Control of 2D scalar conservation laws in the presence of shocks

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Optimal control problem

The objetive function

$$\min_{u^0 \in \mathcal{U}_{ad}} J(u^0) = \int_{\mathbb{R}^2} (u(x,T) - u^d(x))^2 dx$$

$$\left\{ \begin{aligned} \partial_t u + \mathrm{div}_x(f(u)) &= 0, && \text{in } \mathbb{R}^2 \times (0,T) \\ u(x,0) &= u^0(x), && x \in \mathbb{R}^2 \end{aligned} \right.$$

Linearization without shock

$$\delta J(u^0)[\delta u^0] = \int_{\mathbb{R}^2} (u(x,T) - u^d(x))\delta u^T(x)dx$$

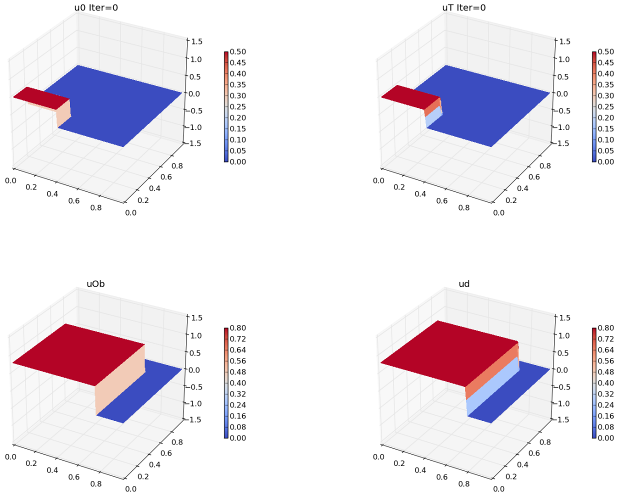
$$\begin{cases} \partial_t \delta u + \mathrm{div}_x(f'(u)\delta u) &= 0, \qquad \text{in } \mathbb{R}^2 \times (0,T) \\ \delta u(x,0) &= \delta u^0(x), \qquad x \in \mathbb{R}^2 \end{cases}$$

Using the adjoint system

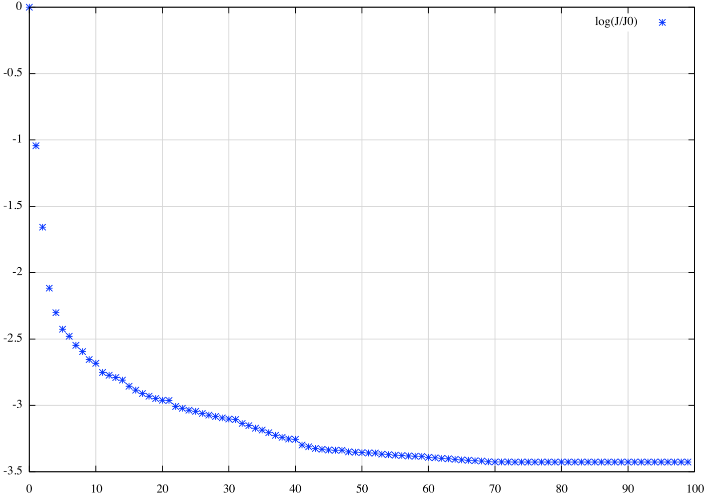
$$\delta J(u^0)[\delta u^0] = \int_{\mathbb{R}^2} p(x,0)\delta u^0(x)dx$$

$$\begin{cases} \partial_t p + f'(u)\cdot\nabla p &= 0, \qquad \text{in } \mathbb{R}^2 \times (0,T) \\ p(x,T) &= u(x,T) - u^d, \qquad x \in \mathbb{R}^2 \end{cases}$$

Example A) $u_t + \left(u^2/2\right)_x + \left(u^4/4\right)_y = 0$, with $T = 0.2$

