Classical and quantum fractal code

; topological order beyond TQFT and asymptotically good quantum LDPC code

[1] Beni Yoshida, Annals of Physics 338, 134 (2013)

[2] Beni Yoshida, Phys. Rev. B 88, 125122 (2013)

Beni Yoshida

Caltech, IQIM

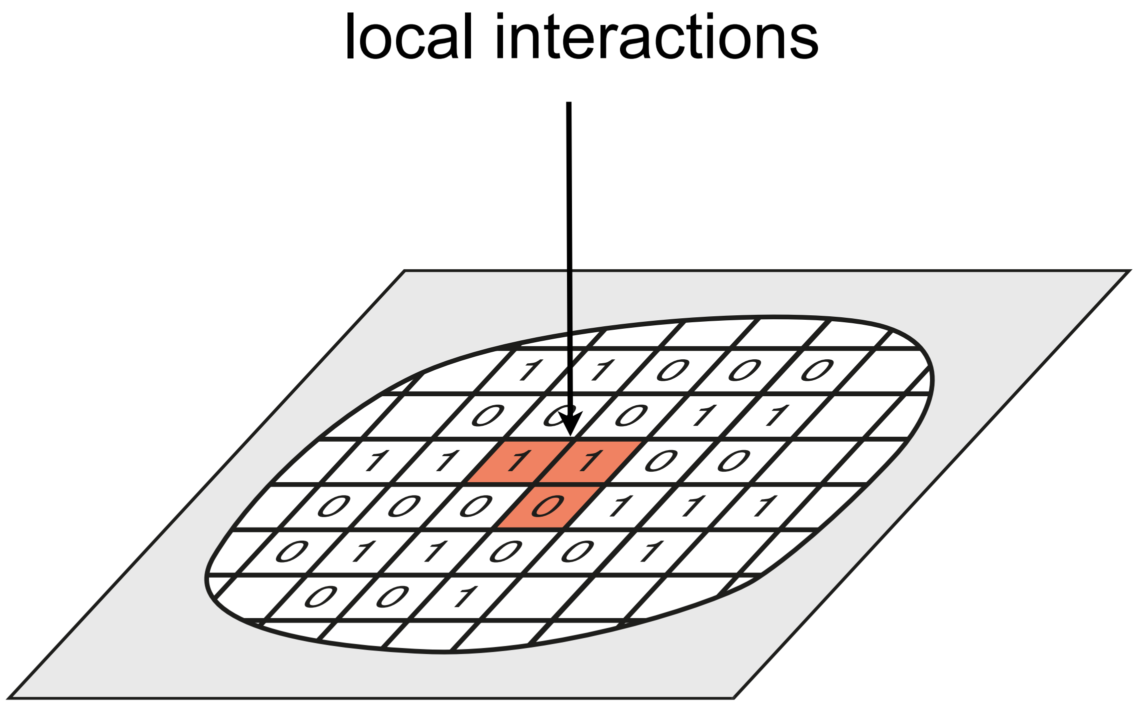
Feb 2014 @ QIP, Barcelona

Question:

- Is there any limit on information storage capacity of physical systems ?

The (classical) local code bound

- Encode information into ground states of a geometrically local Hamiltonian on a D-dim lattice



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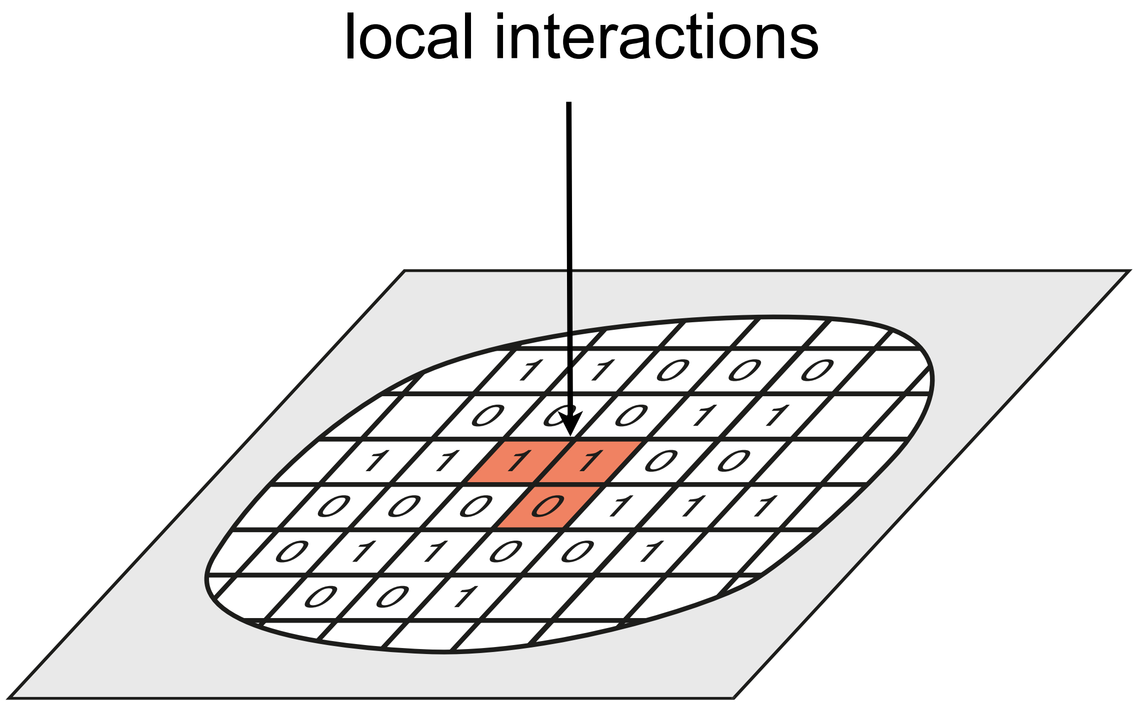


Local Code Bound Bravyi, Terhal and Poulin (2009)

