$\tau^- \to K^-(\eta, \eta')\nu_\tau$ in ChPT with Resonances

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Taller de Altas Energias

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$$BR(\tau \to \nu_\tau hadrons) \sim 65\%$$

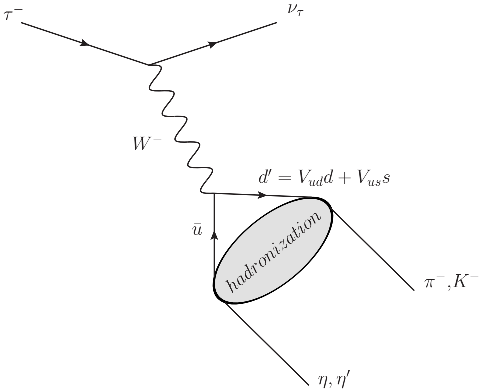
Why $K\eta^{(\prime)}$ modes?

- No previous detailed study
- Sensitive to the vector resonance $K^\star(1410)$
- New data for the $K\eta$ channel from both Belle and BaBar $\Rightarrow$ to determine the $M_{K^\star(1410)}$ and $\Gamma_{K^\star(1410)}$ parameters from fitting data

Purpose: To evaluate the hadronic matrix element and to give reasonable estimates of the invariant mass spectra and branching ratios.

Outline

Decay amplitude



$$\mathcal{M} = \frac{g^2}{8M_W^2} V_{us}\bar{u}(p_{\nu_\tau})\gamma^\mu(1-\gamma^5)u(p_\tau)\langle K^-, \eta^{(\prime)}|\bar{s}\gamma^\mu(1-\underbrace{\not\gamma^5}_{0^-,1^+\not\to 0^+,1^-})u|0\rangle$$

$BR(\tau^- \to K^-(\eta,\eta')\nu_\tau) \propto |V_{us}|^2 \Rightarrow$ CKM suppression

Hadronic matrix element

The hadronic matrix element is generally parametrized as

$$\langle K^-\eta^{(\prime)}|\bar{s}\gamma^\mu u|0\rangle = C^V_{K\eta^{(\prime)}}\left[(p_{\eta^{(\prime)}} - p_K)^\mu F_+^{K\eta^{(\prime)}}(s) - (p_{\eta^{(\prime)}} + p_K)^\mu F_-^{K\eta^{(\prime)}}(s)\right] \quad (1)$$

where $C^V_{K^-\eta^{(\prime)}} = -\sqrt{3/2}$.
Taking the divergence of Eq.(1) we obtain on one hand

$$\langle 0|\partial_\mu(\bar{s}\gamma^\mu u)|K^+\eta^{(\prime)}\rangle = i(m_s - m_u)\langle 0|\bar{s}u|K^+\eta^{(\prime)}\rangle = i\Delta_{K\pi} C^S_{K^-\eta^{(\prime)}} F_0^{K\eta^{(\prime)}}(s) \quad (2)$$

where $\Delta_{K\pi} = M_K^2 - M_\pi^2$, $C^S_{K^-\eta} = -1/\sqrt{6}$ and $C^S_{K^-\eta'} = \frac{2}{\sqrt{3}}$
and on the other hand we get (where $q_\mu = (p_{\eta^{(\prime)}} + p_{K^-})_\mu$ and $q^2 = s$)

$$iq_\mu\langle K^-\eta^{(\prime)}|\bar{s}\gamma^\mu u|0\rangle = iC^V_{K\eta^{(\prime)}}\left[(m^2_{\eta^{(\prime)}} - m^2_{K^-})F_+^{K\eta^{(\prime)}}(s) - sF_-^{K\eta^{(\prime)}}(s)\right] \quad (3)$$

$\Rightarrow$ Vector current not conserved

Hadronic matrix element

Equating Eq.(2) and Eq.(3) allows us to relate $F_-^{K\eta^{(\prime)}}(s)$ with $F_0^{K\eta^{(\prime)}}(s)$ as

