Stability of local quantum dissipative systems

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joint work with
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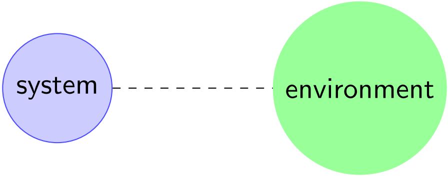
Dissipative quantum systems

Let H be a finite-dimensional Hilbert space.

A dissipative quantum system is given by a 1-parameter continuous semigroup $(T_t)_{t \geqslant 0}$ of completely positive, trace preserving (CPTP) maps (also called quantum channels):

$$T_t : \mathcal{B}(H) \to \mathcal{B}(H)$$



Physically, this models to a system weakly coupled with an environment.

Liouvillian: generator of dissipative evolution

The generator $\mathcal{L}$ of a semigroup of quantum channels is called Liouvillian.

For time-homogeneous dynamics:

$$T_t = e^{t\mathcal{L}} \longleftrightarrow \mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}t} T_t \Big|_{t=0}$$

The properties of T_t force $\mathcal{L}$ to have a very particular structure, called the Lindblad-Kossakowski form:

$$\mathcal{L}(\rho) = i[H, \rho] + \sum_i K_i \rho K_i^\dagger - \frac{1}{2}\{K_i K_i^\dagger, \rho\}$$

[see e.g. M. Wolf, *Quantum Channels & Operations. Guided Tour* for details]

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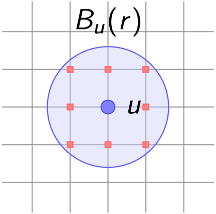
In dissipative dynamics, the Liouvillian plays the analogous role to the Hamiltonian in unitary dynamics (it encodes all the physical properties of the system).

Liouvillians on many-body quantum systems

$$\mathcal{L} = \sum_{u \in \Lambda} \sum_{r \geqslant 0} \mathcal{L}_u(r); \quad \operatorname{supp} \mathcal{L}_u(r) = B_u(r)$$



On many-body quantum systems on a lattice Λ, it is natural to assume locality of the Liouvillian:

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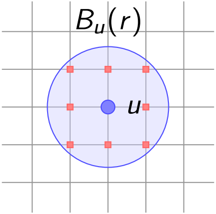

We usually assume either:

Finite range: $\mathcal{L}_u(r) = 0$ for $r > r^*$

Exponential decay: $\|\mathcal{L}_u(r)\|_{1\to 1} \leqslant e^{-\alpha r}$

Power law decay: $\|\mathcal{L}_u(r)\|_{1\to 1} \leqslant (1+r)^{-\alpha}$

For the rest of the talk, just consider exponential decay, but results can be generalised to polynomial decay.

Motivation

Why are they interesting?

- Theoretical models for some kind of open evolutions
  - Modelling of noise
- Dissipative quantum computation
- Dissipative state engineering
  - Theoretical work: [Kraus et al, 2008] [Verstraete, Wolf, Cirac, 2008]
  - Experimental implementations: [Barreiro et al, 2010] [Krauteret al, 2011]

Stability is crucial for applicability

Let O_A be an observable supported on $A \subset \Lambda$
and $O_A(t)$ it's evolution under $\mathcal{L}$ (in the Heisenberg picture).

We consider a perturbed evolution given by $\widetilde{\mathcal{L}} = \sum_{u,r} \widetilde{\mathcal{L}}_u(r)$ such that

$$\left\| \widetilde{\mathcal{L}}_u(r) - \mathcal{L}_u(r) \right\|_{1\to 1} \leqslant \varepsilon \left\| \mathcal{L}_u(r) \right\|_{1\to 1}$$

The problem

Let $\widetilde{O}_A(t)$ be the perturbed observable. Under which conditions can we conclude

$$\forall t \geqslant 0, \quad \left\| O_A(t) - \widetilde{O}_A(t) \right\| \leqslant k_A \, \varepsilon \quad ?$$