$$kd^{1/D} \leq O(n)$$

k : number of logical bits — Amount

d : code distance — Reliability

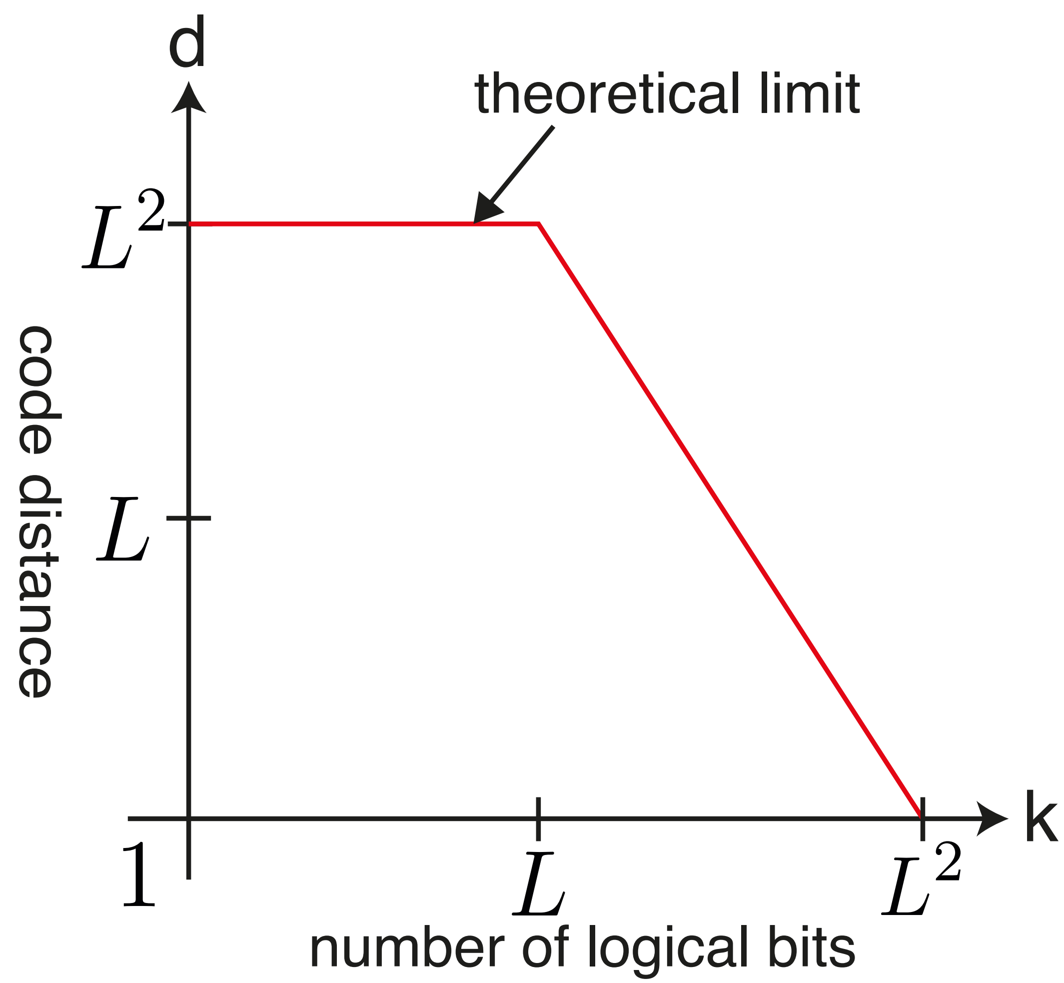
n : total number of spins

Saturation for discrete systems ?

- Previously found systems are far below the bound ...

Bound for D=2

$$kd^{1/2} \leq O(n) \qquad n = L^2$$



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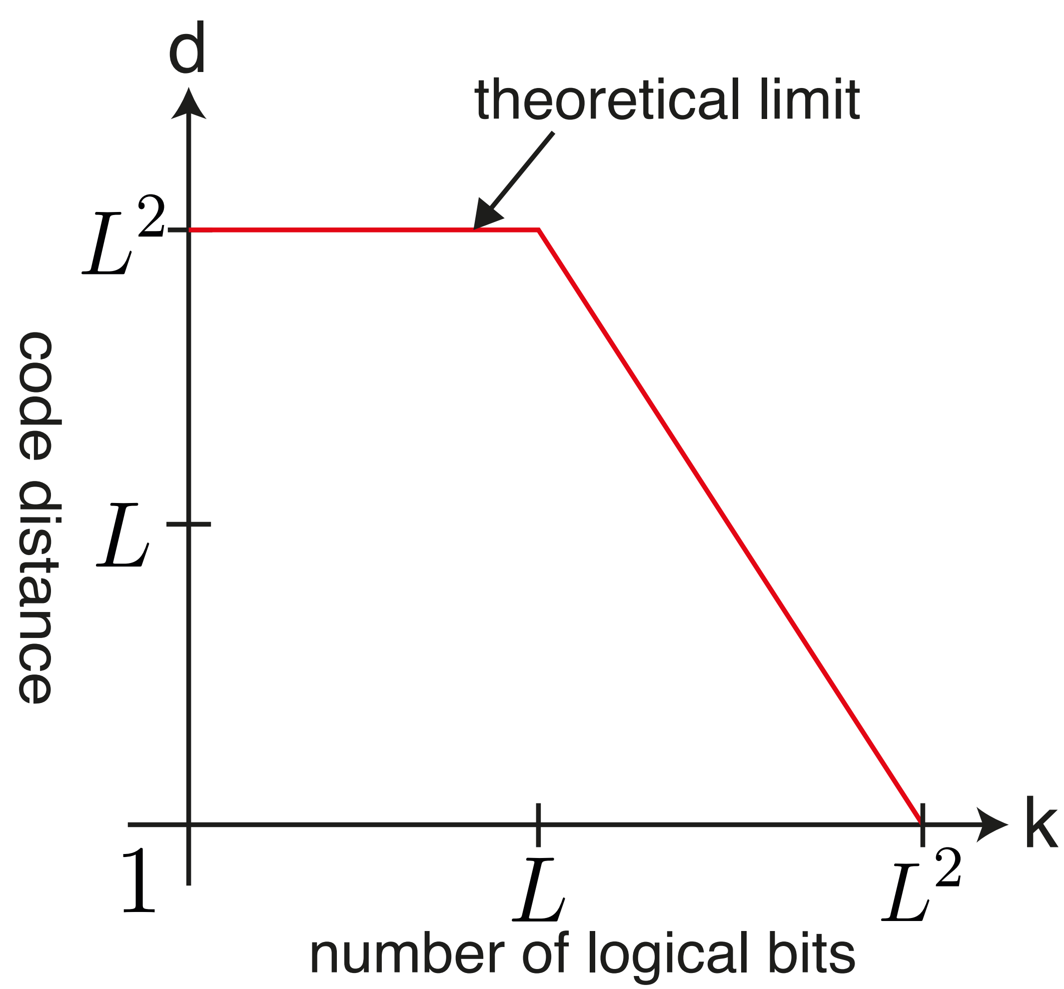
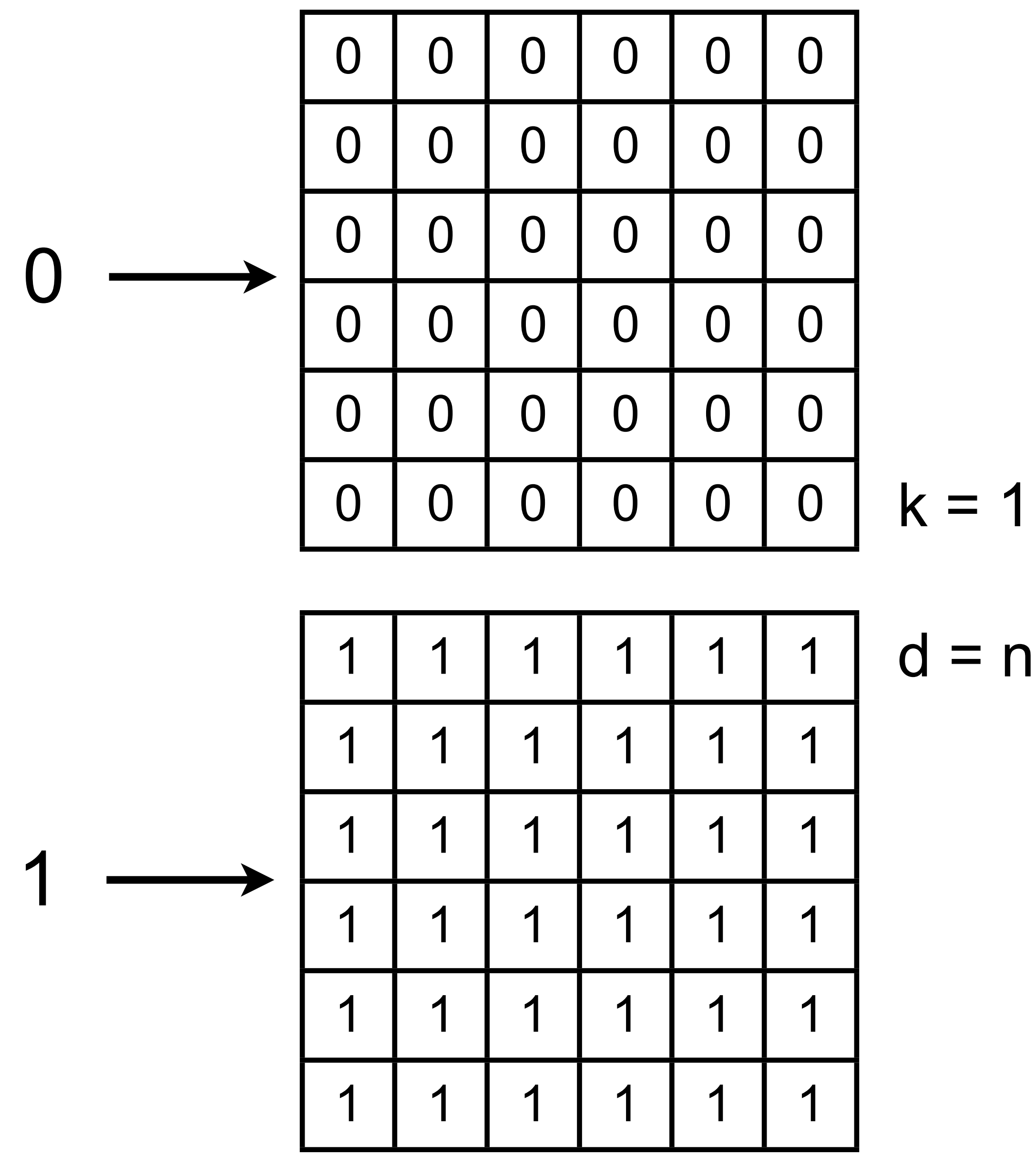


