Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components

joint work with Jean-Michel Coron

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Partial differential equations, optimal design and numerics, Benasque

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Outline

Notations

Ω smooth bounded domain of $\mathbb{R}^3$ and ω open subset of Ω.

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$v \in L^2(\Omega)$ (control).
i-th component of f: f^i.
j-th derivative of g: g_j ($j = 1, 2, 3, t$).

The controlled Navier-Stokes system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p & = (0,0,v1_\omega) \text{ in } Q, \\ \nabla\cdot y & = 0 \text{ in } Q, \\ y(0,\cdot) & = y^0 \text{ in } \Omega, \\ y & \equiv 0 \text{ on } \Sigma.s \end{cases} \qquad \text{(NS-1Cont)}$$

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We act only on the third equation (indirect control).

Non-exhaustive state of the art

- Local exact controllability of Navier-Stokes system and linearized Navier-Stokes systems with a control on each equation: Fernandez-Cara-Guerrero-Imanuvilov-Puel'04,

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- Null-controllability of Stokes system and local null controllability of Navier-Stokes system with a control having a vanishing component: Carreno-Guerrero'12.

Main theorem

Theorem

For every $T > 0$ *and for every* $r > 0$, *there exists* $\eta > 0$ *such that, for every* $y^0 \in V$ *verifying* $||y^0||_{H^1(\Omega)^3} \leqslant \eta$, *there exist a control* $v \in L^2(Q)$ *and a solution* (y, p) *of* (NS-1Cont) *such that*

$$y(T, \cdot) = 0,$$

$$||v||_{L^2(Q)^3} \leqslant r,$$

$$||y||_{L^2((0,T),H^2(\Omega)^3)\cap L^\infty((0,T),H^1(\Omega)^3)} \leqslant r.$$

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Exists geometries for which System (Stokes) is not even approximatively controllable (Lions-Zuazua'96 and Diaz-Fursikov'97).
$\Rightarrow$ Return method

Return method (J.M. Coron)

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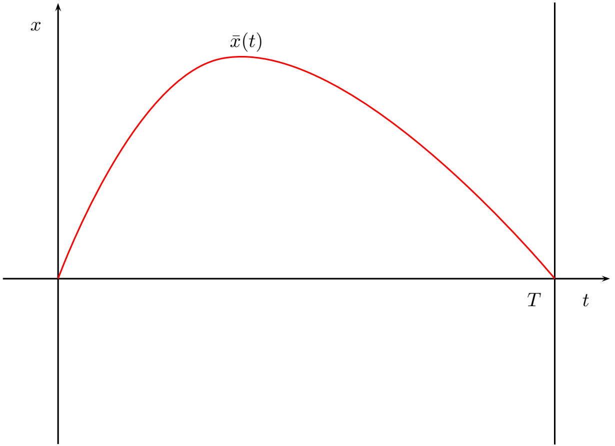

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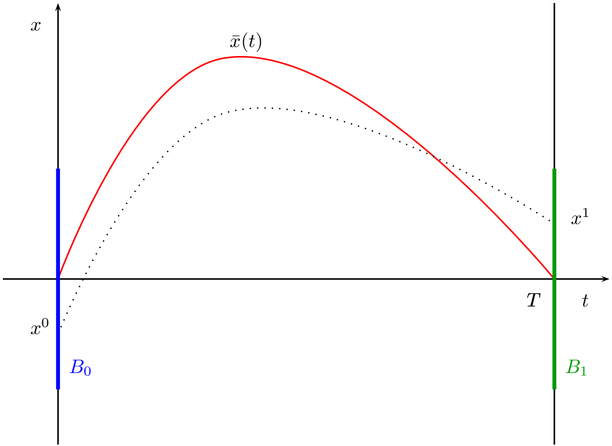

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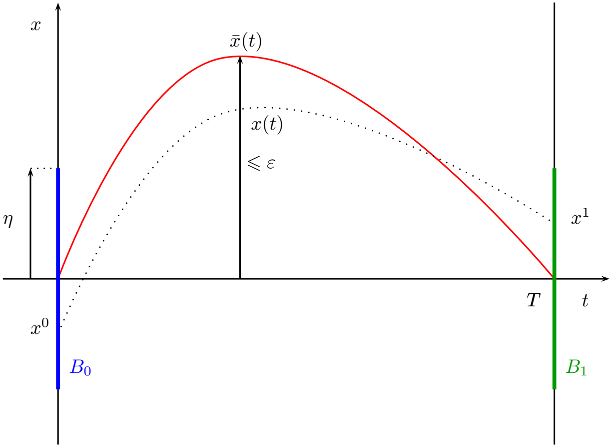