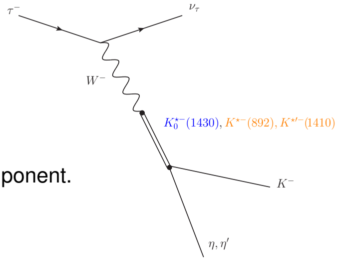
$$F_-^{K\eta^{(\prime)}}(s) = -\frac{\Delta_{K-\eta^{(\prime)}}}{s}\left[\frac{C^S_{K\eta^{(\prime)}}}{C^V_{K\eta^{(\prime)}}}\frac{\Delta_{K\pi}}{\Delta_{K-\eta^{(\prime)}}}F_0^{K\eta^{(\prime)}}(s) + F_+^{K^-\eta^{(\prime)}}(s)\right] \quad (4)$$

The vectorial hadronic matrix element finally reads

$$\langle K^-\eta^{(\prime)}|\bar{s}\gamma^\mu u|0\rangle =$$

$$\left[(p_{\eta^{(\prime)}} - p_K)^\mu + \frac{\Delta_{K-\eta^{(\prime)}}}{s}q^\mu\right]C^V_{K\eta^{(\prime)}}F_+^{K\eta^{(\prime)}}(s) + \frac{\Delta_{K\pi}}{s}q^\mu C^S_{K-\eta^{(\prime)}}F_0^{K\eta^{(\prime)}}(s) \quad (5)$$

Advantages of this decomposition:



$F_0^{K^-\eta^{(\prime)}}(s)$ corresponds to the S-wave projection of the final state, whereas $F_0^{K^-\eta^{(\prime)}}(s)$ is the P-wave component.

Differential decay width: $\tau^- \to K^-(\eta,\eta')\nu_\tau$

$$\frac{d\Gamma\left(\tau^- \to K^-\eta^{(\prime)}\nu_\tau\right)}{d\sqrt{s}} = \frac{G_F^2 M_\tau^3}{32\pi^3 s} S_{EW} \underbrace{|V_{us}}_{\text{suppression}} F_+^{K^-\eta^{(\prime)}}(0)|^2 \left(1-\frac{s}{M_\tau^2}\right)^2$$

$$\left\{\left(1+\frac{2s}{M_\tau^2}\right) q^3_{K\eta^{(\prime)}}(s)|\widetilde{F}_+^{K^-\eta^{(\prime)}}(s)|^2 + \frac{3\Delta^2_{K\eta^{(\prime)}}}{4s} q_{K\eta^{(\prime)}}(s)|\widetilde{F}_0^{K^-\eta^{(\prime)}}(s)|^2\right\} \quad (6)$$

where

$$q_{PQ}(s) = \frac{\sqrt{s^2 - 2s\Sigma_{PQ} + \Delta^2_{PQ}}}{2\sqrt{s}}, \quad \Sigma_{PQ} = m_P^2 + m_Q^2, \quad \Delta_{PQ} = m_P^2 - m_Q^2 \quad (7)$$

and

$$\widetilde{F}_{+,0}^{K^-\eta^{(\prime)}}(s) = \frac{F_{+,0}^{K^-\eta^{(\prime)}}(s)}{F_{+,0}^{K^-\eta^{(\prime)}}(0)}, \quad F_+^{K^-\eta^{(\prime)}}(0) = -\frac{C^S_{K^-\eta^{(\prime)}}}{C^V_{K^-\eta^{(\prime)}}}\frac{\Delta_{K\pi}}{\Delta_{K^-\eta^{(\prime)}}} F_0^{K^-\eta^{(\prime)}}(0) \quad (8)$$

$$F_+^{K^-\eta}(0) = F_+^{K^-\pi}(0)\cos\theta_P, \quad F_+^{K^-\eta'}(0) = F_+^{K^-\pi}(0)\underbrace{\sin\theta_P}_{\text{suppression}} \quad (9)$$

with $\theta = (-13.3 \pm 1.0)^\circ$ being the $\eta - \eta'$ mixing angle and $V_{us} \cdot F_+^{K^-\pi}(0) = 0.21664 \pm 0.00048$ from $K_{l3}^{0,\pm}$

Differential decay width: $\tau^- \to K^-(\eta,\eta')\nu_\tau$

$$\frac{d\Gamma\left(\tau^- \to K^-\eta^{(\prime)}\nu_\tau\right)}{d\sqrt{s}} = \frac{G_F^2 M_\tau^3}{32\pi^3 s} S_{EW} \underbrace{|V_{us}|^2}_{suppression} |F_+^{K^-\eta^{(\prime)}}(0)|^2 \left(1-\frac{s}{M_\tau^2}\right)^2$$

