Benasque

2013

Optimal control problems in coefficients for an elliptic PDE

- Joint work with P. Kogut (submitted 2013).

The equation

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- On a regular bounded domain Ω of $\mathbb{R}^N$ let $f \in H^{-1}(\Omega)$ and A a matrix with L^2 coefficients.

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- We restrict ourselves to the case $A = A_s + A_{skew}$ where $A_{skew} \in L^2$, $A_s \in L^\infty$ and there exists $0 < \alpha \leq \beta$ such that $\alpha \leq A_s \leq \beta$ in the sense of quadratic forms.

The OCP

The Optimal control problem

- We consider the following optimal control problem (OCP). Given $y_d \in L^2(\Omega)$.

$$\min_{(y,A)\in\Xi} \|y - y_d\|_{L^2}^2 + \int_\Omega \nabla y.A_s \nabla y dx, \tag{1.2}$$

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- Actually, by a solution y of **(1.1)** with such a A, we mean that $y \in H^1_0(\Omega)$ satisifies

$$\forall \varphi \in C^\infty_0(\Omega) \int_\Omega \nabla\varphi.A\nabla y dx = \langle f, \varphi\rangle \tag{1.3}$$

The domain $D(A)$

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- In order to deal with our type of A, we define

$$D(A) = \Big\{ y \in H_0^1(\Omega),\ \forall \varphi \in C_0^\infty(\Omega)$$

$$\left| \int_\Omega \nabla\varphi . A_{skew} \nabla y dx \right| \leq c(A_{skew}, y) \left(\int_\Omega \|\nabla\varphi\|^2 dx \right)^{1/2} \Big\}.$$

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$$[y, \varphi] = \int_\Omega \nabla\varphi . A_{skew} \nabla y dx,\ \forall \varphi \in C_0^\infty(\Omega).$$

- It is clear that for $y \in D(A)$, by taking $\varphi_n \in C_0^\infty(\Omega)$ with $\varphi_n \to y$, one can define

$$[y, y] = \lim\, [y, \varphi_n]$$

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- Remark that if y is a solution of **(1.1)** then $y \in D(A)$.
  Indeed one has

$$[y, \varphi] = -\int_\Omega \nabla\varphi . A_s \nabla y + \langle f, \varphi \rangle.$$

  and thus

$$|\,[y, \varphi]\,| \leq (\beta \|y\|_{H_0^1} + \|f\|_{H^{-1}}) \|\varphi\|_{H_0^1}$$

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- Energy estimate: $\int_\Omega \nabla y.A_s\nabla y dx+[y,y]=<f,y>$

- Remark also that if $u\in D(A)$ then $\mathrm{div}(A\nabla u)\in H^{-1}(\Omega)$.

The OCP continued

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- To precise our optimal control problem we define the admissible set Ξ.

The OCP continued

The admissible set for the matrices Ad_m consists of A such that

- $A_s \in L^\infty$ and $\alpha \le A_s \le \beta$.
- $a_{s,ij} \in BV(\Omega)$ and $\exists c \ge 0$ such that $\forall i, j \; TV(a_{s,ij}) \le c$ where
$$TV(a) = \int_\Omega |Da| = \sup_{\varphi \in C_0^1(\Omega, \mathbb{R}^N), \, ||\varphi||_\infty \le 1} \int a \mathrm{div}(\varphi) dx$$
- There exists A^*, such that $A_{skew} \le A^*$ (meaning that $\forall i, j$, $|a_{skew,ij}| \le |a^*_{ij}|$).
- There exists Q a compact convex subset of $L^2(\Omega, Skew)$ such that $A_{skew} \in Q$ and containing 0.

The OCP continued

- The admissible set Ξ for the pair (y, A) consists of those $A \in Ad_m$ and $y \in H_0^1(\Omega)$ such that **(1.1)** is satisfied.
- With that recall the OCP **(1.2)**
$$\min_{(y,A)\in\Xi} \|y - y_d\|_{L^2} + \int_\Omega \nabla y . A_s \nabla y.$$
- Note also that, due to the possibly unbounded skew-symetric part we may face non-uniqueness of solutions of **(1.1)**, this is the main reason why the OCP is settled in (y, A).

The OCP solved ?

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- It is then quite clear that **(1.2)** has a solution: the compactness for the skew-symetric part is an assumption, the compactness for the symetric part comes from the assumption on the total variations of our admissible matrices.
- The main feature of our paper concerns the type of optimal solutions: namely we consider a concept of variational and non-variational solutions.
- Zhikov,....
- Many things are still not known.

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- Definition: we say that $(\hat{y}, \hat{A})$ is a variational solution to the OCP **(1.2)** if it is an optimal pair which can be approximated by a sequence of optimal pairs in a suitable sense of variational limit according to a sequence of approximation $A_k^* \to A^*$ (for example the truncation as above) with $A_k^* \in L^\infty$ and $A_k^* \to A^*$ in L^2. Essentially for the variational limit —but not solely and needs more— we assume the convergence of sequences of minimizers to some minimizers.

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- Theorem: If **(1.2)** has a variational solution $(\hat{y}, \hat{A})$, then $[\hat{y}, \hat{y}] = 0$.