Construction of the particular trajectory (1)

Assume $0 \in \omega$.
$r := \sqrt{x_1^2 + x_2^2}$. $\mathcal{C}_1$ cylinder $r \leqslant r_1$ and $|x_3| \leqslant r_1$, and $\mathcal{C}_2$ cylinder $r \leqslant r_1/2$ and $|x_3| \leqslant r_1/2$ (r_1 small enough such that $\mathcal{C}_1, \mathcal{C}_2 \subset\subset \omega$).

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$$\bar{y}(t,x) := \begin{pmatrix} \varepsilon a(t) b(r^2) c'(x_3) x_1 \\ \varepsilon a(t) b(r^2) c'(x_3) x_2 \\ -2\varepsilon a(t)(b(r^2) + r^2 b'(r^2)) c(x_3) \end{pmatrix}.$$

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$\bar{y}$ written in this general form is such that $\nabla . \bar{y} = 0$ and there exists a pressure $\bar{p}$ and a control $\bar{v}$ such that

$$\bar{y}_t - \Delta \bar{y} + (\bar{y} \cdot \nabla)\bar{y} + \nabla \bar{p} = (0, 0, \bar{v}).$$

Construction of the particular trajectory (2)

$\nu > 0$ numerical constant.

$$\mathrm{Supp}(a) \subset [T/4, T] \text{ and } a(t) = e^{\frac{-\nu}{(T-t)^5}} \text{ in } [T/2, T],$$
$$\mathrm{Supp}(b) \subset (-\infty, r_1^2) \text{ and } b(w) = w,\ \forall w \in (-\infty, r_1^2/4],$$
$$\mathrm{Supp}(c) \subset (-r_1, r_1) \text{ and } c(x_3) = x_3^2 \text{ in } [-r_1/2, r_1/2].$$

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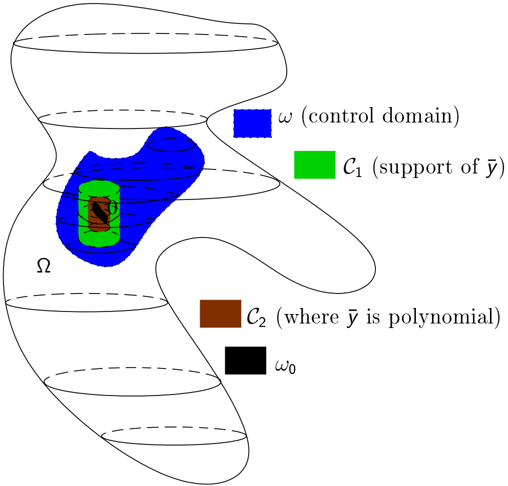
$\bar{y}$ simple form on $\mathcal{C}_2$ (polynomial in space).

Figure representing the different open subsets introduced



The linear control system

$$\begin{cases} y^1_t - \Delta y^1 + (\bar{y}\cdot\nabla)y^1 + (y\cdot\nabla)\bar{y}^1 + p_1 & = 0 \text{ in } Q,\\ y^2_t - \Delta y^2 + (\bar{y}\cdot\nabla)y^2 + (y\cdot\nabla)\bar{y}^2 + p_2 & = 0 \text{ in } Q,\\ y^3_t - \Delta y^3 + (\bar{y}\cdot\nabla)y^3 + (y\cdot\nabla)\bar{y}^3 + p_3 & = v1_\omega \text{ in } Q,\\ \nabla\cdot y & = 0 \text{ in } Q,\\ y & = 0 \text{ on } \Sigma,\\ y(0,\cdot) & = y^0 \text{ in } \Omega. \end{cases} \tag{NS-Lin-1Cont}$$

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Goal: prove a result of null-controllability for this system (with a source term) in suitable weighted Sobolev spaces and application of a local inverse mapping theorem to go back to the nonlinear system.

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Differential operators

Ideas of this section: Gromov (partial differential relations, 1986).
To simplify, C^∞ setting. Q_0 open subset of $\mathbb{R}^n$.

