On the weak solutions to the Landau-Lifshitz equations

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The Landau-Lifshitz equation

In a ferromagnet, the magnetic moment

$$m : [0, +\infty) \times \mathbb{R}^3 \to S^2$$

satisfies the Landau-Lifshitz (LL for short) equation :

$$\partial_t m + m \wedge \partial_t m = 2m \wedge \Delta m.$$

Formal energy identity

We recall

$$LL: \quad \partial_t m + m \wedge \partial_t m = 2m \wedge \Delta m.$$

One takes the inner product of the LL equations with $\partial_t m$ and Δm to get

$$(\partial_t m)^2 = 2(m \wedge \Delta m) \cdot \partial_t m, \tag{1}$$

$$\partial_t m \cdot \Delta m + (m \wedge \partial_t m) \cdot \Delta m = 0. \tag{2}$$

Observe that the combination $(1) - 2(2)$ yields

$$(\partial_t m)^2 - 2\partial_t m \cdot \Delta m = 0.$$

Then integrate by parts in x and in t to obtain the energy identity : for any $T > 0$,

$$\int_{\mathbb{R}^3} |\nabla m|^2(T,x)\, dx + \int_{(0,T)\times\mathbb{R}^3} |\partial_t m|^2\, dx\, dt = \int_{\mathbb{R}^3} |\nabla m|^2(0,x) dx.$$

Existence of weak solutions / No uniqueness

Theorem (Alouges-Soyeur, 91')

- *Let* $m_0 \in L^\infty(\mathbb{R}^3;\mathbb{R}^3)/\ |m_0| = 1$ *a.e. and* $\int_{\mathbb{R}^3} |\nabla m_0|^2\, dx < +\infty$.
- *Then, there exists a corresponding weak solution* $m : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ *to the LL equations s.t.* $|m| = 1$ *a.e., and, for a.e.* $T > 0$,

$$J[m](T) := \Big(\int_{\mathbb{R}^3} |\nabla m|^2(t,\cdot)\, dx \Big)(T) + \int_{(0,T)\times\mathbb{R}^3} |\partial_t m|^2\, dx\, dt$$

satisfies

$$J[m](T) \leqslant \int_{\mathbb{R}^3} |\nabla m_0|^2\, dx.$$

- *There is no uniqueness.*

Weak-Strong uniqueness

Theorem (Dumas-S., 13')

In the previous setting, assume moreover that m_0 is smooth, and assume that

- *m_2 is a global weak solution to the LL equations on $(0, \infty) \times \mathbb{R}^3$ satisfying the energy inequality, as in the theorem by Alouges and Soyeur,*
- *m_1 is a smooth solution to the LL equations up to some time $T > 0$, with the same initial data m_0.*

Then $m_2 = m_1$ on $(0, T) \times \mathbb{R}^3$.

Sketch of proof

We denote $m := m_1 - m_2$ and expand $J[m] := J[m](T)$ into

$$J[m] = J[m_1] + J[m_2] - 2\Big(\int_{\mathbb{R}^3} \nabla m_1 : \nabla m_2 \, dx \Big)(T) - 2 \int_0^T \int_{\mathbb{R}^3} \partial_t m_1 \cdot \partial_t m_2 \, dx \, dt.$$

Using some integration by parts, we have

$$\begin{aligned} \Big(\int_{\mathbb{R}^3} \nabla m_1 : \nabla m_2 \, dx \Big)(T) &= \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i \partial_t m_1) \cdot \partial_i m_2 \, dx \, dt \\ &\quad - \int_0^T \int_{\mathbb{R}^3} (\Delta m_1) \cdot \partial_t m_2 \, dx \, dt \\ &\quad + \int_{\mathbb{R}^3} |\nabla m_0|^2 \, dx. \end{aligned}$$

Sketch of proof

Now, the two solutions satisfy the energy inequality, so that, for almost every $T \geq 0$,

$$J[m] \leqslant K[m_1, m_2],$$

where

$$\begin{aligned} K[m_1, m_2] \quad := \quad & -2 \sum_i \int_0^T \int_{\mathbb{R}^3} (\partial_i \partial_t m_1) \cdot \partial_i m_2 \, dx \, dt \\ & +2 \int_0^T \int_{\mathbb{R}^3} (\Delta m_1) \cdot \partial_t m_2 \, dx \, dt \\ & -2 \int_0^T \int_{\mathbb{R}^3} \partial_t m_1 \cdot \partial_t m_2 \, dx \, dt. \end{aligned}$$

