A mixed formulation for the approximation of the HUM control for the wave equation

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The wave equation with boundary control

We consider the following wave equation:

$$\left\{\begin{array}{lll} y_{tt} - (c(x)y_x)_x + d(x,t)y = 0, & & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = 0, & y(1,t) = v(t), & t \in (0,T) \\ y(x,0) = y_0(x), & y_t(x,0) = y_1(x), & x \in (0,1). \end{array}\right.$$

- $c \in C^3([0,1])$ with $c(x) \geq c_0 > 0$ in $[0,1]$
- $d \in L^\infty((0,1) \times (0,T))$
- $y_0 \in L^2(0,1)$ and $y_1 \in H^{-1}(0,1)$
- We search a control $v = v(t)$ such that

$$y(T) = 0, \qquad y_t(T) = 0. \tag{1}$$

Aim

For a controllability time $T > 0$ large enough and for every y_0, y_1, give a numerical approximation of the control v of minimal L^2-norm.

$$\begin{cases} \text{Minimize} \quad J(y, v) = \dfrac{1}{2} \displaystyle\int_0^T |v(t)|^2 \, dt \\ \text{Subject to} \quad (y, v) \in \mathcal{C}(y_0, y_1; T) \end{cases} \tag{2}$$

where $\mathcal{C}(y_0, y_1; T)$ denotes the linear manifold

$$\mathcal{C}(y_0, y_1; T) = \left\{ \begin{array}{l} (y, v) : v \in L^2(0, T), \; y \text{ solves the wave equation} \\ \text{and satisfies } y(T) = y_t(T) = 0 \end{array} \right\}.$$

Hilbert Uniqueness Method - a brief recall

$$\begin{cases} \text{Minimize} \quad J(y, v) = \dfrac{1}{2}\displaystyle\int_0^T |v(t)|^2 \, dt \\ \text{Subject to} \quad (y, v) \in \mathcal{C}(y_0, y_1; T) \end{cases} \tag{2}$$

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By duality arguments this minimization problem is equivalent to the following one

$$\min_{(\varphi_0, \varphi_1) \in H_0^1(0,1) \times L^2(0,1)} J^\star(\varphi_0, \varphi_1)$$

$$\begin{aligned} J^\star(\varphi_0, \varphi_1) = \quad & \tfrac{1}{2} \textstyle\int_0^T |\varphi_x(1, t)|^2 dt \\ & + \textstyle\int_0^1 y_0(x) \varphi_t(x, 0) dx - \langle y_1, \varphi(\cdot, 0) \rangle_{H^{-1}, H_0^1} \end{aligned}$$

Hilbert Uniqueness Method - a brief recall

Dual minimization problem reads as :

$$\min_{(\varphi_0,\varphi_1)\in H_0^1(0,1)\times L^2(0,1)} J^\star(\varphi_0,\varphi_1)$$

$$\begin{aligned} J^\star(\varphi_0,\varphi_1) = \quad & \tfrac{1}{2}\int_0^T |\varphi_x(1,t)|^2 dt \\ & + \int_0^1 y_0(x)\varphi_t(x,0)dx - \langle y_1, \varphi(\cdot,0)\rangle_{H^{-1},H_0^1} \end{aligned}$$

$$\left\{ \begin{array}{lll} L\varphi = 0 & \text{in} & Q_T, \\ \varphi = 0 & \text{on} & \Sigma_T \\ (\varphi(\cdot,T),\varphi_t(\cdot,T)) = (\varphi_0,\varphi_1), & \text{in} & \Omega. \end{array} \right.$$

$$\min_{(\varphi_0,\varphi_1)\in H_0^1(0,1)\times L^2(0,1)} J^\star(\varphi_0,\varphi_1)$$

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$$\begin{cases} L\varphi = 0 & \text{in} \quad Q_T, \\ \varphi = 0 & \text{on} \quad \Sigma_T \\ (\varphi(\cdot,T),\varphi_t(\cdot,T)) = (\varphi_0,\varphi_1), & \text{in} \quad \Omega. \end{cases}$$

The coercivity of $J^\star$ is the consequence of the following observability estimate : there exists a constant $k_T > 0$ such that

$$\|\varphi(\cdot,0),\varphi_t(\cdot,0)\|_{\boldsymbol{V}}^2 \le k_T^2 \|\varphi_x(1,\cdot)\|_{L^2(0,T)}^2, \quad \forall(\varphi_0,\varphi_1)\in \boldsymbol{V}, \quad (2)$$

where $\boldsymbol{V} = H_0^1(0,1)\times L^2(0,1)$.