Repetition code



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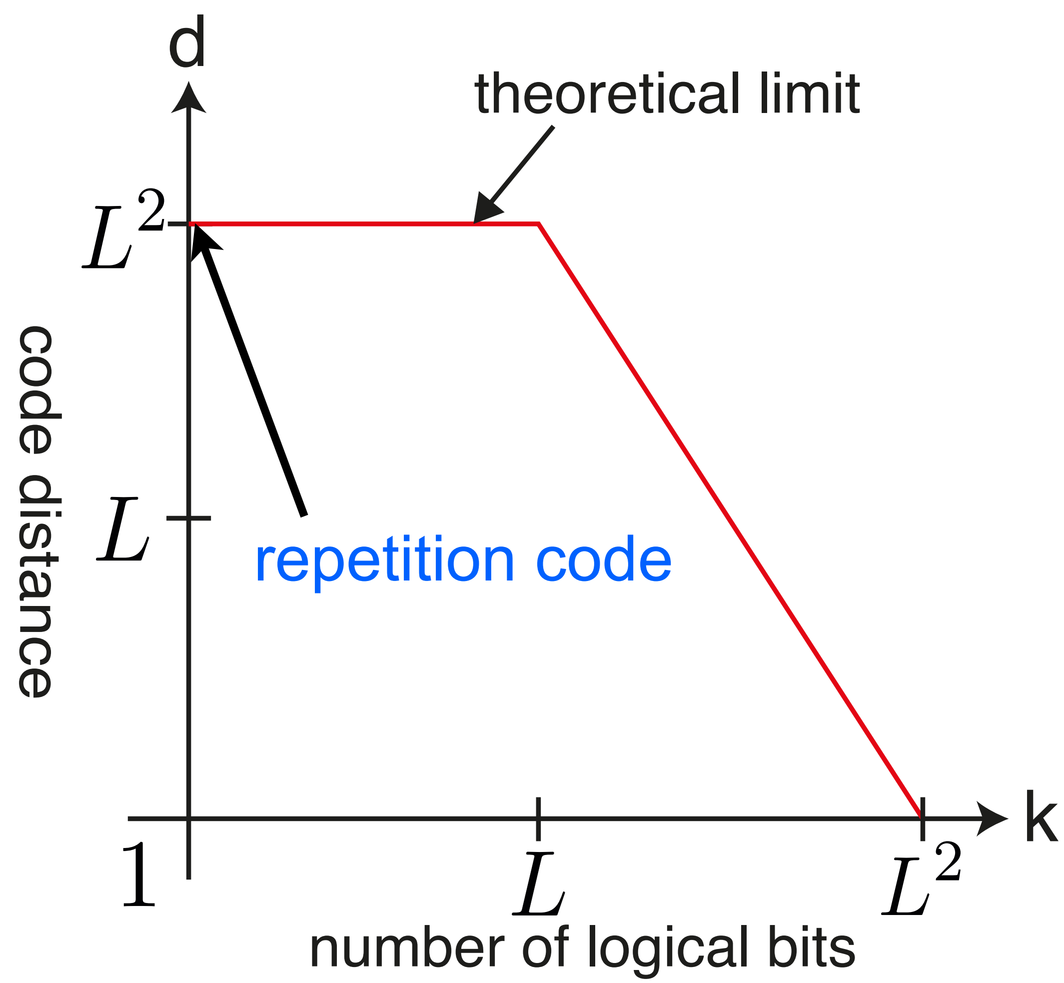