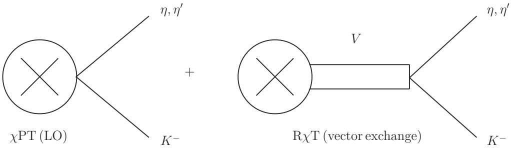
$$\left\{\left(1+\frac{2s}{M_\tau^2}\right) q^3_{K\eta^{(\prime)}}(s)|\widetilde{F}_+^{K^-\eta^{(\prime)}}(s)|^2 + \frac{3\Delta^2_{K\eta^{(\prime)}}}{4s} q_{K\eta^{(\prime)}}(s)|\widetilde{F}_0^{K^-\eta^{(\prime)}}(s)|^2\right\} \quad (10)$$

Our next task:

$\Rightarrow$ to compute the vector and scalar Form Factors $\tilde{F}_+^{K^-\eta^{(\prime)}}(s)$ and $\tilde{F}_0^{K^-\eta^{(\prime)}}(s)$.

Vector Form Factor: Breit-Wigner parametrization

The vector Form Factor is obtained by computing the following set of diagrams



$$\widetilde{F}_+^{K^-\eta}(s) = 1 + \frac{F_V G_V}{F_\pi^2}\frac{s}{M_{K^\star}^2 - s} = \widetilde{F}_+^{K^-\eta'}(s) \quad (11)$$

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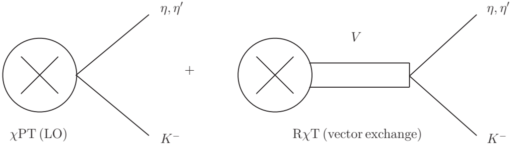

$$\tilde{F}_+^{K^-\eta^{(\prime)}}(s) = 1 + \frac{F_V G_V}{F_\pi^2}\frac{s}{M_{K^\star}^2 - s} + \frac{F_V' G_V'}{F_\pi^2}\frac{s}{M_{K^{\star\prime}}^2 - s} \quad (12)$$

requirement: $\tilde{F}_+^{K^-\eta^{(\prime)}}(s)$ vanish for $s \to \infty \Rightarrow F_V G_V + F_V' G_V' = F_\pi^2$

$$\boxed{\tilde{F}_+^{K^-\eta^{(\prime)}}(s) = \frac{M_{K^\star}^2 + \gamma s}{M_{K^\star}^2 - s - iM_{K^\star}\Gamma_{K^\star}(s)} - \frac{\gamma s}{M_{K^{\star\prime}}^2 - s - iM_{K^{\star\prime}}\Gamma_{K^{\star\prime}}(s)}} \quad (13)$$

where $\gamma = -\frac{F_V' G_V'}{F_\pi^2} = \frac{F_V G_V}{F_\pi^2} - 1 = -0.021 \pm 0.031$.

Vector Form Factor: Breit-Wigner parametrization

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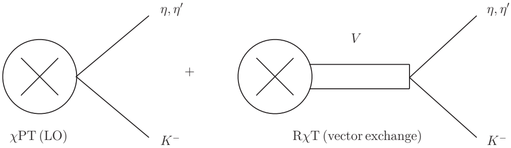

$$\widetilde{F}_+^{K^-\eta^{(\prime)}}(s) = 1 + \frac{F_V G_V}{F_\pi^2}\frac{s}{M_{K^\star}^2 - s} + \frac{F_V' G_V'}{F_\pi^2}\frac{s}{M_{K^{\star\prime}}^2 - s} \quad (12)$$

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$$\widetilde{F}_+^{K^-\eta^{(\prime)}}(s) = \frac{M_{K^\star}^2 + \gamma s}{M_{K^\star}^2 - s - iM_{K^\star}\Gamma_{K^\star}(s)} - \frac{\gamma s}{M_{K^{\star\prime}}^2 - s - iM_{K^{\star\prime}}\Gamma_{K^{\star\prime}}(s)} \quad (13)$$

violates analyticity

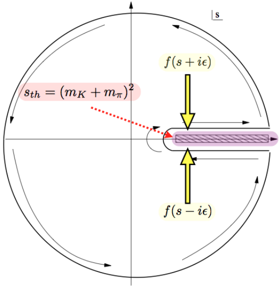
where $\gamma = -\frac{F_V' G_V'}{F_\pi^2} = \frac{F_V G_V}{F_\pi^2} - 1 = -0.021 \pm 0.031$.