Hilbert Uniqueness Method - a reformulation

Since φ is completely and uniquely determined by (φ_0, φ_1), we consider the following extremal problem:

$$\min_{\varphi \in \Phi} \hat{J}^\star(\varphi), \qquad \text{subject to} \quad L\varphi = 0,$$

where

$$\Phi = \left\{ \varphi \in L^2(Q_T),\, \varphi = 0 \text{ on } \Sigma_T \text{ such that } \begin{array}{l} L\varphi \in L^2(Q_T) \\ \varphi_x(1,\cdot) \in L^2(0,T) \end{array} \right\}.$$

Remark

Φ *is an Hilbert space endowed with the inner product*

$$(\varphi, \overline{\varphi})_\Phi = \int_0^T c(1)\varphi_x(1,t)\overline{\varphi}_x(1,t)\,dt + \eta \iint_{Q_T} L\varphi L\overline{\varphi}\,dx\,dt.$$

for any fixed $\eta > 0$.

Hilbert Uniqueness Method - a mixed reformulation

We consider the following mixed formulation : find $(\varphi, \lambda) \in \Phi \times L^2(Q_T)$ solution of

$$\left\{ \begin{array}{rcll} a(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) & = & l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{array} \right. \tag{3}$$

where

$$a : \Phi \times \Phi \to \mathbb{R}, \quad a(\varphi, \overline{\varphi}) = \int_0^T c(1) \varphi_x(1,t) \overline{\varphi}_x(1,t) dt$$

$$b : \Phi \times L^2(Q_T) \to \mathbb{R}, \quad b(\varphi, \lambda) = \iint_{Q_T} L\varphi(x,t) \lambda(x,t) dx dt$$

$$l : \Phi \to \mathbb{R}, \quad l(\varphi) = -\int_0^1 y_0(x)\, \varphi_t(x,0) dx + \langle y_1, \varphi(\cdot, 0) \rangle_{-1,1}.$$

$$
\left\{
\begin{array}{rcll}
a(\varphi,\overline{\varphi})+b(\overline{\varphi},\lambda) & = & l(\overline{\varphi}), & \forall\overline{\varphi}\in\Phi \\
b(\varphi,\overline{\lambda}) & = & 0, & \forall\overline{\lambda}\in L^2(Q_T),
\end{array}
\right.
\tag{3}
$$

Theorem

1. *The mixed formulation* (3) *is well-posed.*
2. *The unique solution* $(\varphi,\lambda)\in\Phi\times L^2(Q_T)$ *is the unique saddle-point of the Lagrangian* $\mathcal{L}:\Phi\times L^2(Q_T)\to\mathbb{R}$ *defined by*
$$\mathcal{L}(\varphi,\lambda)=\frac{1}{2}a(\varphi,\varphi)+b(\varphi,\lambda)-l(\varphi). \tag{4}$$
3. *The optimal function* φ *is the minimizer of* $\hat{J}^\star$ *over* Φ *while the optimal function* $\lambda\in L^2(Q_T)$ *is the state of the controlled wave equation in the weak sense.*

$$
\left\{ \begin{array}{rcll} a(\varphi, \overline{\varphi}) + b(\overline{\varphi}, \lambda) & = & l(\overline{\varphi}), & \forall \overline{\varphi} \in \Phi \\ b(\varphi, \overline{\lambda}) & = & 0, & \forall \overline{\lambda} \in L^2(Q_T), \end{array} \right. \tag{3}
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Theorem