Repetition code

0 →

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

k = 1

d = n

1 →

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

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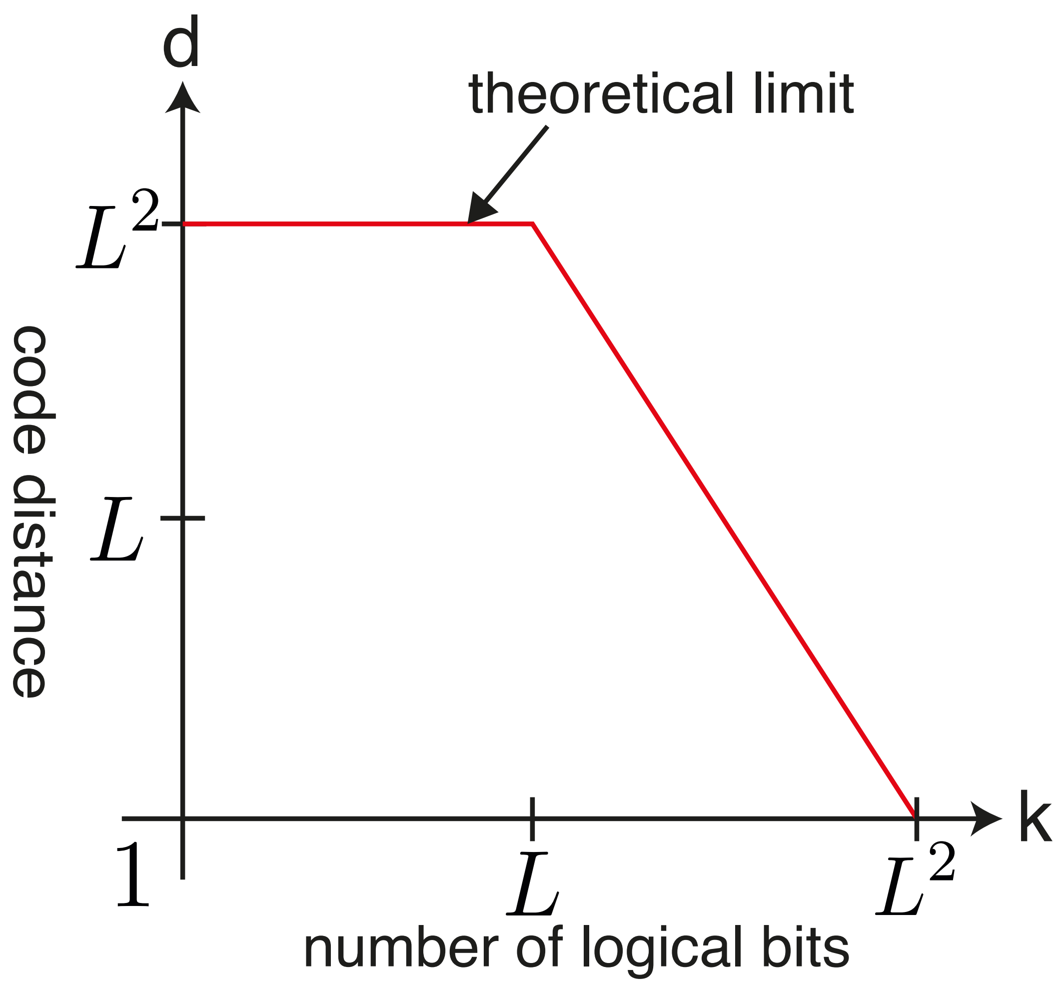
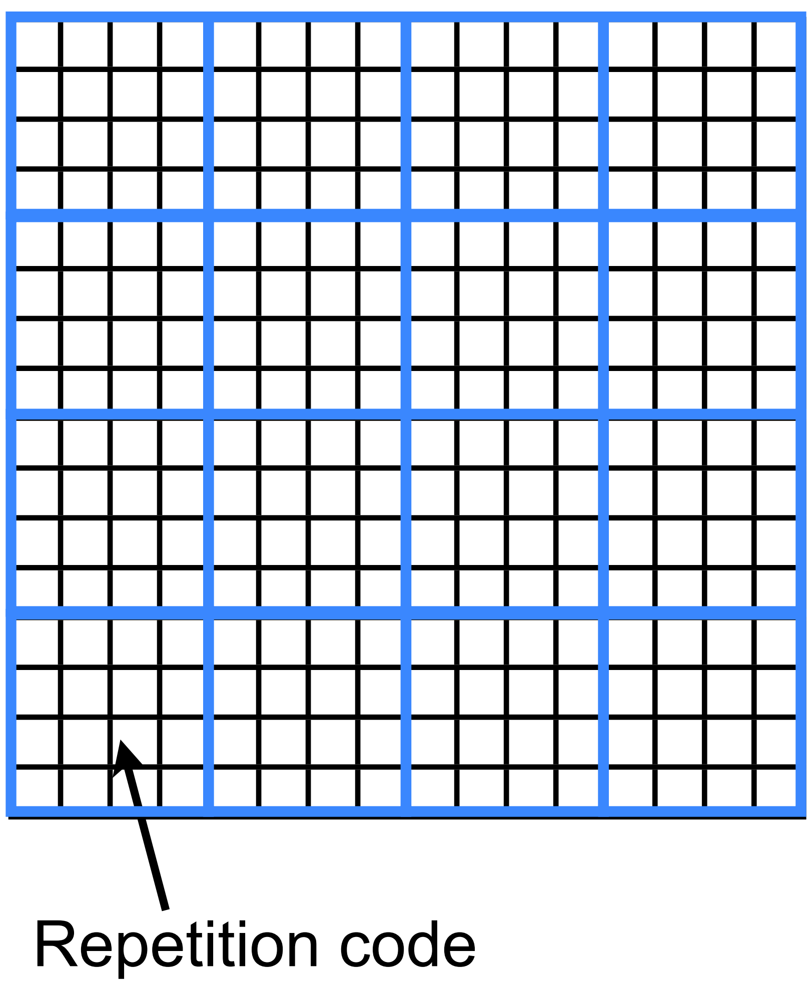


Copies of repetition codes



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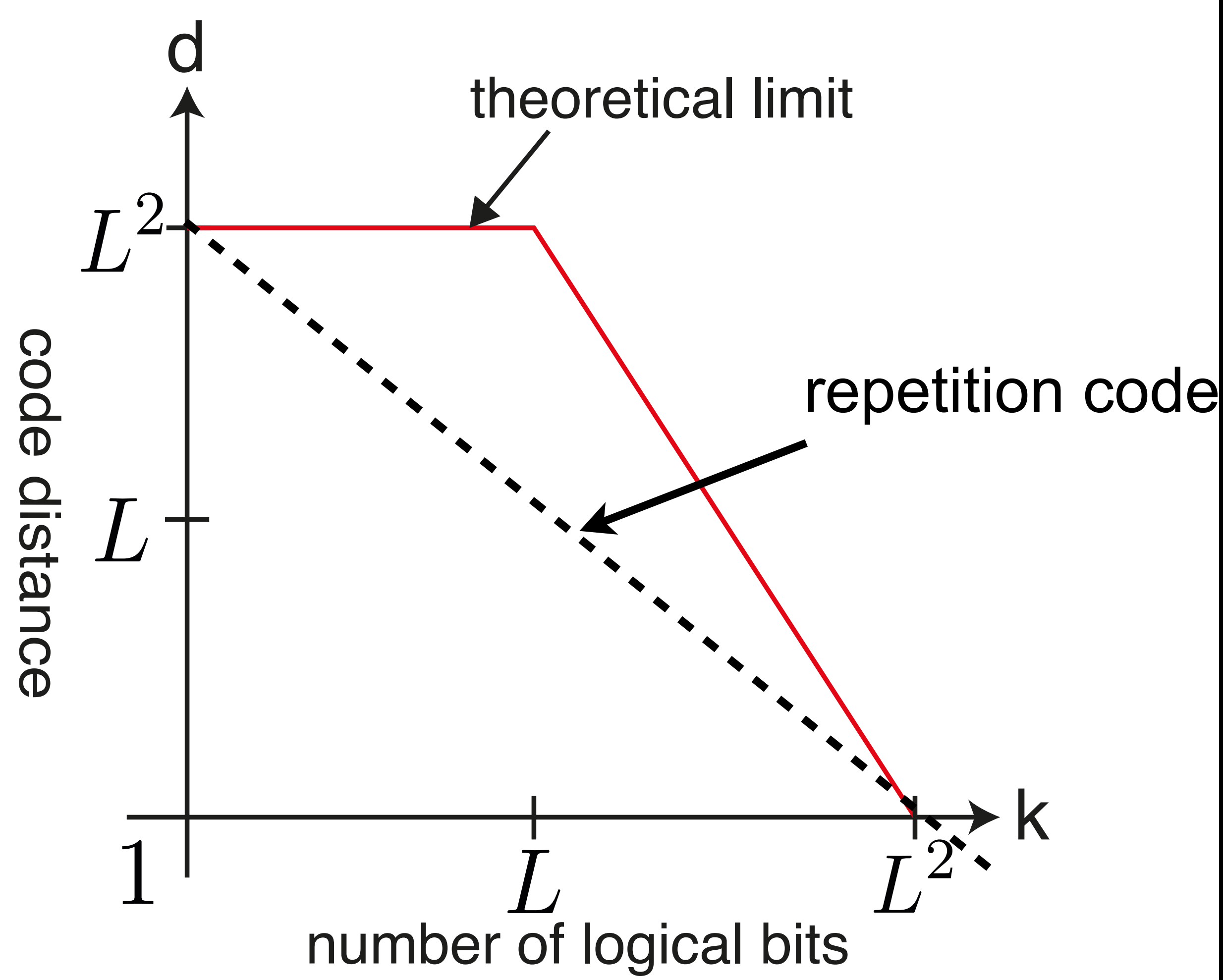