$$\boxed{\widetilde{F}_+^{K^-\eta^{(\prime)}}(s) = \widetilde{F}_+^{K^-\pi}(s)}$$

$\widetilde{F}_+^{K^-\pi}(s)$ well under control from $\tau \to K_S\pi\nu_\tau$

- Jamin-Pich-Portolés: Phys.Lett. B640 (2006)
- Boito-Escribano-Jamin: JHEP 1009 (2010) 031

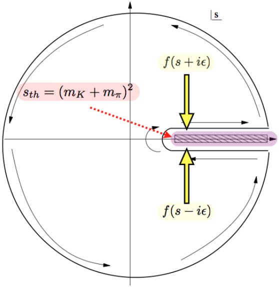
Vector Form Factor: Exponential parametrization (Guerrero-Pich: Phys.Lett. B412 (1997))



analyticity through dispersion relation

$$F_+^{K\pi}(s) = \frac{1}{\pi}\int_{(M_K+M_\pi)^2}^{\infty} ds' \, \frac{\mathrm{Im} f(s')}{s'(s'-s-i\epsilon)}$$

Vector Form Factor: Exponential parametrization (Guerrero-Pich: Phys.Lett. B412 (1997))



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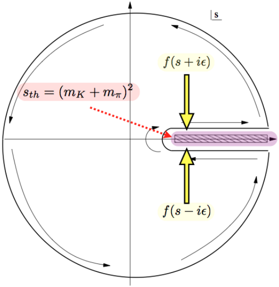
unitarity + watson's theorem

$$\mathrm{Im}f(s') = |f(s')|\sin\delta_1^{1/2}(s') = \tan\delta_1^{1/2}(s')\mathrm{Re}f(s')$$

Vector Form Factor: Exponential parametrization (Guerrero-Pich: Phys.Lett. B412 (1997))



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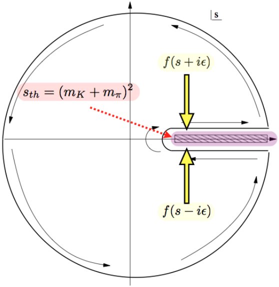
$$\mathrm{Im} f(s') = |f(s')| \sin\delta_1^{1/2}(s') = \tan\delta_1^{1/2}(s') \mathrm{Re} f(s')$$

Omnès solution

$$F_+^{K\pi}(s) = P(s)\exp\left[\frac{s}{\pi}\int_{(M_K+M_\pi)^2}^{\infty} ds' \frac{\delta_1^{1/2}(s')}{s'(s'-s-i0)}\right] \quad (14)$$

Vector Form Factor: Exponential parametrization (Guerrero-Pich: Phys.Lett. B412 (1997))



analyticity through dispersion relation

$$F_+^{K\pi}(s) = \frac{1}{\pi}\int_{(M_K+M_\pi)^2}^{\infty} ds' \frac{\mathrm{Im} f(s')}{s'(s'-s-i\epsilon)}$$

↓ **unitarity + watson's theorem**

$$\mathrm{Im} f(s') = |f(s')| \sin\delta_1^{1/2}(s') = \tan\delta_1^{1/2}(s') \mathrm{Re} f(s')$$

↓ **Omnès solution**

$$F_+^{K\pi}(s) = P(s) \exp\left[\frac{s}{\pi}\int_{(M_K+M_\pi)^2}^{\infty} ds' \frac{\delta_1^{1/2}(s')}{s'(s'-s-i0)}\right] \quad (14)$$

$$F_{K\pi,+}^{ChPT+VMD}(s) \ (Guerrero-Pich: Phys.Lett.\ B412\ (1997)\ and\ P.Roig's\ talk) \quad (15)$$

⇒ Mathching Eq.(14) and Eq.(15) one finds (Jamin-Pich-Portolés: Phys.Lett. B640 (2006))

$$F_+^{K\pi}(s) = \left[\frac{M_{K^\star}^2 + \gamma s}{M_{K^\star}^2 - s - iM_{K^\star}\Gamma_{K^\star}(s)} - \frac{\gamma s}{M_{K^{\star\prime}}^2 - s - iM_{K^{\star\prime}}\Gamma_{K^{\star\prime}}(s)}\right] \exp\left\{\frac{3}{2} Re\left[\widetilde{H}_{K\pi}(s) + \widetilde{H}_{K\eta}(s)\right]\right\}$$