1. *The mixed formulation* (3) *is well-posed.*
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$$
\mathcal{L}_r(\varphi, \lambda) = \frac{1}{2} a_r(\varphi, \varphi) + b(\varphi, \lambda) - l(\varphi). \tag{4}
$$
3. *The optimal function* φ *is the minimizer of* $\hat{J}^\star$ *over* Φ *while the optimal function* $\lambda \in L^2(Q_T)$ *is the state of the controlled wave equation in the weak sense.*

Ingredients of the proof

We easily check that :

- a is continuous over $\Phi \times \Phi$, symmetric and positive
- b is continuous over $\Phi \times L^2(Q_T)$
- assuming $c \in \mathcal{A}$ and $T > T^\star(c)$, l is continuous over Φ (as a direct consequence of an apropriate Carleman estimate see it)

The well-posedness of the mixed formulation is a consequence of the following two properties (see Brezzi and Fortin 1991 book):

- a is coercive on $\mathcal{N}(b)$, where:

$$\mathcal{N}(b) = \{\varphi \in \Phi \text{ such that } b(\varphi, \lambda) = 0 \text{ for every } \lambda \in L^2(Q_T)\}.$$

- b satisfies the usual "inf-sup" condition over $\Phi \times L^2(Q_T)$:

$$\inf_{\lambda \in L^2(Q_T)} \sup_{\varphi \in \Phi} \frac{b(\varphi, \lambda)}{\|\varphi\|_\Phi \|\lambda\|_{L^2(Q_T)}} \geq \delta.$$

Numerical approximation

Let Φ_h and M_h be two finite dimensional spaces parametrized by the variable h such that

$$\Phi_h \subset \Phi, \quad M_h \subset L^2(Q_T), \quad \forall h > 0.$$

We introduce the following approximated problems : find $(\varphi_h, \lambda_h) \in \Phi_h \times M_h$ solution of

$$\left\{ \begin{array}{rcll} a_r(\varphi_h, \overline{\varphi}_h) + b(\overline{\varphi}_h, \lambda_h) & = & l(\overline{\varphi}_h), & \forall \overline{\varphi}_h \in \Phi_h \\ b(\varphi_h, \overline{\lambda}_h) & = & 0, & \forall \overline{\lambda}_h, \in M_h. \end{array} \right.$$

where

$$a_r(\varphi, \varphi) = a(\varphi, \varphi) + r \iint_{Q_T} |L\varphi|^2 \, dx \, dt$$

for any given $r > 0$.

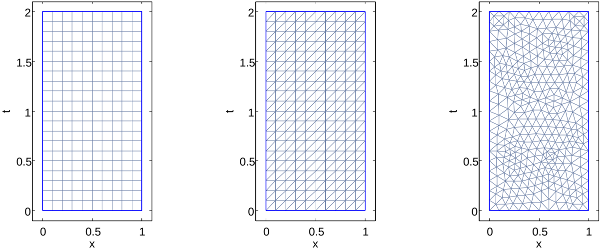
We introduce a triangulation $\mathcal{T}_h$ such that $\overline{Q_T} = \cup_{K \in \mathcal{T}_h} K$ and we assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family. We define

$$\Phi_h = \{\varphi_h \in C^1(\overline{Q_T}) : \varphi_h|_K \in \mathbb{P}(K) \quad \forall K \in \mathcal{T}_h,\ \varphi_h = 0 \text{ on } \Sigma_T\}$$

where $\mathbb{P}(K)$ may be chosen as

- The *Bogner-Fox-Schmit* (BFS for short) C^1 element defined for rectangles.
- The reduced *Hsieh-Clough-Tocher* (HCT for short) C^1 element defined for triangles.

$$M_h = \{\lambda_h \in C^0(\overline{Q_T}) : \lambda_h|_K \in \mathbb{Q}(K) \quad \forall K \in \mathcal{T}_h\},$$



