A tight Landauer Principle with finite-size improvements

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`arXiv:1306.4352 (Landauer's principle)`
`arXiv:1304.0036 (entropy inequalities)`

A common formulation of Landauer's Principle

Suppose a computer "erases" 1 bit of information.

Then: The amount of "heat" "dissipated" into the environment is at least $k_B T \log 2$:

$$\Delta Q \;\geq\; k_B T \log 2 \;,$$

where $T =$ temperature of the environment of the computer.

$$\beta \Delta Q \;\geq\; \Delta S$$ "Landauer bound"

where $k_B \equiv 1 \,,\;\; \beta \equiv 1/T$

Why erasure? E.g. to re-initialize error correcting mechanism.

Existing derivations of Landauer's Principle

- **based on 2^{nd} Law of Thermodyn:** e.g. Landauer '61, . . .
  $\rightarrow$ mix-up of notions (cf. Earman/Norton, Bennett)

- **in specific models:** e.g. 1-particle gas in box
  $\rightarrow$ need to accept thermodyn formalism (e.g. "quasistatic")

- **recently: (more) microscopic**
  - Shizume (1995): *effective* dissipative force (Fokker-Planck)
  - Piechocinska (2000): *Jarzynski equality*
    – assumes: final product state $\rho_S \otimes \rho_R \longmapsto \rho'_S \otimes \rho'_R$
    – assumes: ρ'_S pure
    – assumes: ρ'_R diagonal in energy eigenbasis $\rightarrow$ *quantum?*
  - Sagawa/Ueda (2009): need system Hamiltonian H_S, . . .

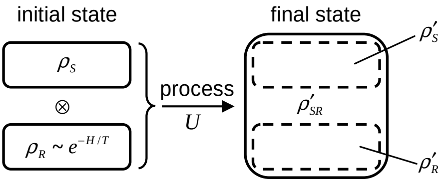
- **claimed "violations" of LP:**
  $\rightarrow$ Nieuwenhuizen '01, Orlov '12, . . .

our work: rigorous and minimal formulation & proof of LP

Overview

1. Minimal formulation & proof
   - $\beta\Delta Q \;=\; \Delta S + I(S' : R') + D(\rho'_R \| \rho_R)$
2. Finite-size effects
   - $\beta\Delta Q \;\geq\; \Delta S + \frac{(\Delta S)^2}{7\log^2 d}$
   - entropy inequalities
3. Applications
   - energy–time tradeoff in achieving $\beta\Delta Q \to \Delta S$
   - Carnot bound

Minimal setup for Landauer's Principle



(0) system S, reservoir R: $\mathcal{H}_{SR} = \mathcal{H}_S \otimes \mathcal{H}_R$

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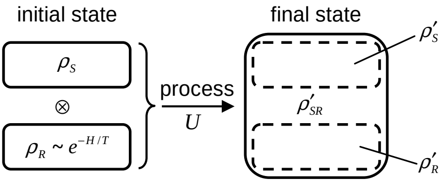

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(1) initially uncorrelated: $\rho_{SR} = \rho_S \otimes \rho_R$

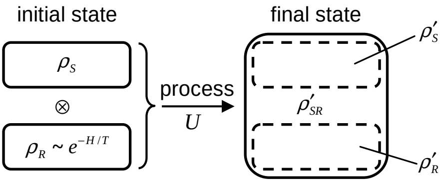
Otherwise: e.g. $\rho_{SR} = \sum_i p_i |i\rangle_S\langle i| \otimes |i\rangle_R\langle i|$.

and $U : |i\rangle_S|i\rangle_R \mapsto |0\rangle_S|i\rangle_R$

$\Rightarrow U\rho_{SR}U^\dagger = |0\rangle_S\langle 0| \otimes \sum_i p_i |i\rangle_R\langle i|$

$\Rightarrow \rho_S$ pure, $\rho'_R = \rho_R$ $\Rightarrow \beta\Delta Q \not\geq \Delta S$

Minimal setup for Landauer's Principle



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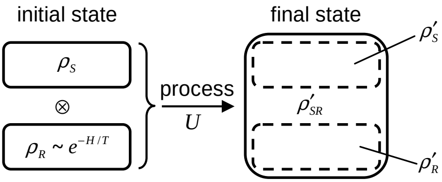
(2) $\rho_R = \frac{e^{-\beta H}}{\mathrm{tr}[e^{-\beta H}]}$ (R-Hamiltonian H, R-temperature $T \equiv 1/\beta$)