Vector Form Factor: Dispersive representation

(Boito-Escribano-Jamin: JHEP 1009 (2010) 031)

3-times subtracted dispersion relations → helps the convergence of the form factor

$$\widetilde{F}_+(s) = \exp\left[\alpha_1 \frac{s}{m_\pi^2} + \frac{1}{2}\alpha_2 \frac{s^2}{m_\pi^4} + \frac{s^3}{\pi}\int_{s_{K\pi}}^{s_{cut}} ds' \frac{\delta(s')}{(s')^3(s'-s-i0)}\right] \quad (16)$$

The input phase, $\delta(s)$, is obtained as

analyticity and unitarity

$$\delta(s) = \tan^{-1}\left[\frac{\mathrm{Im}\widetilde{f}_+(s)}{\mathrm{Re}\widetilde{f}_+(s)}\right], \quad (17)$$

where $\widetilde{f}_+(s)$ resums the real part of the two-point loop function in the denominator:

$$\widetilde{f}_+(s) = \frac{m_{K^\star}^2 - \kappa_{K^\star}\widetilde{H}_{K\pi}(0) + \gamma s}{D(m_{K^\star}, \gamma_{K^\star})} - \frac{\gamma s}{D(m_{K^{\star\prime}}, \gamma_{K^{\star\prime}})}. \quad (18)$$

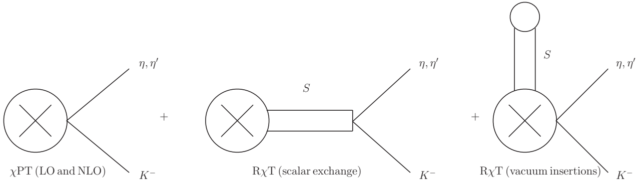
The denominators in eq. (18) are $D(m_n, \gamma_n) \equiv m_n^2 - s - \kappa_n Re\left[H_{K\pi}(s)\right] - i m_n \gamma_n(s)$, where

$$\kappa_n = \frac{192\pi F_K F_\pi}{\sigma^3(m_n^2)}\frac{\gamma_n}{m_n}, \quad \gamma_n(s) = \gamma_n \frac{s}{m_n^2}\frac{\sigma_{K\pi}^3(s)}{\sigma_{K\pi}^3(m_n^2)}\sigma(m_P^2), \quad \sigma_{PP}(s) = \sqrt{1 - \frac{4m_P^2}{s}} \quad (19)$$

⋆ No available data of $K^-\pi^0$ mode ⇒ refit $K_S\pi^-$ using the K^- and π^0 masses

Scalar Form Factor: Breit-Wigner

$$(m_s - m_u)\langle K^-\eta^{(\prime)}|\bar{s}u|0\rangle = \Delta_{K\pi} C^S_{K^-\eta^{(\prime)}} F_0^{K^-\eta^{(\prime)}}(s) =$$



$$\widetilde{F}_0^{K^-\eta^{(\prime)}}(s) = 1 + \frac{c_d c_m}{F_\pi^2}\frac{s}{M^2_{K_0^\star} - s} \quad (20)$$

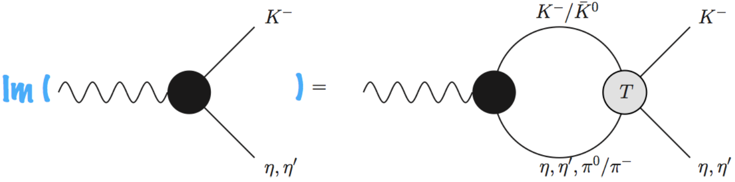
requirement: $\widetilde{F}_0^{K^-\eta^{(\prime)}}(s)$ vanish for $s \to \infty \Rightarrow 4c_d c_m = F_\pi^2,$ *and* $c_d - c_m = 0$

$$\boxed{\widetilde{F}_0^{K^-\eta^{(\prime)}}(s) = \frac{M^2_{K_0^\star}}{M^2_{K_0^\star} - s - iM_{K_0^\star}\Gamma_{K_0^\star}(s)}} \quad (21)$$

violates analyticity

Scalar Form Factor: Coupled channels dispersion relations

$$F_0^{K\eta^{(\prime)}}(s) = \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\mathrm{Im} f(s')}{s'(s'-s-i\epsilon)} \qquad (22)$$