Mixed formulation as a linear system

Let $n_h = \dim \Phi_h, m_h = \dim M_h$ and let the real matrices $A_{r,h} \in \mathbb{R}^{n_h,n_h}$, $B_h \in \mathbb{R}^{m_h,n_h}$ and $L_h \in \mathbb{R}^{n_h}$ be defined by

$$\begin{cases} a_r(\varphi_h, \overline{\varphi_h}) =< A_{r,h}\{\varphi_h\}, \{\overline{\varphi_h}\} >_{\mathbb{R}^{n_h},\mathbb{R}^{n_h}}, & \forall \varphi_h, \overline{\varphi_h} \in \Phi_h, \\ b(\varphi_h, \lambda_h) =< B_h\{\varphi_h\}, \{\lambda_h\} >_{\mathbb{R}^{m_h},\mathbb{R}^{m_h}}, & \forall \varphi_h \in \Phi_h, \forall \lambda_h \in M_h, \\ l(\varphi_h) =< L_h, \{\varphi_h\} >, & \forall \varphi_h \in \Phi_h \end{cases}$$

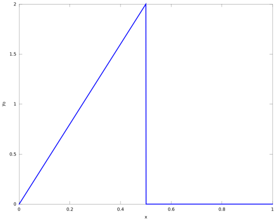
With these notations, the finite dimensional mixed problem reads as follows : find $\{\varphi_h\} \in \mathbb{R}^{n_h}$ and $\{\lambda_h\} \in \mathbb{R}^{m_h}$ such that

$$\begin{pmatrix} A_{r,h} & B_h^T \\ B_h & 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h,n_h+m_h}} \begin{pmatrix} \{\varphi_h\} \\ \{\lambda_h\} \end{pmatrix}_{\mathbb{R}^{n_h+m_h}} = \begin{pmatrix} L_h \\ 0 \end{pmatrix}_{\mathbb{R}^{n_h+m_h}}.$$

$$v_h(t) = c(1)\pi_{\Delta t}(\phi_{h,x}(1,\cdot)).$$

A numerical example

$$y_0(x) = 4x\ 1_{(0,1/2)}(x), \qquad y_1(x) = 0, \qquad T = 2.4.$$



The control of minimal L^2-norm is known exactly :

$$v(t) = 2(1-t)\mathbf{1}_{1/2,\ 3/2}(t).$$

A numerical example

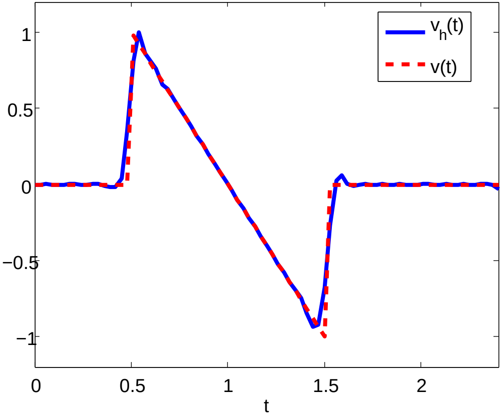


Figure: Control of minimal L^2-norm v and its approximation v_h on $(0, T)$.

A numerical example

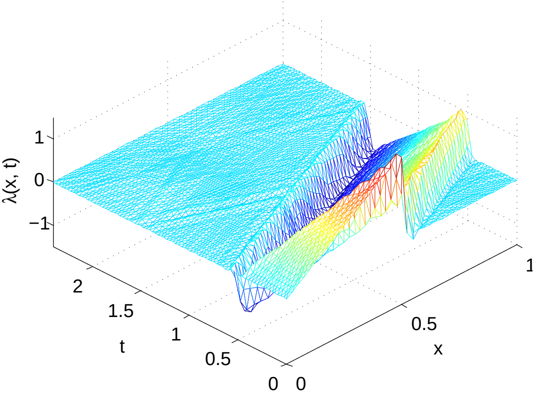


Figure: The primal variable λ_h in Q_T; $h = 2.46 \times 10^{-2}$; $r = 10^{-2}$.