- parameter β in Landauer's bound
- "cheaply available" states

Minimal setup for Landauer's Principle



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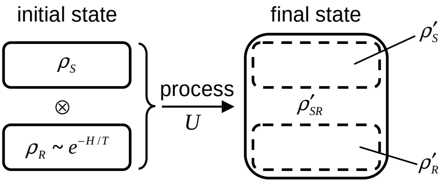
(3) unitary evolution: $\rho'_{SR} = U \rho_{SR} U^\dagger$

- microscopic laws of nature
- no hidden entropy sinks

Minimal setup for Landauer's Principle



$$\Delta S := S(\rho_S) - S(\rho'_S)$$

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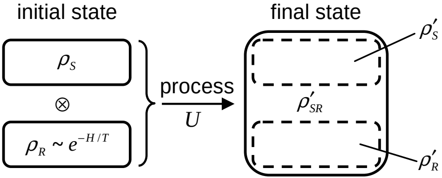

$$\Delta S := S(\rho_S) - S(\rho'_S)$$
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- no Hamiltonian, no temperature for S needed
- ρ'_{SR} may be correlated, ρ'_S need not be pure
- ΔS positive or negative
- classical and quantum ($[H, \rho'_R] \neq 0$)

Equality form of Landauer's Principle

Theorem 1

Let $\rho_{SR} = \rho_S \otimes \rho_R$ be a product state,

with $\rho_R = e^{-\beta H}/\mathrm{Tr}\left[e^{-\beta H}\right]$ thermal state of Hamiltonian H,

and let $\rho'_{SR} := U\rho_{SR}U^\dagger$ with a unitary U.

Then:

$$\begin{aligned} \beta\Delta Q &= \Delta S + I(S' : R') + D(\rho'_R \| \rho_R) \\ &\geq \Delta S\,. \end{aligned}$$

Proof:

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step 1: $\Delta \equiv S(\rho'_R) - S(\rho_R) \overset{(1),\,(3)}{=\!=} \Delta S + I(S' : R')$

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Proof:

step 1: $\Delta \equiv S(\rho'_R) - S(\rho_R) \overset{(1),(3)}{=\!=} \Delta S + I(S' : R')$

step 2: $\beta\Delta Q \equiv \mathrm{Tr}\left[\beta H(\rho'_R - \rho_R)\right] \overset{(2)}{=\!=} \Delta + D(\rho'_R \| \rho_R)$ □

Extensions of $\beta\Delta Q = \Delta S + I(S' : R') + D(\rho'_R \| \rho_R)$

- additional memory system M [del Rio et al. 2011]
  - $\rightarrow$ use $\Delta S_{cond} := S(S|M) - S(S'|M')$,
  - $\rightarrow$ extra term $+[S(M) - S(M')]$
- initial S–R correlations ["violations of LP"]
  - $\rightarrow$ extra term $-I(S : R)$
- deviations from thermal ρ_R
  - $\rightarrow$ extra term $-D\big(\rho_R \| e^{-\beta H}/\mathrm{Tr}[e^{-\beta H}]\big)$
- non-increasing entropy in $\rho_{SR} \mapsto \rho'_{SR}$ (e.g. positive unital map)
  - $\rightarrow$ *in*equality $\beta\Delta Q \geq \Delta S + \ldots$

Equality cases in Landauer's bound

Recall: $\beta \Delta Q \;=\; \Delta S + I(S' : R') + D(\rho'_R \| \rho_R)$

- $D(\rho'_R \| \rho_R) = 0 \quad \Rightarrow \quad \rho'_R \;=\; \rho_R$
- $I(S' : R') = 0 \quad \Rightarrow \quad \rho'_{SR} \;=\; \rho'_S \otimes \rho'_R \;=\; U(\rho_S \otimes \rho_R)U^\dagger$

Thus: $\beta \Delta Q = \Delta S \quad \Leftrightarrow \quad \Delta S \;=\; \Delta Q \;=\; 0$

$\Leftrightarrow$ process is trivial

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- $D(\rho'_R \| \rho_R) = 0 \quad \Rightarrow \quad \rho'_R = \rho_R$
- $I(S' : R') = 0 \quad \Rightarrow \quad \rho'_{SR} = \rho'_S \otimes \rho'_R = U(\rho_S \otimes \rho_R)U^\dagger$