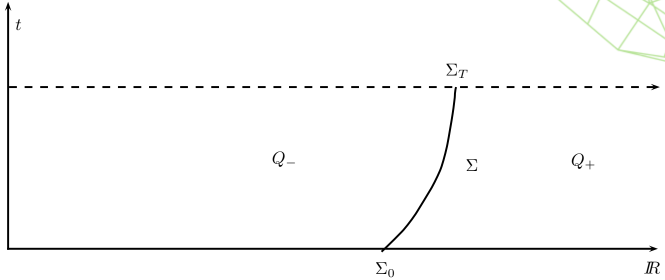


The functional $J_{\#}$ in example A)

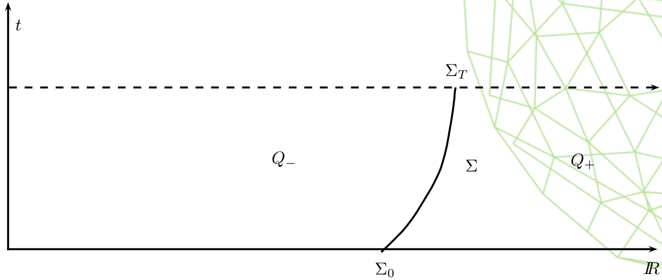


C. Castro, F. Palacios, and E. Zuazua.

An alternating descent method for the optimal control of the inviscid burgers equation in the presence of shocks.
Mathematical Models and Methods in Applied Sciences, 18(03):369–416, 2008.



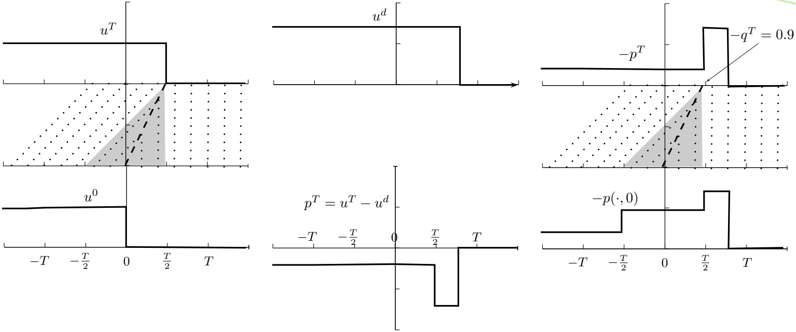
$$\begin{aligned} u_t + (f(u))_x &= 0, \quad \text{in } Q_- \cup Q_+ \\ n_\Sigma^t [u]_{\Sigma^t} + n_\Sigma^x [f(u)]_{\Sigma^t} &= 0, \quad \text{on } \Sigma \end{aligned}$$

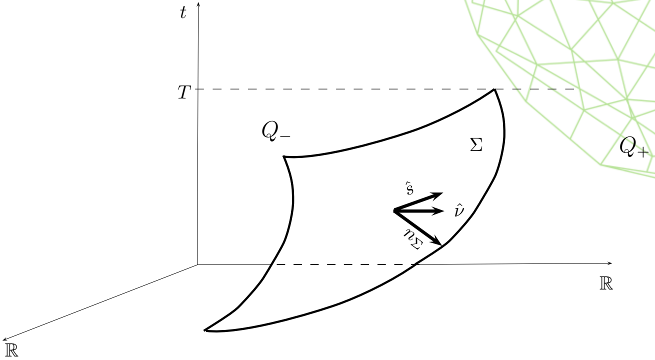


$$
\begin{aligned}
p_t + f'(u)p_x &= 0, && \text{in } Q_- \cup Q_+ \\
[p]_{\Sigma^t} &= 0, && \text{on } \Sigma \\
q(t) &= p(\varphi(t), t), && \\
\dot{q} &= 0, && \\
p(x,T) &= u(x,T) - u^d(x), && x \in \mathbb{R} \setminus \Sigma^T \\
q(T) &= \frac{[(u(\cdot,T) - u^d)^2/2]_{\Sigma^T}}{[u]_{\Sigma^T}}. &&
\end{aligned}
$$

$$
\delta J(u^0)[\delta u^0, \delta\varphi^0] = \int_{\mathbb{R}} (u(x,T) - u^d(x))\delta u^0(x)dx - [u]_{\Sigma 0} q(0)\delta\varphi^0
$$