A numerical example

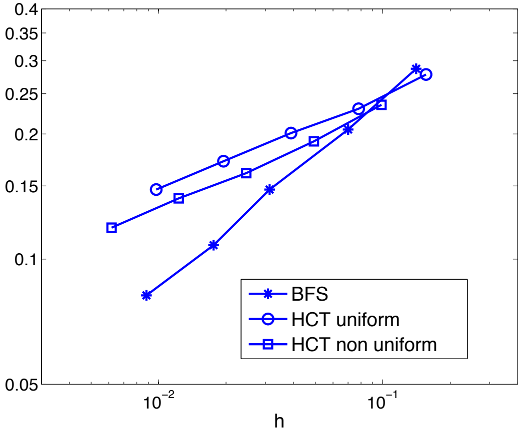


Figure: Evolution of $\|v - v_h\|_{L^2(0,T)}$ w.r.t. h for BFS finite element ($\star$), HCT-uniform mesh ($\circ$) and HCT- non uniform mesh ($\square$); $r = 1$.

A numerical example - mesh adaptation

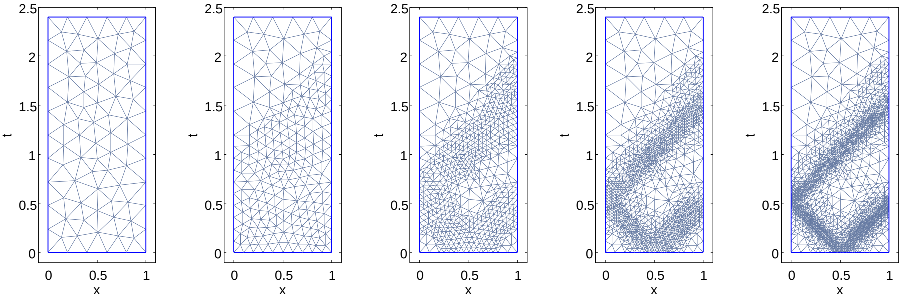


Figure: Iterative refinement of the triangular mesh over Q_T with respect to the variable λ_h: $142, 412, 1\ 154, 2\ 556, 4\ 750$ triangles; $r = 2 \times 10^{-3}$.

A numerical example - mesh adaptation

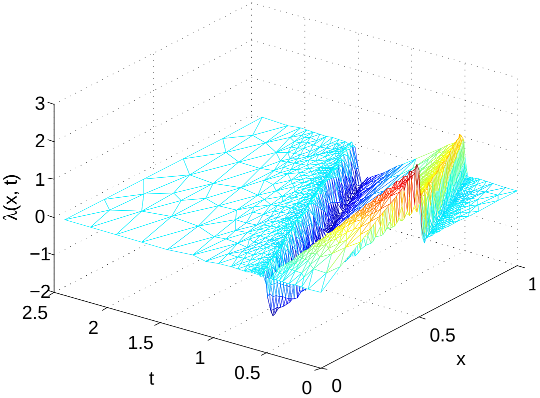


Figure: Primal variable λ_h in Q_T.

Another numerical example - distributed control case

$$y_0(x) = e^{-500(x-0.2)^2}, \qquad y_1(x) = 0$$

$$T = 2.2, \qquad \omega = (0.2, 0.4)$$

and a non-constant function $c = c(x) \in C^1([0,1])$ with

$$c(x) = \begin{cases} 1. & x \in [0, 0.45] \\ \in [1., 5.] \quad (c'(x) > 0), & x \in (0.45, 0.55) \\ 5. & x \in [0.55, 1]. \end{cases} \tag{5}$$