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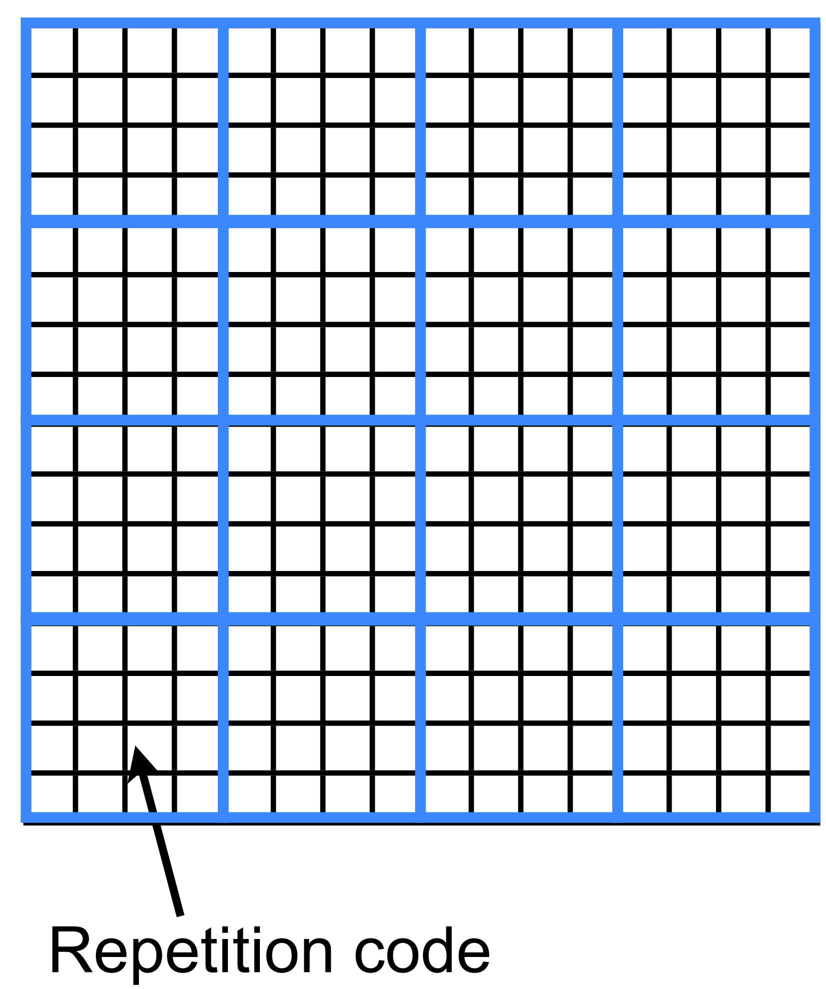

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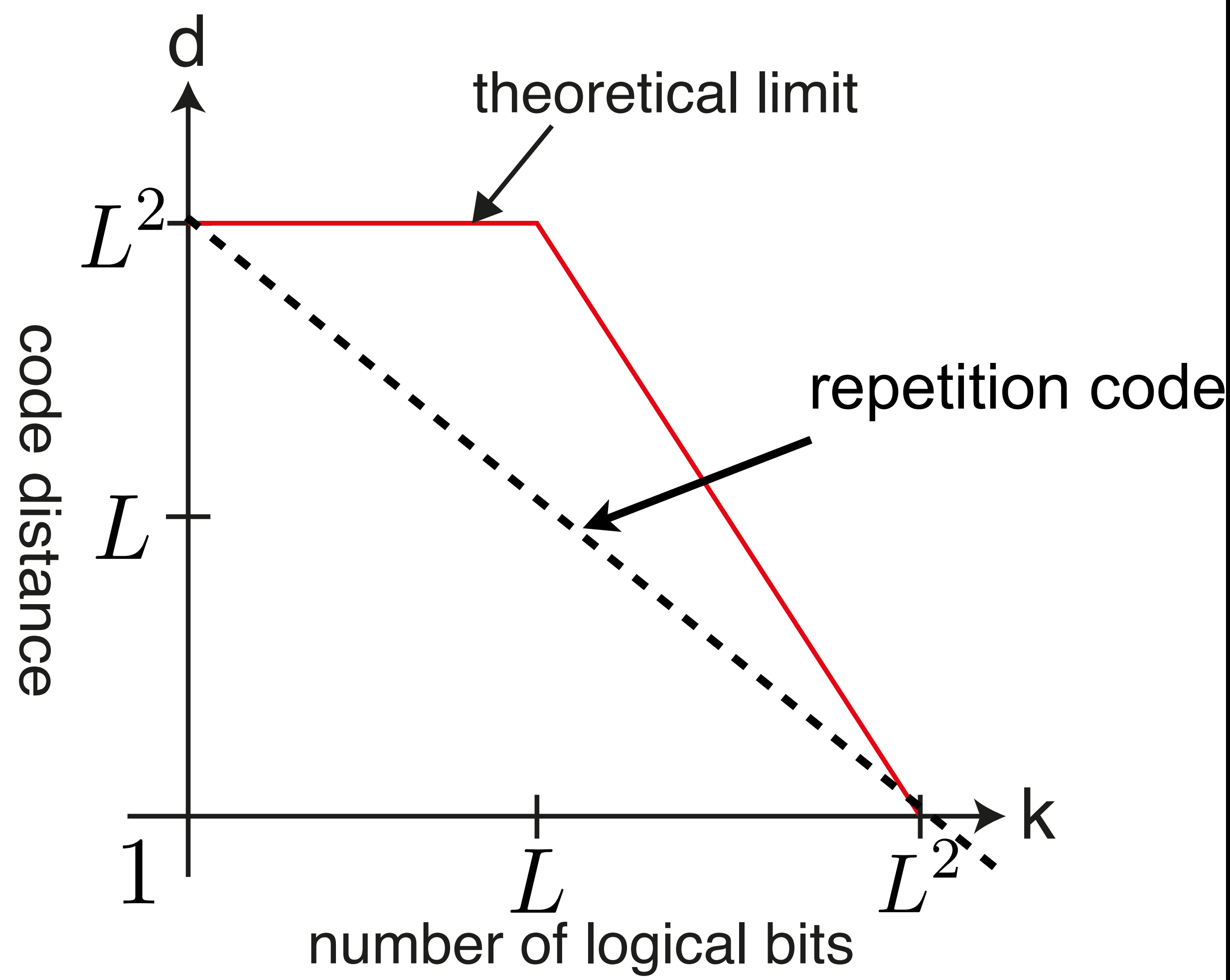
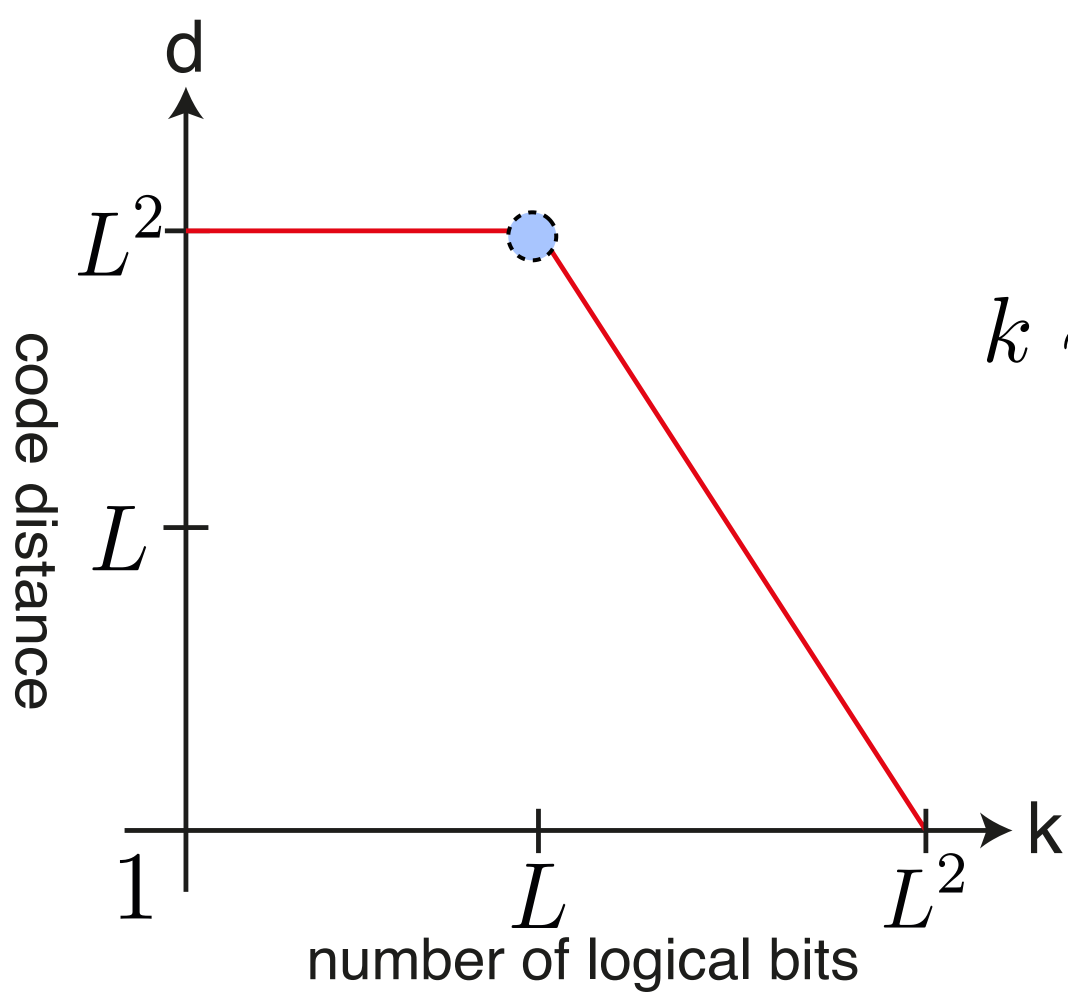