Thus: $\beta \Delta Q = \Delta S \quad \Leftrightarrow \quad \Delta S = \Delta Q = 0$

$\Leftrightarrow$ process is trivial

$\Rightarrow$ explicitly improve Landauer's bound for $\Delta S \neq 0$

But have to assume finite reservoir size $d \equiv \dim(\mathcal{H}_R) < \infty$:

- e.g. *small* error-correcting mechanism
- e.g. *short* interaction time S–R $\rightarrow$ effectively small d

Finite-size effects

Recall: $$\beta \Delta Q \;\geq\; \Delta S + D(\rho'_R \| \rho_R)$$

Lemma A:
$$D(\rho'_R \| \rho_R) \;\geq\; \frac{\Delta^2}{3 \log^2 d} \qquad \big(\Delta \equiv S(\rho'_R) - S(\rho_R)\big)$$
$$\geq\; \frac{(\Delta S)^2}{3 \log^2 d} \qquad \text{if } \Delta S \geq 0\,.$$

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stat mech: $$D(\rho'_R\|\rho_R) \;\geq\; \frac{(\beta\Delta Q)^2}{2\max_{T'} C_H(T')} \qquad \text{if } \beta\Delta Q \leq 0$$

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Lemma B: $$\geq\; \frac{(\beta\Delta Q)^2}{2\log^2 d}$$

re-insert: $$\beta\Delta Q \;\geq\; \Delta S \;+\; \frac{(\Delta S)^2}{7\log^2 d}$$

Relative entropy vs. entropy difference

Lemma A

Let σ, ρ on $\mathbb{C}^d$. Denote $\Delta := S(\sigma) - S(\rho)$.

Then: $$D(\sigma\|\rho) \;\geq\; M(\Delta, d) \;\geq\; \frac{\Delta^2}{3\log^2 d}\,,$$

where
$$M(\Delta, d) := \min_{0\leq s,r\leq 1}\left\{D_2(s\|r)\,\middle|\,H_2(s)-H_2(r)+(s-r)\log(d-1) = \Delta\right\}.$$

- $M(\Delta, d)$: tight bound, effectively computable, strictly convex

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Proof:

step 1: $D(\sigma\|\rho) = \mathrm{Tr}\left[(-\log\rho)\sigma\right] - S(\sigma)$ at fixed ρ, fixed $S(\sigma)$

$\Rightarrow\;\sigma = \rho^\gamma/\mathrm{Tr}\left[\rho^\gamma\right]$ $\quad\rightarrow$ *commuting*

step 2: Lagrange multipliers, discrete optimization $\square$

Intermission: Further QI applications

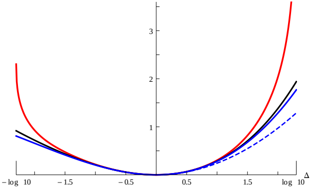
- hypothesis testing, universal source coding:

$$\text{dist}(E,\rho) \;:=\; \inf_{\sigma:\,S(\sigma)>R} D(\sigma\|\rho) \;\geq\; \frac{\big(R-S(\rho)\big)^2}{3\log^2 d}$$

- Shannon capacity of $T_{Y|X} = \begin{pmatrix}\vec{y}_1 & \ldots & \vec{y}_n\end{pmatrix} \in \mathbb{R}^{|Y|\times|X|}$:

$$C(T) \;=\; \min_{\vec{y}} \max_{x} D(\vec{y}_x\|\vec{y}) \;\geq\; \frac{\big(S(\vec{y}_{x'}) - S(\vec{y}_{x''})\big)^2}{12\log^2|Y|}$$

- better than Pinsker + Fannes-Audenaert



$D(\sigma\|\rho) \;\geq\; M(\Delta, d=10)$

- tight bound
- asymmetric $\Delta \leftrightarrow (-\Delta)$
- $\mathcal{O}(\Delta^2)$ for $|\Delta| \to 0$

Maximum heat capacity in dimension $d < \infty$

$$C_H(T) \;=\; \frac{d}{dT}\,\mathrm{tr}\Big[H\,\underbrace{\frac{e^{-H/T}}{\mathrm{tr}[e^{-H/T}]}}_{=:\;\rho_T}\Big] \;=\; \underbrace{\mathrm{Tr}\left[\rho_T\left(\log\rho_T + S(\rho_T)\right)^2\right]}_{=:\;\mathrm{var}_{\rho_T}[\log\rho_T]}$$

