $E \subset \mathbb{R}^2$ be a set with finite perimeter. We say that E is *calibrable* if there exists $\sigma \in L^2(E;\mathbb{R}^2)$ (*calibration*) such that

$$\|\sigma\|_\infty \leq 1 \quad , \quad -\mathrm{div}\sigma = h_E \quad , \quad [\sigma \cdot \nu_E] = -1 \quad \mathcal{H}^1 - \text{a.e. on } \partial E$$

Proposition

Let $E \subset \mathbb{R}^2$ be a bounded domain with finite perimeter. Then

$$E \textit{ calibrable} \iff E \textit{ is Cheeger set of itself}$$

The proof follows from divergence Theorem and the fact that:

$$h_E = \max\{\lambda \in \mathbb{R} \ : \ \exists \sigma \in L^2(E;\mathbb{R}^2)\,,\ \|\sigma\|_\infty \leq 1\,,\ -\mathrm{div}\sigma = \lambda\}\ .$$

Remark: If E is convex, then E calibrable $\iff \|H_{\partial E}\|_\infty \leq \frac{|\partial E|}{|E|}$

Revisited free boundary problem

Looking for special solutions amounts to find a "plateau" $\Omega \subset D$ (smooth enough) so that

- Ω is **calibrable**
- There exits a solution $u \in H_0^1(D)$ to the **overdetermined** problem

$$\begin{cases} -\triangle u = h_\Omega \quad , \quad |\nabla u| > 1 & \text{in } D \setminus \Omega \\ |\nabla u| = 1 & \text{on } \partial\Omega \\ u \text{ constant on each connected part of} & \partial\Omega \end{cases}$$

Vanishing volume fraction

$$\lim_{k\to+\infty}\frac{\phi(k)}{\sqrt{2k}} = \inf\left\{C^{\lim}(\mu)+\frac{1}{2}\int d\mu \ :\ \mu\in\mathcal{M}^+(Q)\right\}$$
$$= \min_{\sigma\in\mathcal{M}(Q;\mathbb{R}^2)}\left\{\int|\sigma|\ :\ \mathrm{div}_{x'}\sigma=0\,,\ \int_D(x_1d\sigma_2-x_2d\sigma_1)=\gamma(x_3)\right\}$$

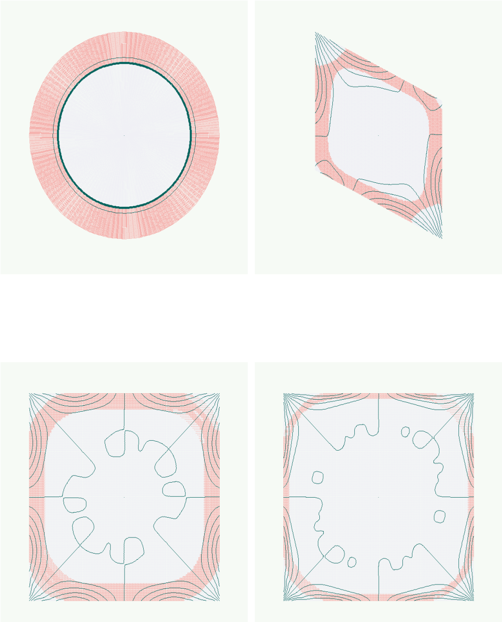
Then ν optimal $\iff$ $\nu=(-\partial_2 u,\partial_1 u)$, with u optimal for

$$\min\left\{\int|Du|\ :\ u\in BV(\mathbb{R}^2)\ ,\ u=0 \text{ in } \mathbb{R}^2\setminus\overline{D}\ ,\ \int_D u=1\right\} = h_D$$

Thus if D is convex, the optimal stress concentrates on ∂C_D:

Material concentrates on the boundary of the Cheeger set of D

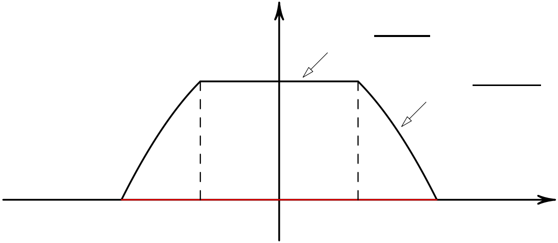
Some numerical computations



Proposition (radial case)

Let D be the ball $B(0,R)$. For every $s \in \mathbb{R}$ there exists a special solution u for $m(s)$.