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- Use lower semi-continuity to prove that $[\hat{y}, \hat{y}] \geq 0$.
- Use the variational limit concept to improve this and obtain $[\hat{y}, \hat{y}] = 0$.

Some possible converse situations

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- We can prove that for our OCP if for all $A \in Ad_m$, for all $y \in D(A)$ one has $[y, y] = 0$ then there exist variational solutions (i.e. there can be approximated by the suitable procedure of, say, a trunctation).

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- We are able to construct an admissible A_0 and a $y_0 \in D(A_0)$ such that $[y_0, y_0] < 0$. And with that we are able to construct some (other) OCP for which a minimum is attained at some non-variational solution (A_0, y_0).

$$A_{0,skew}(x) = \begin{pmatrix} 0 & a(x) & 0 \\ -a(x) & 0 & b(x) \\ 0 & -b(x) & 0 \end{pmatrix}, \; a(x) = -\frac{x_1}{2||x||^2}, \; b(x) = -\frac{x_3}{2||x||^2}$$

$$y_0 = \frac{\sqrt{2\alpha}}{\pi^2}(1 - ||x||^5)\sqrt{1 - \mathrm{atan}_2(\frac{x_1}{||x||}, \frac{x_2}{||x||})}$$

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- Is there a way to approximate non variational solutions ?

Non variational solutions

- Under some added assumptions on A^*:

$$\exists (p_1, ..., p_q) \in \Omega \quad \text{s.t.} \quad A^* \in C^\infty(\Omega \setminus \{p_1, ..., p_q\})$$

we can attempt to consider a way of approximating our non variational solutions.

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- Take $q = 1$. We perforate in Ω around p according to A^* of order ε (precised later). Denote Ω_ε the perforated domain. For technical reasons we need, at the present time, to have restrictions on this perforation for which our A_0 is not convenient.

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- We consider an OCP with a fictitious control:

$$(1.4)_\varepsilon \qquad \min_{(y,v,A)\in\Xi_\varepsilon} \|y - y_d\|_{L^2(\Omega_\varepsilon)} + \int_{\Omega_\varepsilon} \nabla y . A_s \nabla y dx + \frac{1}{\varepsilon^\sigma} \|v\|_{H^{-1/2}(\Gamma_\varepsilon)}.$$

  σ small enough .

Non variational solutions

- Our admissible set Ξ_ε takes into account the perforation and (y, v, A) are related by

$$\begin{cases} -\mathrm{div}(A\nabla y) = f \text{ in } \Omega_\varepsilon \\ y \in H_0^1(\Omega_\varepsilon, \partial\Omega) \\ \dfrac{\partial y}{\partial \nu_A} = v \text{ on } \Gamma_\varepsilon. \end{cases}$$

With this, we have

Non variational solutions

- Theorem:
  Assume (technical) that A^* satisfies, at first $\partial\Omega_\varepsilon$ lipschitz,

$$\begin{cases} |\partial\{x \in \Omega \sup |a^*_{ij}|(x)| \geq \varepsilon\}| = o(\varepsilon) \\ \forall A \preceq A^*,\ \forall y \in D(A),\ \exists c(h)\text{s.t.}| \int_{\Omega\backslash\Omega_\varepsilon} \nabla\varphi.A_{skew}\nabla y dx| \leq c(h)\frac{\sqrt{|\Omega - \Omega_\varepsilon|}}{\varepsilon}||\nabla\varphi||_{L^2(\Omega\backslash\Omega_\varepsilon)} \end{cases}$$

  Assume that there are some $(y, A) \in \Xi$ such that $[y, y] \neq 0$ if $A_{skew} = A^*$, then **(1.2)** is the variational limit on **(1.4)**$_\varepsilon$.

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  Assume that there are some $(y, A) \in \Xi$ such that $[y, y] \neq 0$ if $A_{skew} = A^*$, then **(1.2)** is the variational limit on **(1.4)**$_\varepsilon$.

- And if we require stronger assumptions, we can also pass to the limit in optimality conditions.

Some references

Some references

- Buttazzo & Kogut, Casas & Fernandez, Fanjiang & Papanicolau, Fursikov, Kogut & Leugering, Ioffe & Tichomirov, Jin & Mazya & VanSchaftinger, Serrin, Zhikov,...

Some further extensions or questions

- Necessary and sufficient conditions in order to understand which solutions you're going to obtain ?
- What happens if A_s has also some degeneracy, work in progress (but just started) with Peter.
- What are the fine structures of $D(A)$ and $[.,.]$ on $D(A)$?
- Does it make sense to study the heat equation with such A and initial data close to non-variational solutions ?

Thank you for your attention

Variational convergence.

- We recall some consequences of the variational convergence (see the book of Kogut-Leugering).
- Theorem: Assume that we have functionals I_0 I_ε defined on variable Banach spaces (we need to speak of convergence in variable spaces).
- Some Ξ_0 and Ξ_ε admissible sets.
- We assume that $\inf\limits_{\Xi_\varepsilon} I_\varepsilon$ and $\inf\limits_{\Xi_0} I_0$ are achieved.
- Then if we have variational convergence of this problems, then compact sequences u_ε of minimizers of I_ε with respect to ε converge up to subsequence to some optimal solution of $\inf_{\Xi_0} I_0$ and $I_\varepsilon(u_\varepsilon) \to I_0(u_0)$.