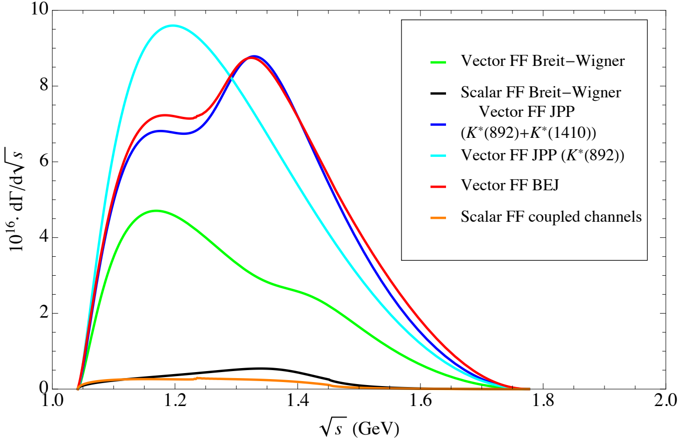
$$F_0^i(s) = \frac{1}{\pi}\sum_{j=1}^{3}\int_{s_i}^{\infty} ds' \frac{\sigma_j(s')F_0^j(s')T_0^{i\to j}(s')^\star}{(s'-s-i0)} \quad (23)$$

Analytic and Unitary ✓

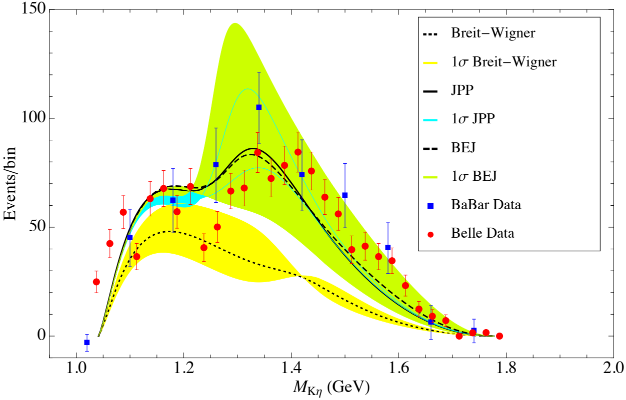
(Jamin-Oller-Pich: Nucl.Phys. B622 (2002))

Decay spectrum of $\tau \to K\eta\nu_\tau$

- Decay dominated by the vectorial sector ($\sim$ 96% of the BR)
- Destructive Interference between $K^\star(892)$ and $K^\star(1410)$