Copies of repetition codes

Main Result : Asymptotic saturation

- We give a construction of local codes which "asymptotically" saturate the bound. (BY 2011)

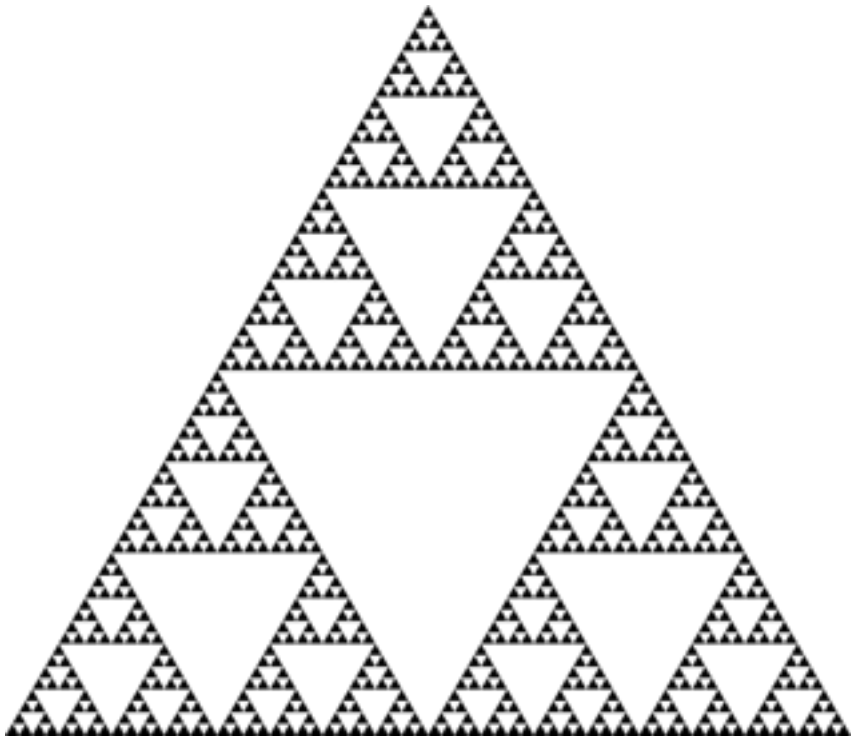


$$k \sim O(L^{D-1}), \qquad d \sim O(L^{D-\epsilon}),$$

Beni Yoshida, Annals of Physics 338, 134 (2013)

Key Idea:

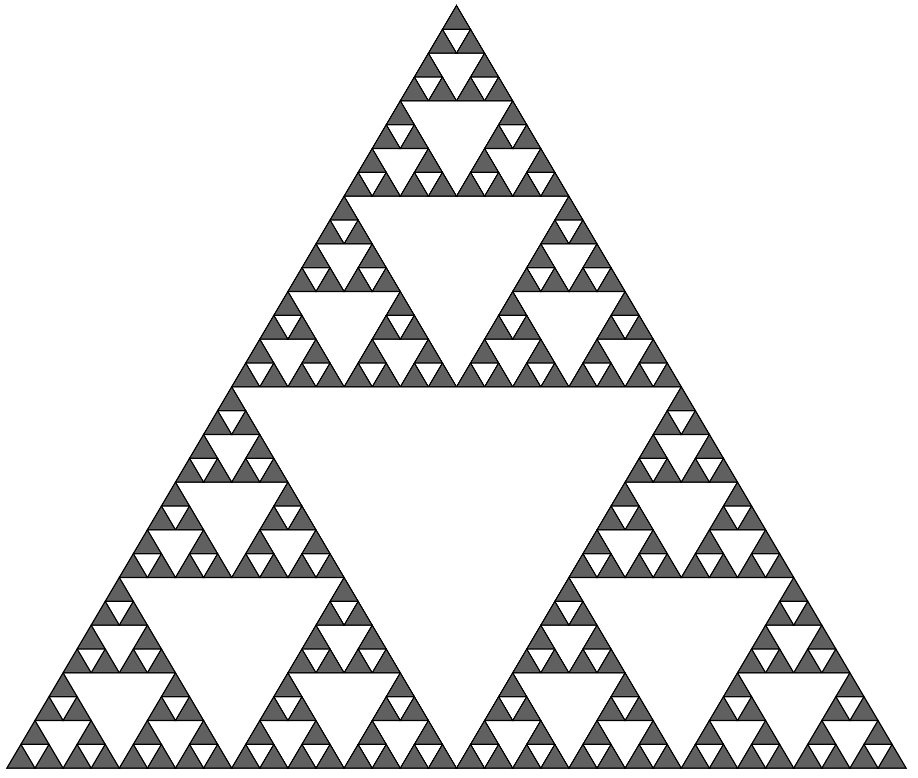
- Use fractal geometry in the Sierpinski triangle.