Definition

$\mathcal{M} : C^\infty(Q_0)^k \to C^\infty(Q_0)^s$ is called linear partial differential operator of order m if, for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| := \alpha_1 + \alpha_1 + \ldots + \alpha_n \leqslant m$, there exists $A_\alpha \in C^\infty(Q_0; \mathcal{L}(\mathbb{R}^k; \mathbb{R}^s))$ such that

$$(\mathcal{M}\varphi)(\xi) = \sum_{|\alpha| \leqslant m} A_\alpha(\xi)\partial^\alpha\varphi(\xi), \forall \xi \in Q_0, \forall \varphi \in C^\infty(Q_0)^k.$$

Algebraic solvability of differential systems

$\mathcal{L}: C^\infty(Q_0)^m \to C^\infty(Q_0)^s, \mathcal{B}: C^\infty(Q_0)^k \to C^\infty(Q_0)^s$ linear partial differential operators. We consider equation

$$\mathcal{L}y = \mathcal{B}f, \tag{Gen-Dif-Syst}$$

where the unknown is y.

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Definition

Equation (Gen-Dif-Syst) *is algebraically solvable if there exists a linear partial differential operator* $\mathcal{M} : C^\infty(Q_0)^k \to C^\infty(Q_0)^5$ *such that, for every* $f \in C^\infty(Q_0)^k$, $\mathcal{M}f$ *is a solution of* (Gen-Dif-Syst), *i.e. such that*

$$\mathcal{L} \circ \mathcal{M} = \mathcal{B}. \qquad \text{(LcompM=B)}$$

Formal adjoint

For every linear partial differential operator $\mathcal{M} : C^\infty(Q_0)^k \to C^\infty(Q_0)^l$, $\mathcal{M} = \sum_{|\alpha| \leqslant m} A_\alpha \partial^\alpha$, associate (formal) adjoint

$$\mathcal{M}^* : C^\infty(Q_0)^l \to C^\infty(Q_0)^k$$

defined by

$$\mathcal{M}^* \psi := \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \partial^\alpha (A_\alpha^{\mathrm{tr}} \psi), \ \forall \psi \in C^\infty(Q_0)^l.$$

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$\mathcal{M}^{**} = \mathcal{M}$ and, if $\mathcal{M} : C^\infty(Q_0)^k \to C^\infty(Q_0)^l$ and $\mathcal{N} : C^\infty(Q_0)^l \to C^\infty(Q_0)^m$ are two linear partial differential operators, then $(\mathcal{N} \circ \mathcal{M})^* = \mathcal{M}^* \circ \mathcal{N}^*$.

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Hence, (LcompM=B) is equivalent to

$$\mathcal{M}^*\circ\mathcal{L}^* = \mathcal{B}^*.$$

A simple case (1)

$f \in C_0^\infty(\mathbb{R})$. find x_1, x_2 in $C_0^\infty(\mathbb{R})$ verifying

$$a_1 x_1 - a_2 {x_1}' + a_3 x_1'' + b_1 x_2 - b_2 {x_2}' + b_3 {x_2}'' = f.$$

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Under the form $\mathcal{L}(x_1, x_2) = \mathcal{B} f$ with $\mathcal{B} = Id_{C^\infty(\mathbb{R})}$ and

$$\mathcal{L} = \begin{pmatrix} a_1 - a_2\partial_t + a_3\partial_{tt} & b_1 - b_2\partial_t + b_3\partial_{tt} \end{pmatrix}.$$

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Find $\mathcal{M}$ such that $\mathcal{L} \circ \mathcal{M} = Id \Leftrightarrow$ find $\mathcal{N}$ such that $\mathcal{N} \circ \mathcal{L}^* = Id$.
This implies necessarily that $\mathcal{L}^* x = 0 \Rightarrow N \circ \mathcal{L}^* x = x = 0$.

A simple case (2)

We compute

$$\mathcal{L}^* = \begin{pmatrix} a_1 + a_2\partial_t + a_3\partial_{tt} \\ b_1 + b_2\partial_t + b_3\partial_{tt} \end{pmatrix}.$$

Let us solve $\mathcal{L}^* x = 0$. (System now analytically overdetermined).

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$$\left\{ \begin{array}{l} a_1 x + a_2 x' + a_3 x'' = 0 \\ b_1 x + b_2 x' + b_3 x'' = 0 \end{array} \right.$$

We differentiate.