$$
\begin{array}{rll}
p_t + f'(u)p_x & = 0, & \text{in } Q_- \cup Q_+ \\
[p]_{\Sigma^t} & = 0, & \text{on } \Sigma \\
q(t) & = p(\varphi(t), t), & \\
\dot{q} & = 0, & \\
p(x, T) & = u(x, T) - u^d(x), & x \in \mathbb{R} \setminus \Sigma^T \\
q(T) & = \frac{[(u(\cdot,T) - u^d)^2/2]_{\Sigma^T}}{[u]_{\Sigma^T}}. &
\end{array}
$$





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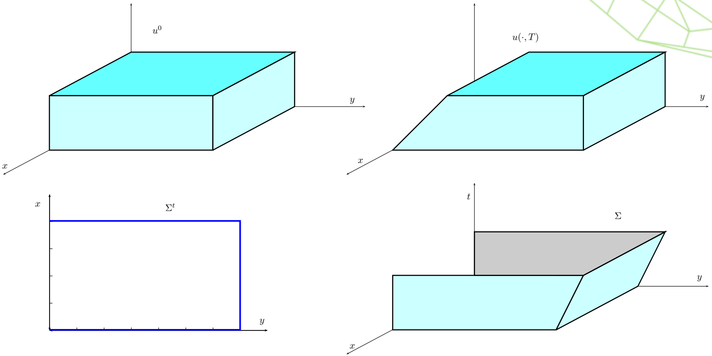
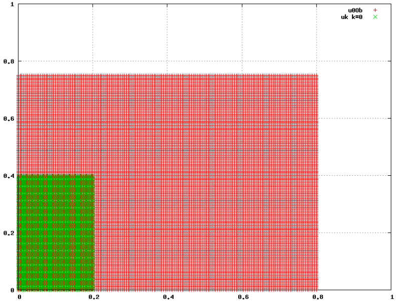
$$u_t + \left(\frac{u^2}{2}\right)_y = 0, \quad \mathbb{R}^2 \times (0,T),$$

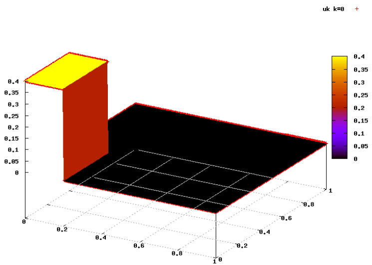
$$u^0(x,y) = \begin{cases} 1 & (x,y) \in (0,1)\times(0,1), \\ 0 & \text{otherwise.} \end{cases}$$



Example A)

$$u_t + (u^2/2)_x + (u^4/4)_y = 0$$





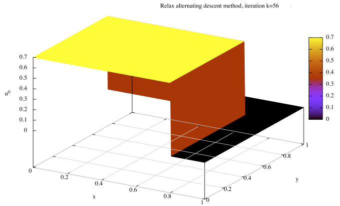


Figure : u^0, alternating descent method, iteration $k = 56$

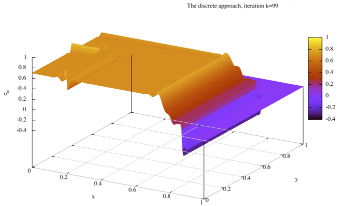


Figure : u^0, the discrete approach, iteration $k = 99$

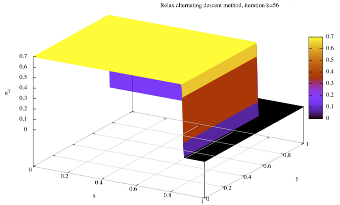


Figure : u^T, alternating descent method, iteration $k = 56$

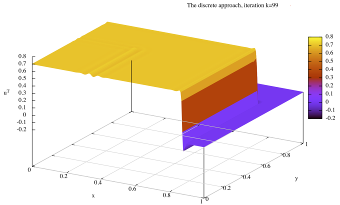


Figure : u^T, the discrete approach, iteration $k = 99$

THANK YOU!