It is not just standard perturbation theory

The problem

$$\frac{\left\|\widetilde{\mathcal{L}}_u(r) - \mathcal{L}_u(r)\right\|_{1\to 1}}{\|\mathcal{L}_u(r)\|_{1\to 1}} \leqslant \varepsilon \quad \overset{?}{\Longrightarrow} \quad \left\|O_A(t) - \widetilde{O}_A(t)\right\| \leqslant k_A\, \varepsilon, \quad \forall t$$

Remark

ε *is the microscopic strength of the perturbation, not its global norm:*

$$\frac{\left\|\widetilde{\mathcal{L}}_u(r) - \mathcal{L}_u(r)\right\|_{1\to 1}}{\|\mathcal{L}_u(r)\|_{1\to 1}} \leqslant \varepsilon \quad \textit{but} \quad \left\|\mathcal{L} - \widetilde{\mathcal{L}}\right\|_{1\to 1} \to \infty$$

Conditions for stability

Conditions for stability:

- unique fixed point (not necessary of full rank) and no periodic points
- rapid mixing

- bulk interactions are defined independently of the system size

Rapid mixing

Let $T_t = e^{t\mathcal{L}}$. We define the contraction of T_t the number

$$\eta(T_t) = \frac{1}{2} \sup_{\rho} \| T_t(\rho) - T_\infty(\rho) \|_1 .$$

We say that $\mathcal{L}$ satisfies rapid mixing if

$$\eta(T_t) \leqslant \text{poly}(|\Lambda|) e^{-\gamma t}.$$

Equivalently:

$$t_{mix}(\varepsilon) \leqslant O(\log N/\varepsilon).$$

Rapid mixing

We say that $\mathcal{L}$ satisfies rapid mixing if

$$\eta(T_t) \leqslant \text{poly}(|\Lambda|)e^{-\gamma t}.$$

Recent work has generalized Logarithmic Sobolev inequalities to the quantum setting [Kastoryano, Temme, 2012].

A size-independent log-Sobolev constant implies exactly the type of convergence required by rapid mixing (but it is not needed, i.e. rapid mixing is well defined if the fixed point is pure).

Stability theorem

Let $\mathcal{L}$ be a local Liouvillian with a unique fixed point, that satisfies rapid mixing.

Let $E = \sum_u \sum_r E_u(r)$ a local perturbation: $\|E_u(r)\|_{1\to 1} \leqslant \varepsilon e(r)$, and

$$\widetilde{\mathcal{L}}_u(r) = \mathcal{L}_u(r) + E_u(r)$$

Then of all observables O_A supported on $A \subset \Lambda$ we have that

$$\forall t \geqslant 0, \quad \left\| O_A(t) - \widetilde{O}_A(t) \right\| \leqslant \text{poly}(|A|)\, \|O_A\|\, \varepsilon$$

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Exponential decay of correlations/mutual information

The fix point of $\mathcal{L}$ satisfies:

$$I(A:B) \leqslant \text{poly}(|A| + |B|) e^{-\gamma d_{AB}}$$

Sketch of the proof

Proof

Decompose using the integral representation:

$$O_A(t) - \widetilde{O}_A(t) = \sum_u \sum_r \int_0^t \widetilde{T}^*_{t-s} E^*_u(r) O_A(s) \, \mathrm{d}s$$

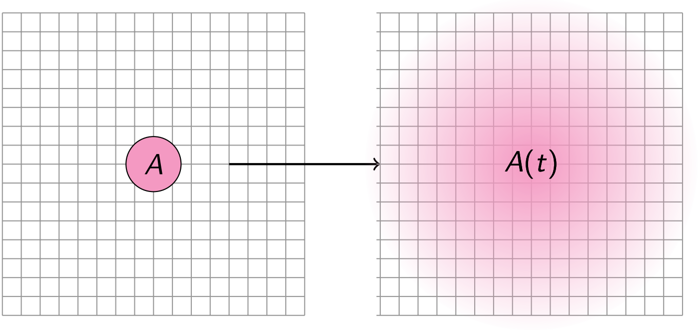
Take norms

$$\left\| O_A(t) - \widetilde{O}_A(t) \right\| \leqslant \sum_u \sum_r \int_0^t \| E^*_u(r) O_A(s) \| \, \mathrm{d}s$$

Lieb-Robinson Bounds

In many-body systems (Hamiltonian, dissipative) there is a finite speed of propagation of information. This is given by the Lieb-Robinson bound.