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A simple case (3)

If we see this equations as a purely algebraic equation, we obtain a system with 4 equations and 4 “unknowns” (at the beginning : 2 equations and 3 “unknowns”), that we write under the form $C(x, x', x'', x''') = 0$ with

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$$C = \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

A simple case (4)

We see that under some conditions on the coefficients, necessarily $x \equiv 0$. Moreover, one can see C^{-1} (which acts on x, x', x'', x''') as a linear partial differential operator $\mathcal{N}$ acting only on x.

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Equality $C^{-1}C = Id_{\mathbb{R}^4}$ can be written in differential form $\mathcal{N} \circ \mathcal{L}^* x = x$ so that $\mathcal{M} = \mathcal{N}^*$ gives $\mathcal{L} \circ \mathcal{M} = Id$.

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- $\mathcal{L}y = \mathcal{B}f$ underdetermined $\Rightarrow$ generically algebraically solvable. Moreover supports of functions conserved.

Solving a differential system (1)

We apply what we did on our problem. To simplify everything is C^∞ and no source term.
We call $Q_0 := (T/2, T) \times \omega_0$ with ω_0 open subset of $\mathcal{C}_2$.
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We call $Q_0 := (T/2, T) \times \omega_0$ with ω_0 open subset of $\mathcal{C}_2$.
$\mathcal{B} = (\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3)$ linear partial differential operator.

$$\begin{cases} y^1_t - \Delta y^1 + (\bar{y}\cdot\nabla)y^1 + (y\cdot\nabla)\bar{y}^1 + p_1 & = \mathcal{B}^1 u \text{ in } Q, \\ y^2_t - \Delta y^2 + (\bar{y}\cdot\nabla)y^2 + (y\cdot\nabla)\bar{y}^2 + p_2 & = \mathcal{B}^2 u \text{ in } Q, \\ y^3_t - \Delta y^3 + (\bar{y}\cdot\nabla)y^3 + (y\cdot\nabla)\bar{y}^3 + p_3 + v1_\omega & = \mathcal{B}^3 u \text{ in } Q, \\ \nabla\cdot y & = 0 \text{ in } Q, \\ y & = 0 \text{ on } \Sigma, \\ y(0,\cdot) & = y^0 \text{ in } \Omega, \end{cases} \tag{NS-lin-Bcont}$$

Unknowns: y, p, v. Datum: $u \in C^\infty(Q)^k$, with support in Q_0.
Under the form $\mathcal{L}(y,p,v) = (\mathcal{B}u, 0)$. Underdetermined system.

Link between controllability and algebraic solvability (1)

Crucial Proposition:

Proposition

Assume:

- *System* (NS-lin-Bcont) *is algebraically solvable*
- *We can control the linearized Navier-Stokes system with a control (having* 3 *components) being in the image of* $\mathcal{B}$*.*

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- *We can control the linearized Navier-Stokes system with a control (having* 3 *components) being in the image of $\mathcal{B}$.*

Then we can control with one component (i.e. we can control (NS-Lin-1Cont)*.)*

Link between controllability and algebraic solvability (2)

Proof:

1. We control with 3 components in the image of $\mathcal{B}$, with $\widehat{u}$ supported in Q_0, i.e. we can find $(\widehat{y},\widehat{p},\widehat{u})$ verifying

$$\begin{cases} \widehat{y}_t-\Delta\widehat{y}+(\bar{y}\cdot\nabla)\widehat{y}+(\widehat{y}\cdot\nabla)\bar{y}+p & =\mathcal{B}\widehat{u} \text{ in } Q,\\ \nabla\cdot\widehat{y} & =0 \text{ in } Q,\\ \widehat{y} & =0 \text{ on } \Sigma,\\ \widehat{y}(0,\cdot) & =y^0 \text{ in } \Omega,\\ \widehat{y}(T,.) & =0 \text{ in } \Omega. \end{cases}$$

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2. Algebraic resolution: there exists $(\tilde{y}, \tilde{p}, \tilde{u})$ solution on Q_0 of $\mathcal{L}(\tilde{y}, \tilde{p}, \tilde{u}) = \mathcal{B}\widehat{u}$. $\tilde{y}, \tilde{p}, \tilde{u}$ vanishing at times $t = 0$ and $t = T$ (support still included in Q_0).

Link between controllability and algebraic solvability (3)

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3. We set $(y, p) = (\widehat{y} - \tilde{y}, \widehat{p} - \tilde{p})$. Then (y, p) verifies $\mathcal{L}(y, p, v) = 0$ with initial condition y^0 and final condition 0, which is what we wanted.