Proof: If $s > 0$, let u be defined as follows:



where $r \in (0,R)$ is the unique solution of $s = \frac{\pi}{4r}(R^4 - r^4)$.

Here $\Omega(u) = \{|x| < r\}$. The dual solution $\sigma = -\frac{x}{r}$ satisifies $|\sigma| \leq 1$ on $\Omega(u)$. □

Is the disk the unique domain ?

Recall: existence of a special solution is equivalent to existence of optimal shape.

- The answer is *yes* (among C^1 domains) for a similar variational problem, corresponding to maximizing the torsional rigidity of rods with a given cross-section D by mixing two linearly elastic materials in fixed proportions. [Murat, Tartar]

- But.... the answer is *no* (even among analytic domains) for our problem!

Reason why: Our integrand $\varphi(z)$ is *not differentiable at* $z = 0$ (it would be C^1 if the void is replaced by a weak material)

Special solutions for D not a ball

Theorem

There exists a domain D (different from a ball) and a parameter $s \in \mathbb{R} \setminus \{0\}$ such that $m(s)$ admits a special solution u.
Moreover D and the plateau $\Omega(u)$ is convex with analytic boundary.

Sketch of proof: We need to construct a bounded analytic domain D such that there exist

- a function $u \in H_0^1(D)$ with

$$\begin{cases} \nabla u = 0 \quad \text{in a convex set} \quad \Omega \subset D \\ |\nabla u| > 1 \quad \text{in } D \setminus \Omega \\ \displaystyle\int_D u = s \,, \ \text{for some } s \in \mathbb{R} \setminus \{0\} \,, \end{cases} \tag{1}$$

- a field $\sigma \in L^2(D; \mathbb{R}^2)$ with

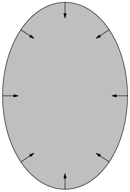
$$\begin{cases} |\sigma| \leq 1 \quad \text{in } \Omega \,, \\ \sigma = \nabla u \quad \text{in } D \setminus \Omega \,, \\ -\text{div}\sigma = \lambda \quad \text{in } D \,, \ \text{for some } \lambda \in \mathbb{R} \,. \end{cases} \tag{2}$$

Steps of the proof

- **Step1**

We consider Ω bounded, convex, with analytic boundary, and such that $\|H_{\partial\Omega}\| < |\partial\Omega|/|\Omega|$.

(known fact) $\Rightarrow$ Ω is Cheeger set of itself, i.e. it is calibrable.



Let $\sigma_1 \in L^2(\Omega;\mathbb{R}^2)$ be a calibration for Ω, then

$$\begin{cases} \|\sigma_1\|_\infty \le 1 & \text{in } \Omega \\ -\mathrm{div}\sigma_1 = h_\Omega & \text{in } \Omega \\ [\sigma_1 \cdot \nu_\Omega] = -1 & \mathcal{H}^1 - \text{a.e. on } \partial\Omega \end{cases}$$

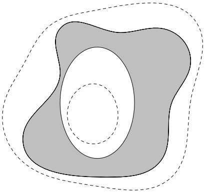
• **Step2**

$\partial\Omega$ analytic $\xRightarrow{\textit{Cauchy}-\textit{Kowalevskaya}}$ $\exists$ v analytic solution of

$$\begin{cases} -\triangle v = h_\Omega & \text{in } \mathcal{N} \\ v = 1\,,\ -v_\nu = 1 & \text{on } \partial\Omega \end{cases}$$

in a neighbourhood $\mathcal{N}$ of $\partial\Omega$.

Moreover there exists a curve $\gamma \subset \mathcal{N}$ analytic that is the boundary of some domain $D \supset \Omega$, such that



$$\begin{cases} -\triangle v = h_\Omega & \text{in } D \setminus \Omega \\ |\nabla v| > 1 & \text{in } D \setminus \Omega \\ v = 1\,,\ v_\nu = -1 & \text{on } \partial\Omega \\ v = 1 - \varepsilon & \text{on } \partial D \end{cases}$$

for some $0 < \varepsilon < 1$.