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Lemma B

For any state ρ on $\mathbb{C}^d$:

$$\mathrm{var}_\rho[\log\rho] \;\leq\; N(d) \;\leq\; \log^2 d\,.$$

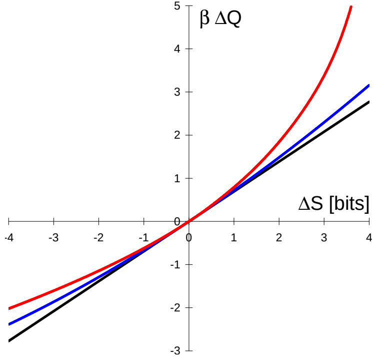
- $N(d)$: tight bound, effectively computable
- "super-extensive": $\log^2 d \simeq$ (#particles)^2
- tight for: $H = \mathrm{diag}(-1, 0, \ldots, 0)$ $\rightarrow$ *strongly interacting*

Finite-size effects in Landauer's Principle

Theorem 2

Let the reservoir R have dimension $d = \dim(\mathcal{H}_R)$.

Then:
$$\beta\Delta Q \;\geq\; \Delta S + \frac{(\Delta S)^2}{7\log^2 d}\,.$$



4-(qu)bit reservoir: $d = 16$

red curve = our best bound (tight for $\Delta S \geq 0$)

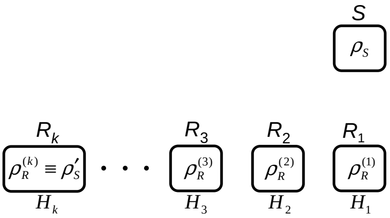
Example: erase $\Delta S = 1$ bit

$\Rightarrow$ at least 14% more heat production

Tradeoff in attaining the bound $\beta \Delta Q \geq \Delta S$

ρ_S, ρ'_S given.

Want $\beta \Delta Q \to \Delta S$
(as $k \to \infty$)



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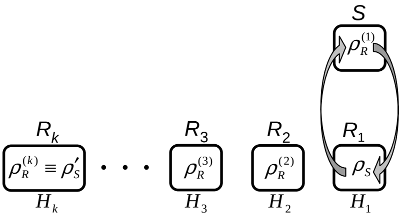

$$\beta \Delta Q = \left[S(\rho_S) - S(\rho_R^{(1)})\right] + D(\rho_S \| \rho_R^{(1)})$$

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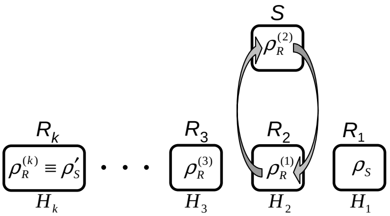

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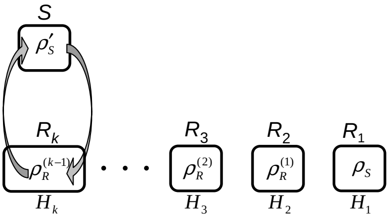

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\beta\Delta Q \;=\;& \left[S(\rho_S) - S(\rho_R^{(1)})\right] + D(\rho_S \| \rho_R^{(1)}) \\
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&+ \ldots \\
&+ \left[S(\rho_R^{(k-1)}) - S(\rho'_S)\right] + D(\rho_R^{(k-1)} \| \rho'_S) \\
=\;& \Delta S + \sum_{i=1}^{k} D\left(\rho_R^{(i-1)} \| \rho_R^{(i)}\right)
\end{aligned}$$

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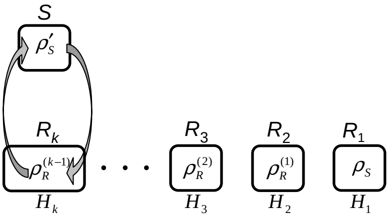