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What we have to do:

- Find $\mathcal{B}$ such that one can solve algebraically (NS-lin-Bcont).
- Find controls in the image of $\mathcal{B}$, regular enough such that it has a sense to apply operator $\mathcal{M}$.

Choice of $\mathcal{B}$ and reformulation of the problem

$\mathcal{B} = Id$ does not work. In fact $\mathcal{B}$ partial differential operator of order 1:

$$\mathcal{B}(f^1, f^2, f^3, f^4, f^5, f^6, f^7) := \begin{pmatrix} f^1_1 + f^2_2 + f^3_3 \\ f^4_1 + f^5_2 + f^6_3 \\ f^7 \end{pmatrix}.$$

We can see that we are led to consider system $\mathcal{L}^*(z, q) = 0$ on Q_0 with moreover $z^3 = 0$ on Q_0. We have to prove (by the previous method)

$$z^1_1 = z^2_2 = z^3_3 = z^2_1 = z^2_2 = z^2_3 = 0.$$

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To have more equations than unknowns, one needs to differentiate at least 19 times the equations: this brings to 30360 equations and 29900 unknowns, so it cannot be done by hand!

Using the computer (1)

Different steps:

- Differentiate the PDE (C^{++}) and applying at a particular point. We stock the result under the form of a sparse matrix A.

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- Differentiate the PDE (C^{++}) and applying at a particular point. We stock the result under the form of a sparse matrix A.
- See if matrix A is invertible or not. unfortunately not the case. $\Rightarrow$ find some suitable submatrix of A which is invertible, i.e. of maximal rank.
- Rank: a lot of time to compute on a computer. We use instead the notion of structural rank, which only depend on the coefficients of the matrix that are equal to 0 or not and is fast to compute. Moreover, there exists an algorithm (Dulmage-Mendelsohn decomposition) that rearrange the matrix in a nice way.

Using the computer (2)

- Find then a submatrix of A (called P) containing the unknowns we want (i.e. $z_1^1, z_2^2, z_3^3, z_1^2, z_2^2, z_3^2$) and being of maximal structural rank, and then verify that it is of full rank. P of size 7321×7321!

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- Use a genericity argument to see that P is invertible everywhere in Q_0 and deduce a differential operator $\mathcal{M}$ such that $\mathcal{L} \circ \mathcal{M} = \mathcal{B}$ by inverting P.

Controlling in the image of $\mathcal{B}$ (1)

We use a suitable Carleman inequality coming from Gueye'12.

Lemma

Let ω^* an open subset of Ω. For K_1, ν large enough, for ε small enough, for every $g \in L^2((0,T)\times\Omega)^3$ and for every solution z of the adjoint of the linearized Navier-Stokes system

$$\begin{cases} -z_t - \Delta z - (\overline{y}\cdot\nabla^t)z - (z\cdot\nabla)\overline{y} + \nabla\pi = g & \text{in } Q, \\ \nabla\cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } [0,T]\times\partial\Omega, \end{cases}$$

one has

$$||e^{\frac{-K_1}{2r(T-t)^5}}z||^2_{L^2((T/2,T),H^1(\Omega)^3)} + ||z(T/2,\cdot)||^2_{L^2(\Omega)^3}$$
$$\leqslant C\left(\int_{(T/2,T)\times\Omega} 1_{\omega^*} e^{-\frac{K_1}{(T-t)^5}}|\nabla\wedge z|^2 + \int_{(T/2,T)\times\Omega} e^{-\frac{K_1}{(T-t)^5}}|g|^2\right).$$

This Carleman inequality gives controls under the form $\nabla\wedge((\nabla\wedge u)1_{\omega^*})$, sum of derivatives, so in image of $\mathcal{B}$. From this inequality, one can create as regular controls as we want so that we can apply operator $\mathcal{M}$.

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$$(\nabla\wedge u)\in L^2((T/2,T),H^{53}(\Omega)^3)\cap H^{27}((T/2,T),H^{-1}(\Omega)^3).$$

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Perspectives

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- Local, global controllability along any trajectory,

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- Global controllability around 0,
- Local, global controllability along any trajectory,
- Other coupled systems, for example hyperbolic systems (non-linear systems of wave equations).

Reference

Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components, Jean-Michel Coron and Pierre Lissy, submitted.

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Thank you for your attention!