- **Step3**

The functions

$$u(x) := \begin{cases} \varepsilon & \text{in } \Omega \\ v - (1-\varepsilon) & \text{in } D \setminus \Omega \end{cases}, \quad \sigma(x) := \begin{cases} \sigma_1 & \text{in } \Omega \\ \nabla v & \text{in } D \setminus \Omega \end{cases}$$

satisfy the conditions (1) and (2), moreover Ω is convex and D, Ω have analytic boundary. $\square$

5. Further properties of special solutions

- **Property 1:** If the value function $m(s)$ is affine on some $[\alpha, \beta]$, then no special solution exists for $\alpha < s < \beta$.

- **Property 2:** If $m(s)$ is *stricly convex* on $[\alpha, \beta]$, then there exists a unique solution for $\alpha < s < \beta$.

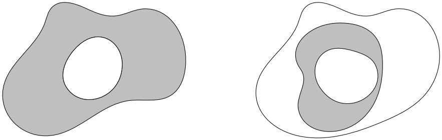
- **Property 3:** Let s be positive and sufficiently small. Then any solution u to problem $m(s)$ satisfies

$$\mathrm{spt}(u) \cap \partial D \neq \emptyset$$

- **Property 4:** Let D be convex and assume that u is a special solution with smooth connected $\Omega(u)$ such that $\Omega(u) \subset\subset D$. Then $\Omega(u)$ is convex.
(proof uses P-functions and Hopf's Lemma)

- **Property 5:** Assume that D is *not* Cheeger set of itself, and let $s_\varepsilon \searrow 0$. Then problem $m(s_\varepsilon)$ cannot admit for every ε a special solution u_ε with $\Omega(u_\varepsilon) \subset\subset D$.

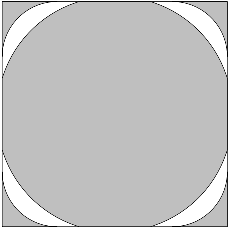
- **Property 6:** Assume that u is a special solution with smooth $\Omega(u)$. Then each connected component of $D \setminus \Omega(u)$ meets the boundary ∂D. ($\Omega(u)$ in dark)



Open problems

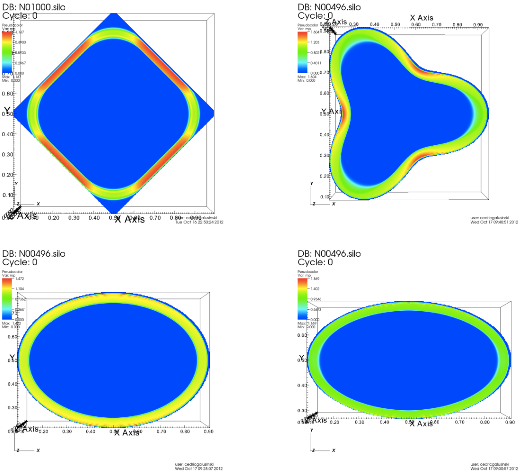
- Regularity of the free boundary
  [Caffarelli, Petrosyan, Salazar, Shahgholian]

- Non-existence of special solutions ? *e.g.* in case of the square
  Kawohl, Stara, Wittum and more recently C. Galusinski, E. Oudet



A possible plateau for a special solution on the square.

Some numerics by C. Galusinski (IMATH-Toulon)



green: zones with homogenization , blue: zones $u = cte$

Conjectures

- There exits a special solution if D is a convex C^2 subset such that $\|H_{\partial D}\|_\infty \leq \frac{|\partial D|}{|D|} = h_D$.

- For a larger class of domains (including convex domains), there exists s^* such that:
  a special solution exists for $m(s)$ for all $s \geq s^*$

References

- ▷ Alibert, Bouchitté, Fragalà, Lucardesi, *Interfaces Free Bound.* (2013)
- ▷ Bouchitté, Fragalà, Lucardesi, Seppecher, *SIAM J. Math. Anal.* (2012).
- ▷ Bouchitté, Fragalà, Seppecher, *Arch. Rat. Mech. Anal.* (2011).