Sierpinski's triangle

The Sierpinski triangle

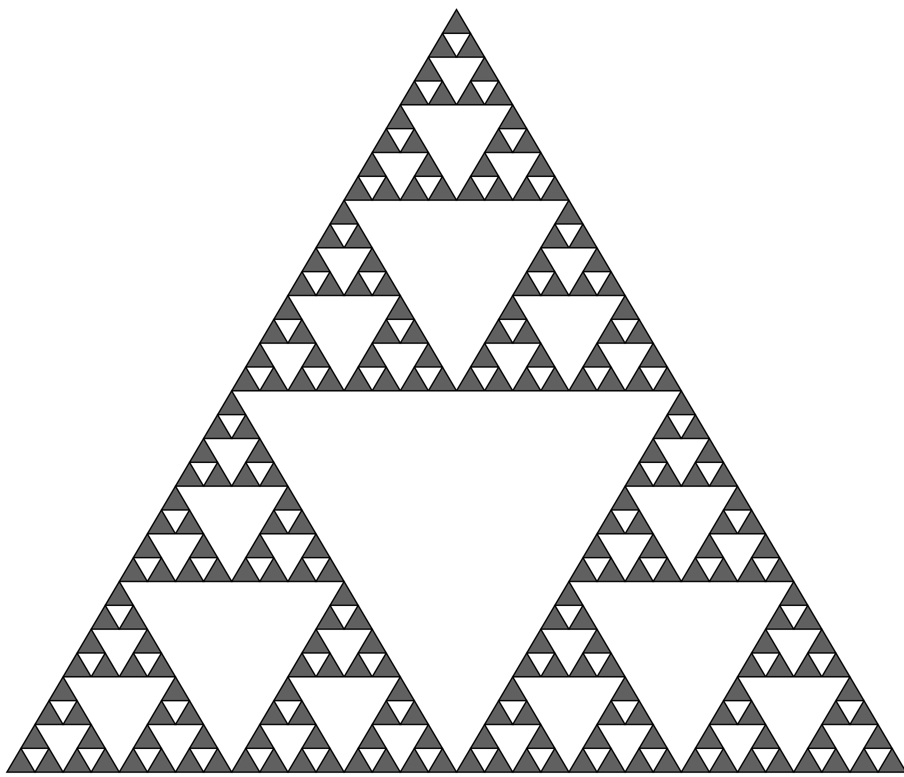
- Fractal geometry with self-similar properties



Fractal dimension $\frac{\log 3}{\log 2} \sim 1.585$

The Sierpinski triangle

- Physical realization ? ("Window Glass model" by Newman and Moore)



1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
1	1	1	1	0	0	0	0
1	0	0	0	1	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1

⋮ ⋱

The Sierpinski triangle

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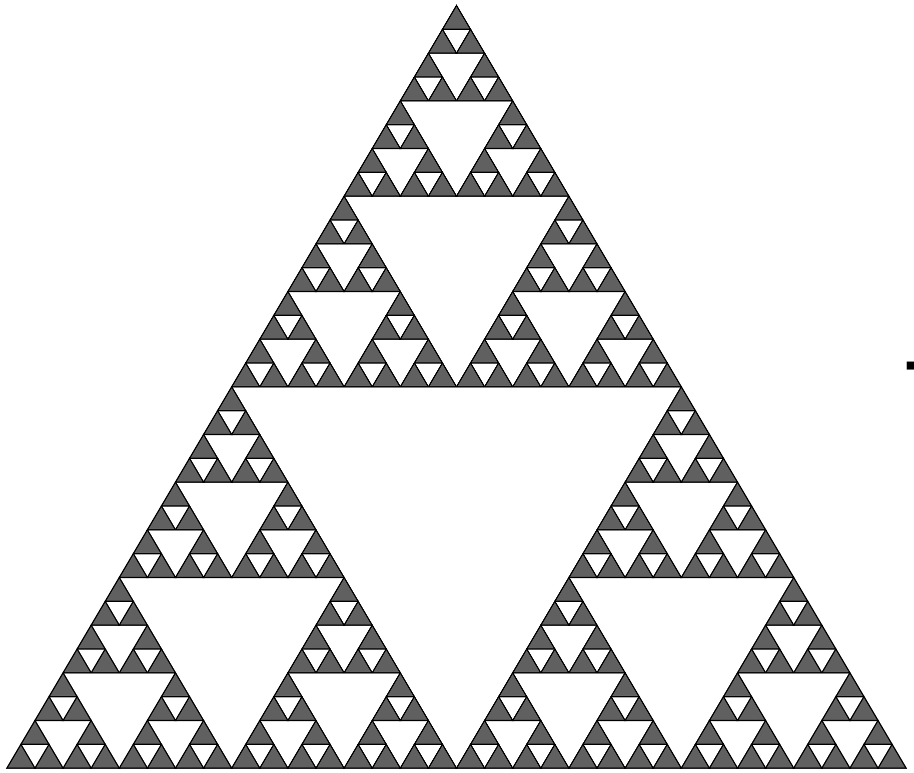

1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
1	1	1	1	0	0	0	0
1	0	0	0	1	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1

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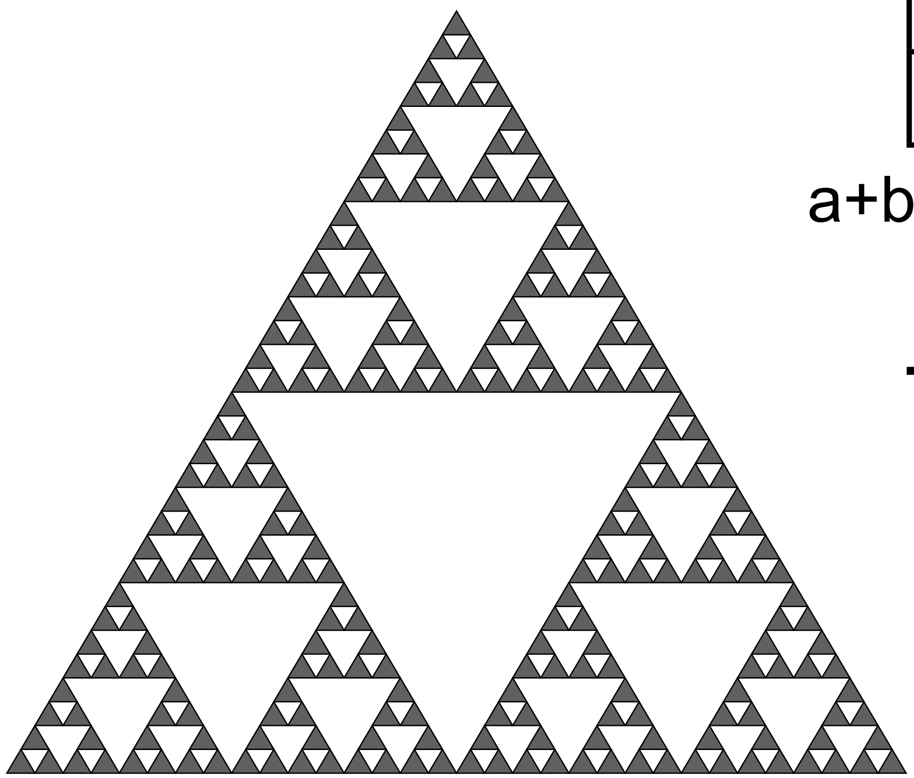

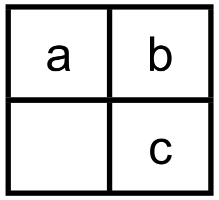

$a+b=c \pmod 2$

1	0	0	0	0	0	0	0	
1	1	0	0	0	0	0	0	
1	0	1	0	0	0	0	0	
1	1	1	1	0	0	0	0	
1	0	0	0	1	0	0	0	
1	1	0	0	1	1	0	0	
1	0	1	0	1	0	1	0	
1	1	1	1	1	1	1	1	
			⋮					⋱