Another numerical example - distributed control case

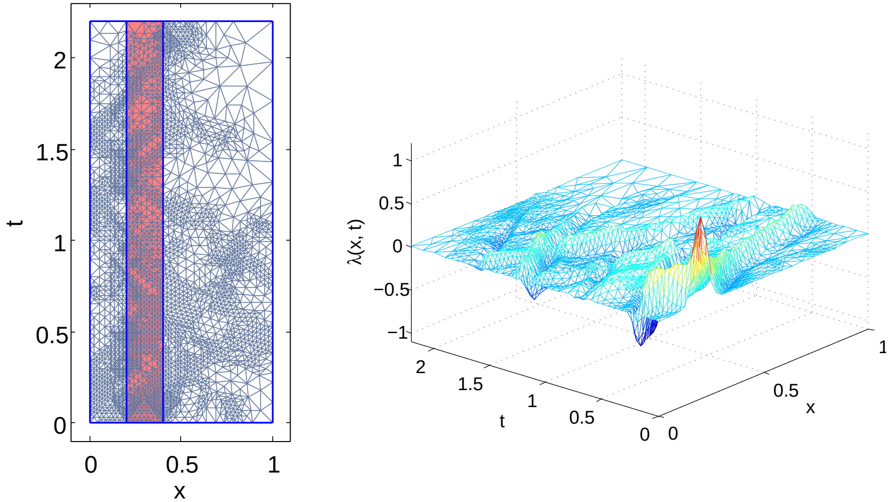
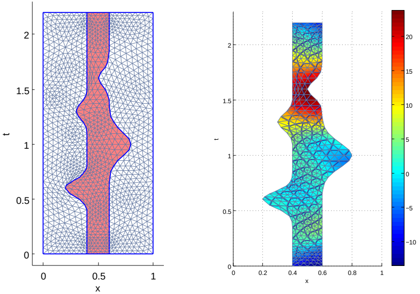


Figure: **Left :** Triangular mesh of q_T and of Q_T. **Right :** The primal variable λ_h in Q_T.

- Optimization in space-time of the support of the control



Some perspectives

$$\inf_{\lambda_h \in M_h} \sup_{\varphi_h \in \Phi_h} \frac{b(\varphi_h, \lambda_h)}{\|\varphi_h\|_{\Phi_h} \|\lambda_h\|_{M_h}} \geq \delta_h.$$

- Optimization in space-time of the support of the control
- "inf-sup" condition is uniform with respect to h?

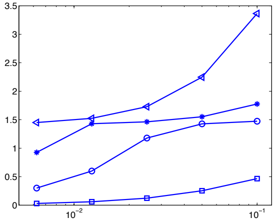


Figure: BFS finite element - Evolution of the inf-sup constante δ_h with respect to h for $r = 10^{-4}$ ($<$), $r = 10^{-3}$ ($\star$), $r = 10^{-2}$ ($\circ$) and $r = 10^{-1}$ ($\square$).

N. Cîndea, A. Munch, *A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations*, submitted August 2013.

Thank you for the attention!

Proposition (C, Fernandez-Cara, Münch, ESAIM COCV 2013)

Let $x_0 < 0$, $c_0 > 0$ and $c \in \mathcal{A}(x_0, c_0)$. Let $\beta > 0$ and let us consider the function $\phi(x,t) := |x - x_0|^2 - \beta t^2 + M_0$ and $g(x,t) := e^{\lambda \phi(x,t)}$. Finally, let us assume that

$$T > \frac{1}{\beta} \max_{[0,1]} c(x)^{1/2}(x - x_0).$$

Then there exist positive constants s_0 and M, such that, for all $s > s_0$, one has

$$s \int_{-T}^{T} \int_0^1 e^{2sg} \left(|w_t|^2 + |w_x|^2\right) dx\, dt + s^3 \int_{-T}^{T} \int_0^1 e^{2sg} |w|^2 \, dx\, dt \\ \leq M \int_{-T}^{T} \int_0^1 e^{2sg} |Lw|^2 \, dx\, dt + Ms \int_{-T}^{T} e^{2sg} |w_x(1,t)|^2 \, dt.$$

$$\mathcal{A}(x_0, c_0) = \Big\{ c \in C^3([0,1]) : c(x) \geq c_0 > 0, \\ -\min_{[0,1]} \Big(c(x) + (x - x_0) c_x(x) \Big) < \min_{[0,1]} \Big(c(x) + \frac{1}{2}(x - x_0) c_x(x) \Big) \Big\},$$