The support of a local observables spreads linearly in time (in the Heisenberg picture), up to an exponentially-small error.



[Nachtergaele, Vershynina, Zagrebnov, 2011]

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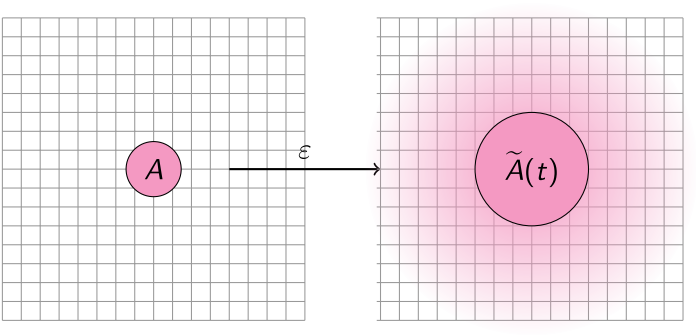
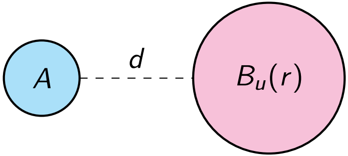


[Nachtergaele, Vershynina, Zagrebnov, 2011]

Short times



Proof

For "short" times $t \leqslant t_0$ we apply Lieb-Robinoson bounds

$$\int_0^{t_0} \left\| E_u^*(r) O_A(s) \right\| \mathrm{d}s \leqslant \varepsilon e(r) \, |A| \, e^{v t_0} e^{-\mu d}$$

where $d = \mathrm{dist}(A, B_u(r))$.

Long times

Proof

For "long" times $t \geqslant t_0$ we insert the fixed point (since $E_u^*(r)\mathbb{1} = 0$):

$$\int_{t_0}^{t} \|E_u^*(r) O_A(s)\| \, \mathrm{d}s = \int_{t_0}^{t} \|E_u^*(r)[O_A(s) - O_A(\infty)]\| \, \mathrm{d}s$$

$$\leqslant \|E_u(r)\|_{1\to 1} \int_{t_0}^{\infty} \|O_A(s) - O_A(\infty)\| \, \mathrm{d}s$$

We are looking for a bound on $\|O_A(s) - O_A(\infty)\|$ independent of the system size.

Local rapid mixing

Definition

Let $A \subset \Lambda$, $T_t = e^{t\mathcal{L}}$. We define the contraction of T_t relative to A the quantity

$$\eta^A(T_t) = \frac{1}{2} \sup_{\rho} \left\| \operatorname{tr}_{A^c} \left[T_t(\rho) - T_\infty(\rho) \right] \right\|_1 .$$

We say that $\mathcal{L}$ satisfies local rapid mixing if for all $A \subset \Lambda$

$$\eta^A(T_t) \leqslant \operatorname{poly}(|A|) e^{-\gamma t}.$$

Remark

η^A in general depends on the whole system, but we are asking for the prefactor to be independent of global system size.