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=\; & \Delta S + \sum_{i=1}^{k} D\big(\rho_R^{(i-1)} \| \rho_R^{(i)}\big) \;\geq\; \Delta S + \frac{(\Delta S)^2}{k \cdot 3 \log^2 d_S}
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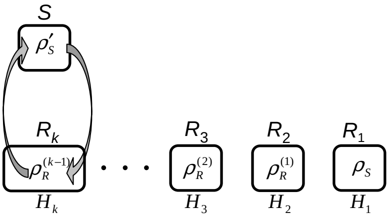
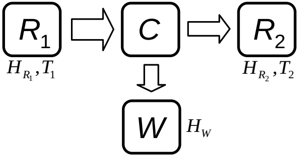


$$\Delta S + \underbrace{\frac{(\Delta S)^2}{k \cdot 3\log^2 d_S}}_{\text{previous slide}} \leq \beta\Delta Q \leq \Delta S + \underbrace{\frac{D(\rho_S \| \rho'_S) + D(\rho'_S \| \rho_S)}{k}}_{\text{[Anders et al.][Skrzypczyk et al.]}}$$

- thermodyn reversibility in the limit $k \to \infty$ (note: $d = d_S^k$)
- $k \simeq$ time duration of process
  $\Rightarrow$ "degree of irreversibility" $\simeq 1/\text{time}$
- $\to$ energy–time tradeoff

Carnot bound



$$\rho \equiv \rho_{WS} \otimes \rho_{R_1} \otimes \rho_{R_2} \quad \longmapsto \quad \rho' \equiv \rho'_{WSR_1R_2}$$

- $S(\rho') \geq S(\rho)$,
- $\mathrm{Tr}\left[(H_W + H_{R_1} + H_{R_2})(\rho' - \rho)\right] \leq 0$.

Then: $$\Delta W \leq \left(1 - \frac{T_2}{T_1}\right) \Delta Q_1 - T_2 \left(S(\rho_{WC}) - S(\rho'_{WC}) \right) ,$$

where $$\Delta W := \mathrm{Tr}\left[H_W(\rho'_W - \rho_W)\right] ,$$

$$\Delta Q_1 := \mathrm{Tr}\left[H_{R_1}(\rho_{R_1} - \rho'_{R_1})\right] .$$

Proof: $\beta H := \beta_1 H_{R_1} + \beta_2 H_{R_2}$ [Pusz/Woronowicz] □

unitary → equality version: $-D(\rho'_{R_1R_2} \| \rho_{R_1R_2}) - I(W'C'|R_1R_2)$

→ finite-size corrections: $$-\frac{\left(S(\rho_{WC}) - S(\rho'_{WC})\right)^2}{7\log(d_1 d_2)}$$

Conclusion

- minimal assumptions: $\rho_S \otimes e^{-\beta H} \stackrel{U}{\longmapsto} \rho'_{SR}$
- LP equality: $\beta \Delta Q \;=\; \Delta S \,+\, I(S' : R') \,+\, D(\rho'_R \| \rho_R)$
- finite-size effects: $\beta \Delta Q \;\geq\; \Delta S + \frac{(\Delta S)^2}{7 \log^2 d}$
  - 14% for reservoir of 4 (qu-)bits
  - model for energy-time tradeoff

- finite-size bounds in other thermodynamic situations ?
- $\beta \Delta Q \,\geq\, F(\rho_S, \rho'_S, d) \,\geq\, S(\rho_S) - S(\rho'_S)$?
- tight bound for $\Delta S < 0$?

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Thank you!

Appendix: “pureness” of final state

$$\lambda_{min}(\rho'_S) \;\geq\; \sum_{i=1}^{d} \lambda_i^{\uparrow}(\rho'_{SR}) \;=\; \sum_{i=1}^{d} \lambda_i^{\uparrow}(\rho_S \otimes \rho_R) \;\geq\; d\,\lambda_{min}(\rho_S)\,\lambda_{min}(\rho_R)$$

$$\lambda_{min}(\rho_R) \;=\; \frac{e^{-\beta H_{max}}}{\mathrm{Tr}\left[e^{-\beta H}\right]} \;\geq\; \frac{e^{-\beta H_{max}}}{d\, e^{-\beta H_{min}}}$$

$$\Rightarrow \quad \frac{\lambda_{min}(\rho'_S)}{\lambda_{min}(\rho_S)} \;\geq\; e^{-\beta(H_{max}-H_{min})} \;\geq\; e^{-2\beta\|H\|}$$

$\rightarrow$ “To erase 1 qubit”, need:

- zero-temperature reservoir ($\beta = \infty$)
- formally $H_{max} = +\infty$ ($\Rightarrow \Delta Q = \infty$)

Analogous: lower bounds on “pureness” $S(\rho'_S)$ via majorization.