Sketch of proof

With a few manipulations, we recast $K[m_1, m_2]$ as follows :

$$\begin{aligned} K[m_1, m_2] &= 4 \sum_i \int_0^T \int_{\mathbb{R}^3} (m \wedge \partial_i \Delta m_1) \cdot \partial_i m \, dx \, dt \\ &\quad - 2 \sum_i \int_0^T \int_{\mathbb{R}^3} (m \wedge \partial_t \partial_i m_1) \cdot \partial_i m \, dx \, dt \\ &\quad - 2 \int_0^T \int_{\mathbb{R}^3} (m \wedge \partial_t m) \cdot \Delta m_1 \, dx \, dt \\ &\quad + \int_0^T \int_{\mathbb{R}^3} (m \wedge \partial_t m) \cdot \partial_t m_1 \, dx \, dt. \end{aligned}$$

Sketch of proof

Since m vanishes at initial time, Poincaré's inequality yields

$$\int_0^T \int_{\mathbb{R}^3} |m|^2 \, dx \, dt \leqslant o(T) \int_0^T \int_{\mathbb{R}^3} |\partial_t m|^2 \, dx \, dt.$$

Thus, for T small enough, one gets

$$|K[m_1, m_2]| \leqslant \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} |\partial_t m|^2 \, dx \, dt + C \int_0^T \int_{\mathbb{R}^3} |\nabla m|^2 \, dx \, dt.$$

We then use Gronwall lemma to conclude that m vanishes, first for small time, but the argument can be repeated as many times as necessary.

Local energy identity

Formally, one has the following local energy identity :

$$\partial_t e + d + \operatorname{div} f = 0,$$

where

$$\begin{aligned} e &:= |\nabla m|^2, \\ d &:= |\partial_t m|^2, \\ f &:= -2(\partial_t m \cdot \partial_i m)_{i=1,2,3}. \end{aligned}$$

Regularization of quadratic terms

Let $\psi \in C_c^\infty(\mathbb{R}^3;\mathbb{R})$ be nonnegative, and such that $\int_{\mathbb{R}^3} \psi(x)\,dx = 1$.

For all $\varepsilon \in (0,1)$, we define the usual mollifier $\psi^\varepsilon := \varepsilon^{-3}\psi(\cdot/\varepsilon)$.

Then, for any function ϕ on $\mathbb{R}^3$, we set

$$\phi_\varepsilon(x) = (\psi^\varepsilon * \phi)(x) = \int_{\mathbb{R}^3} \psi^\varepsilon(y)\phi(x-y)dy.$$

For all $\varepsilon \in (0,1)$ and functions ϕ^1, ϕ^2, we also define

$$\mathcal{B}^\varepsilon[\phi^1, \phi^2] := (\phi^1 \wedge \phi^2)_\varepsilon - \phi^1_\varepsilon \wedge \phi^2_\varepsilon.$$

Anomalous dissipation

We have the following result.

Theorem (Dumas-S., 13')

Let m be a weak solution to the LL equations. Let

$$d^{\mathfrak{a}} := \partial_t e + d + \operatorname{div} f.$$

Let

$$d^{\mathfrak{a},\varepsilon} := -\mathcal{B}^{\varepsilon}[m, \partial_t m - 2\Delta m] \cdot (\partial_t m_\varepsilon - 2\Delta m_\varepsilon).$$

Then,

$$d^{\mathfrak{a},\varepsilon} \to d^{\mathfrak{a}} \text{ in } \mathcal{D}'\big((0,\infty) \times \mathbb{R}^3; \mathbb{R}\big) \text{ when } \varepsilon \to 0,$$

and this holds true whatever is the mollifier chosen.

Some Besov type conditions

Our goal is to provide some sufficient conditions, which rule out anomalous dissipation.
Let $T > 0$, $\alpha \in (0,1)$ and $p, r \in [1, \infty]$.
For every function u on $(0, T) \times \mathbb{R}^3$ we define, for $(t, y) \in (0, T) \times (\mathbb{R}^3 \setminus \{0\})$,

$$f_{\alpha,p}[u](t,y) := \frac{\|u(t, \cdot - y) - u(t, \cdot)\|_{L^p(\mathbb{R}^3)}}{|y|^\alpha}.$$

We denote

- by $\widetilde{L}^r(0, T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))$, the space of functions u on $(0, T) \times \mathbb{R}^3$, which satisfy

$$\sup_y \|f_{\alpha,p}[u](\cdot, y)\|_{L^r(0,T)} < \infty.$$

- by $\widetilde{L}^r(0, T; \dot{B}^{\alpha+1}_{p,\infty}(\mathbb{R}^3))$, the subspace of the u in $\widetilde{L}^r(0, T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))$ which satisfy, for $i = 1, 2, 3$, $\partial_i u \in \widetilde{L}^r(0, T; \dot{B}^\alpha_{p,\infty}(\mathbb{R}^3))$.