Long times

Proof

By local rapid mixing:

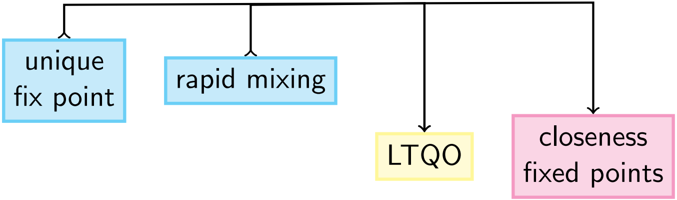
$$\int_{t_0}^{\infty} \|O_A(s) - O_A(\infty)\| \, \mathrm{d}s \leqslant \mathrm{poly}\,|A| \int_{t_0}^{\infty} e^{-\gamma s} \, \mathrm{d}s$$

Putting short and long times toghether yields:

$$\int_0^t \|E_u^*(r) O_A(s)\| \, \mathrm{d}s \leqslant \varepsilon e(r) \, \mathrm{poly}\,|A| \left(e^{vt_0} e^{-\mu d} + e^{-\gamma t_0} \right)$$

We are left to choose $t_0 = t_0(d)$ such that the r.h.s. is summable over Λ.

Proving local rapid mixing



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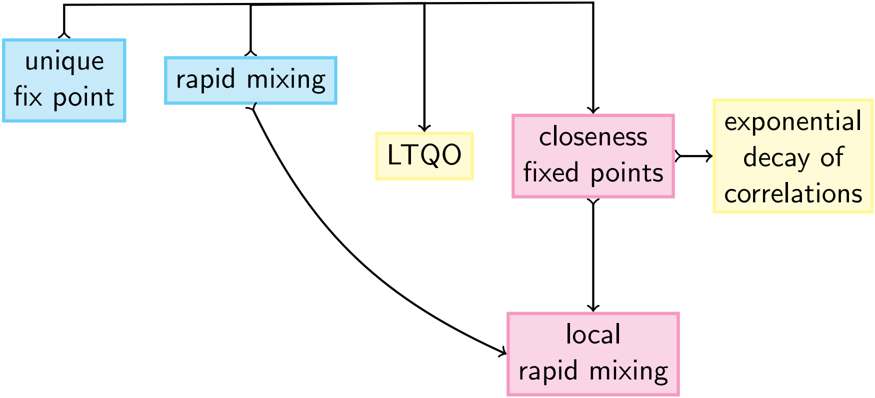
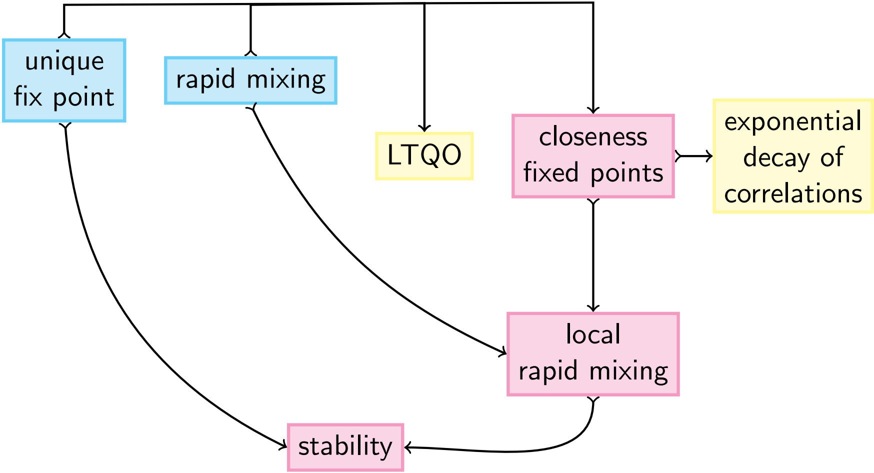


Proving local rapid mixing



Application: classical Glauber dynamics

Glauber dynamics is a classical Markov process sampling from the Gibbs distribution of a finite-range, translationally-invariant classical Hamiltonian on a lattice.

It is the equivalent of the Metropolis-Hastings algorithm in continuous time

It is generated by the following:

$$Qf(\sigma) = \sum_{x \in \Lambda} c(x, \sigma)[f(\sigma^x) - f(\sigma)].$$

$c(x, \sigma)$ are called transition rates, and are chosen to satisfy detailed balance.

We can embed classical Glauber dynamics into a quantum dissipative system, having the same mixing time and fixed points.

The end!

Thank you for your attention

For further reading: arXiv:1303.4744