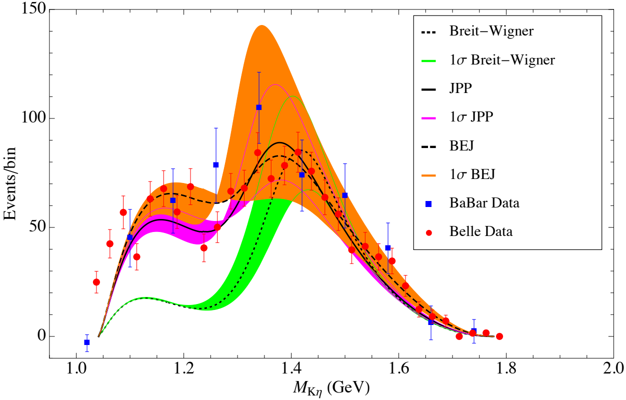


Predictions of $\tau \to K\eta\nu_\tau$

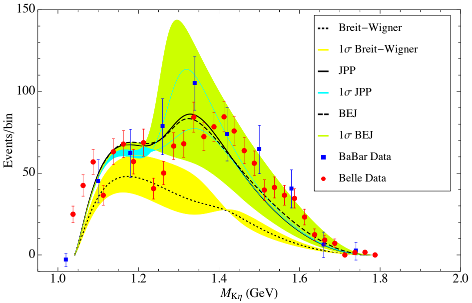


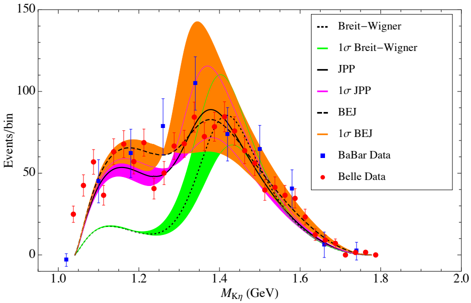
Source	Branchig ratio	χ^2/dof
Dipole Model (BW)	$\left(0.78^{+0.17}_{-0.10}\right) \cdot 10^{-4}$	8.3
JPP	$\left(1.47^{+0.14}_{-0.08}\right) \cdot 10^{-4}$	1.9
BEJ	$(1.49 \pm 0.05) \cdot 10^{-4}$	1.5
Experimental value	$(1.52 \pm 0.08) \cdot 10^{-4}$	-

Fitting the Belle and BaBar data of $\tau \to K\eta\nu_\tau$



Source	Branchig ratio	χ^2/dof
Dipole Model (BW) (Fit γ)	$\left(0.96^{+0.21}_{-0.15}\right)\cdot 10^{-4}$	5.0
Dipole Model (BW) (Fit γ, $M_{K^{\star\prime}}$, $\Gamma_{K^{\star\prime}}$)	Unphysical result	-
JPP (Fit γ)	$\left(1.50^{+0.19}_{-0.11}\right)\cdot 10^{-4}$	1.6
JPP (Fit γ, $M_{K^{\star\prime}}$, $\Gamma_{K^{\star\prime}}$)	$(1.42 \pm 0.04)\cdot 10^{-4}$	1.4
BEJ (Fit γ)	$\left(1.59^{+0.22}_{-0.16}\right)\cdot 10^{-4}$	1.2
BEJ (Fit γ, $M_{K^{\star\prime}}$, $\Gamma_{K^{\star\prime}}$)	$(1.55 \pm 0.08)\cdot 10^{-4}$	0.8
Experimental value	$(1.52 \pm 0.08)\cdot 10^{-4}$	-





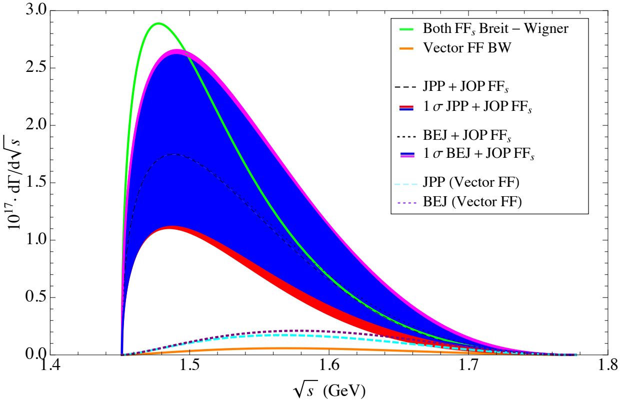
Source	$BR\cdot10^4$	χ^2/dof	Source	$BR\cdot10^4$	χ^2/dof
BW	$0.78^{+0.17}_{-0.10}$	8.3	BW (Fit γ)	$0.96^{+0.21}_{-0.15}$	5.0
			BW (Fit γ, $M_{K^{\star\prime}}$, $\Gamma_{K^{\star\prime}}$)	Unphysical result	-
JPP	$1.47^{+0.14}_{-0.08}$	1.9	JPP (Fit γ)	$1.50^{+0.19}_{-0.11}$	1.6
			JPP (Fit γ, $M_{K^{\star\prime}}$, $\Gamma_{K^{\star\prime}}$)	1.42 ± 0.04	1.4
BEJ	1.49 ± 0.05	1.5	BEJ (Fit γ)	$1.59^{+0.22}_{-0.16}$	1.2
			BEJ (Fit γ, $M_{K^{\star\prime}}$, $\Gamma_{K^{\star\prime}}$)	1.55 ± 0.08	0.8
Experimental value	1.52 ± 0.08		Experimental value	1.52 ± 0.08	-