Anomalous dissipation

If we take the inner product of the regularized version of the equations with Δm_ε we face in particular the expression

$$\int_{(0,T)\times\mathbb{R}^3} (m \wedge \Delta m)_\varepsilon \cdot \Delta m_\varepsilon \, dx\, dt. \tag{3}$$

Let M, X, T be respectively some units for magnetic moment, length and time. Then the quantity in (3) has a dimension equal to $X^{-1}\, TM^3$.

We would like to control the term (3) by $\|f_{\alpha,p}[m](\cdot, y)\|^3_{L^r(0,T)}$ which has a dimension equal to $X^{\frac{9}{p}-3\alpha}\, T^{\frac{3}{r}}\, M^3$, which provides $r = 3$ and $p = \frac{9}{3\alpha-1}$.

Anomalous dissipation

Theorem (Dumas-S., 13')

- *Let m be a weak solution to the LL equations.*
- *Assume furthermore that m belongs to* $\widetilde{L}^3(0,T;\dot{B}^{\alpha}_{p,\infty}(\mathbb{R}^3))_{\mathrm{loc}}$ *for some* $\alpha \in (3/2,2)$ *and* $p > \frac{9}{3\alpha-1}$.
- *Then the local anomalous energy dissipation* $d^{\mathfrak{a}}$ *vanishes.*

In the proof we use that $\mathcal{B}^{\varepsilon}[\phi^1,\phi^2]$ may be written

$$\mathcal{B}^{\varepsilon}[\phi^1,\phi^2] = r^{\varepsilon}[\phi^1,\phi^2] - (\phi^1 - \phi^1_{\varepsilon}) \wedge (\phi^2 - \phi^2_{\varepsilon}),$$

where

$$r^{\varepsilon}[\phi^1,\phi^2](x) := \int_{\mathbb{R}^3} \psi^{\varepsilon}(y)\delta_y\phi^1(x) \wedge \delta_y\phi^2(x)\,dy,$$

with $\delta_y\phi(t,x) = \phi(t,x-y) - \phi(t,x)$.

Local sign of the dissipation

Alouges and Soyeur obtained some weak solutions to the LL equations by passing to the limit, for $\varepsilon \to 0$, the penalized equations :

$$\partial_t m^\varepsilon - m^\varepsilon \wedge \partial_t m^\varepsilon = 2\Big(\Delta m^\varepsilon - \frac{1}{\varepsilon}(|m^\varepsilon|^2 - 1)m^\varepsilon\Big).$$

Theorem

- *Let m be a weak solution to the LL equations obtained as a limit point of the sequence m^ε as considered above.*
- *Assume moreover that, up to a subsequence, for $i = 1, 2, 3$, $\partial_t m^\varepsilon \cdot \partial_i m^\varepsilon$ converge respectively to $\partial_t m \cdot \partial_i m$ in the sense of distributions.*
- *Then there exist two non negative distributions $d^{\mathfrak{a},1}$ and $e^{\mathfrak{a}}$ such that $d^{\mathfrak{a}} = d^{\mathfrak{a},1} + \partial_t e^{\mathfrak{a}}$.*

Commentaries

Let us stress that :

$$\left(\nabla m^\varepsilon \to \nabla m \text{ in } L^2_{\mathrm{loc}}\right) \Rightarrow \left(\text{ for } i = 1, 2, 3, \quad \partial_t m^\varepsilon \cdot \partial_i m^\varepsilon \to \partial_t m \cdot \partial_i m \text{ in } \mathcal{D}'\right).$$

When this strong convergence holds, the proof reveals that $d^{\mathfrak{a},1}$ vanishes and that $e^{\mathfrak{a}}$ is only due to the possible lack of strong convergence of the energy density $e^\varepsilon_{\mathrm{GL}} := \frac{1}{2\varepsilon}\left(|m^\varepsilon|^2 - 1\right)^2$ associated with $\mathcal{E}^\varepsilon_{\mathrm{GL}}$.

It would be interesting to investigate the existence of another way to construct weak solutions to the LL equations for which the distribution $e^{\mathfrak{a}}$ vanishes as well.

Thank you for your attention !