Source	$M_{K^{\star\prime}}$ (MeV)	$\Gamma_{K^{\star\prime}}$ (MeV)	γ
$K\pi$ mode	1277^{+35}_{-41}	218^{+95}_{-66}	$-0.049^{+0.019}_{-0.016}$
this work ($K\eta$ mode)	1330^{+27}_{-41}	217^{+68}_{-122}	$-0.065^{+0.025}_{-0.050}$

$\Rightarrow$ *th. vs exp.* ✓ $M_{K^\star}(1410)$ *parameters from* $K\pi$ *and* $K\eta$ *are in accordance* ✓

Predictions for $\tau^- \to K^-\eta'\nu_\tau$

Decay dominated by the scalar sector ($\sim 85 - 90\%$ of the BR)



Source	Branchig ratio
Dipole Model (BW) (Fit)	$(1.45^{+3.80}_{-0.87}) \cdot 10^{-6}$
JPP (Fit)	$(1.00^{+0.37}_{-0.29}) \cdot 10^{-6}$
BEJ (Fit)	$(1.03^{+0.37}_{-0.29}) \cdot 10^{-6}$
Experimental bound	$< 2.4 \cdot 10^{-6}$ at 90% C.L.

Summary

- The $K\eta$ mode provides an alternative way of determining the parameters of the $M_{K^\star}(1410)$ which are in accordance with other authors results from the widely investigated $K\pi$ channel
- To remark the importance of the final-state interactions
- Agreement between theory and the data from Belle and BaBar at σ level
- Our BR predictions for the $K\eta'$ mode respects the current experimental upper bound from BaBar

Thank you for your attention!

Width of the resonance propagators

Defining $\sigma_{PQ}(s) = \frac{2q_{PQ}(s)}{\sqrt{s}}\theta[s-(m_P+m_Q)^2]$ and $\sigma_P(s) = \sqrt{1-\frac{4m_P^2}{s}}\theta(s-4m_P^2)$ togehter with q_{PQ} defined in Eq.(7) we have:

$$\Gamma_{K^\star}(s) = \Gamma_{K^\star}(M_{K^\star}^2)\frac{s}{M_{K^\star}^2}\frac{\sigma_{K\pi}^3(s)+\cos^2\theta_{\rm P}\sigma_{K\eta}^3(s)+\sin^2\theta_{\rm P}\sigma_{K\eta'}^3(s)}{\sigma_{K\pi}^3(M_{K^\star}^2)+\cos^2\theta_{\rm P}\sigma_{K\eta}^3(M_{K^\star}^2)+\sin^2\theta_{\rm P}\sigma_{K\eta'}^3(M_{K^\star}^2)} \tag{25}$$

$$\Gamma_\kappa(s) = \Gamma_\kappa(M_\kappa^2)\left(\frac{s}{M_\kappa^2}\right)^{3/2}\times$$

$$\frac{\frac{3}{2}\sigma_{K\pi}(s)+\frac{1}{6}\sigma_{K\eta}(s)\left[{\rm c}\theta_{\rm P}(1+\frac{3\Delta_{K\pi}+\Delta_{K\eta}}{s})+2\sqrt{2}{\rm s}\theta_{\rm P}(1+\frac{\Delta_{K\eta}}{s})\right]^2+\frac{4}{3}\sigma_{K\eta'}(s)\left[{\rm c}\theta_{\rm P}(1+\frac{\Delta_{K\eta'}}{s})-\frac{{\rm s}\theta_{\rm P}}{2\sqrt{2}}(1+\frac{3\Delta_{K\pi}+\Delta_{K\eta'}}{s})\right]^2}{\frac{3}{2}\sigma_{K\pi}(M_\kappa^2)+\frac{1}{6}\sigma_{K\eta}(M_\kappa^2)\left[{\rm c}\theta_{\rm P}(1+\frac{3\Delta_{K\pi}+\Delta_{K\eta}}{M_\kappa^2})+2\sqrt{2}{\rm s}\theta_{\rm P}(1+\frac{\Delta_{K\eta}}{M_\kappa^2})\right]^2+\frac{4}{3}\sigma_{K\eta'}(M_\kappa^2)\left[{\rm c}\theta_{\rm P}(1+\frac{\Delta_{K\eta'}}{M_\kappa^2})-\frac{{\rm s}\theta_{\rm P}}{2\sqrt{2}}(1+\frac{3\Delta_{K\pi}+\Delta_{K\eta'}}{M_\kappa^2})\right]^2} \tag{26}$$