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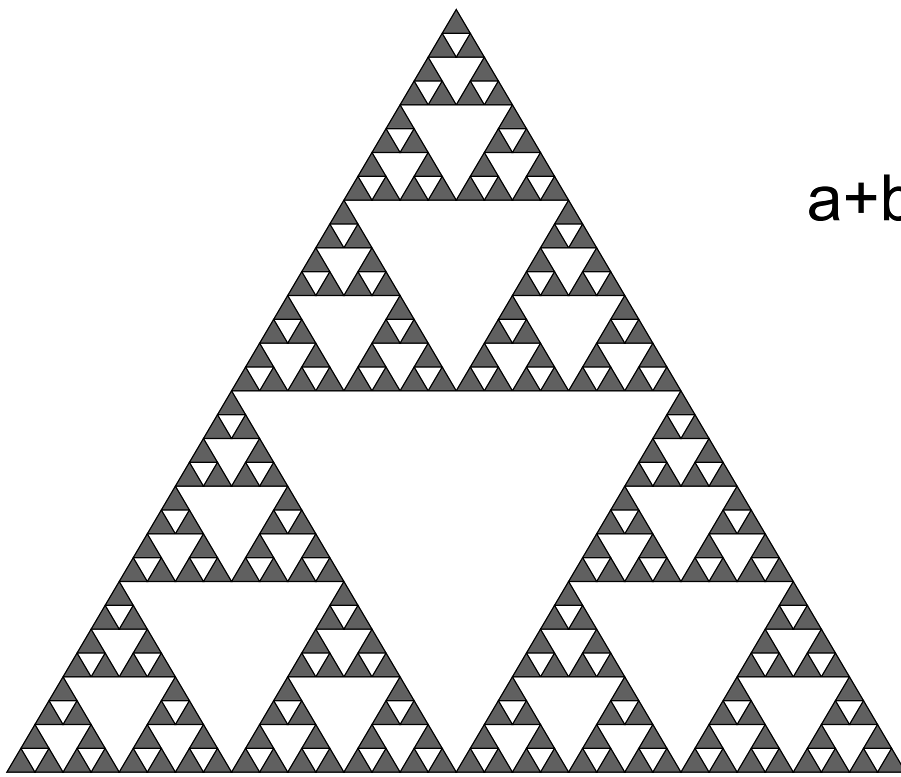
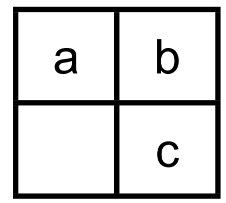
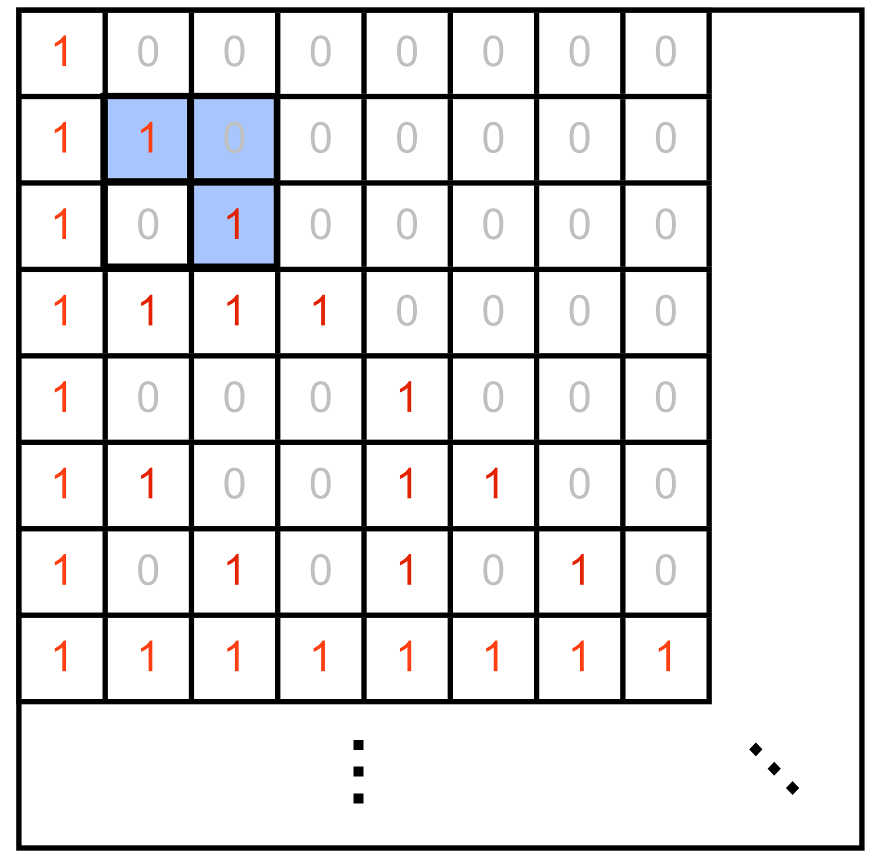




a+b=c (mod 2)



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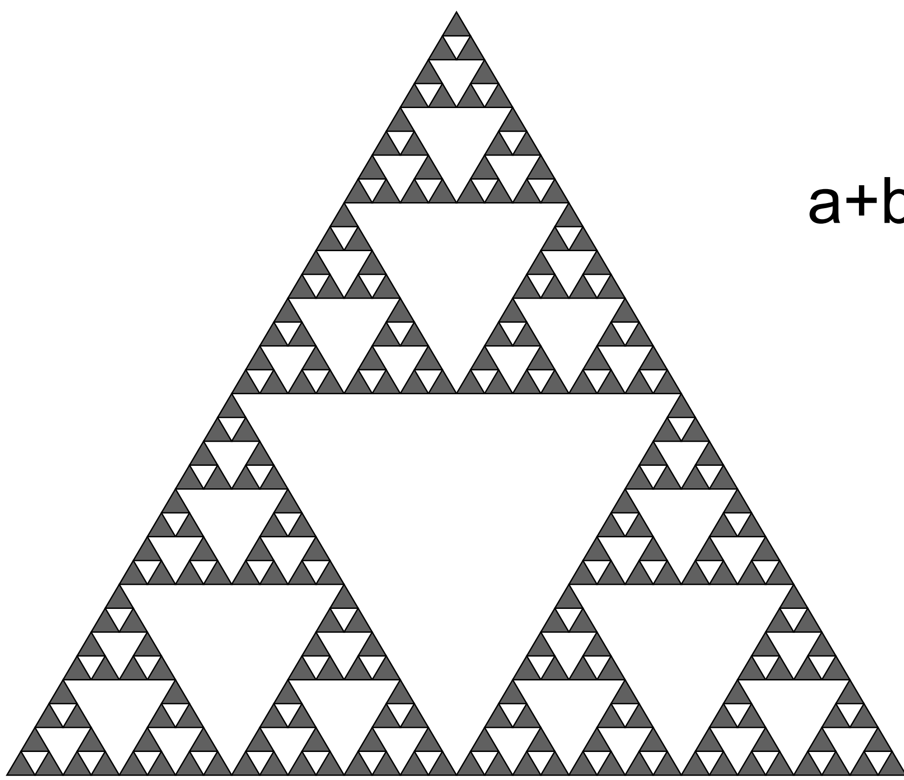

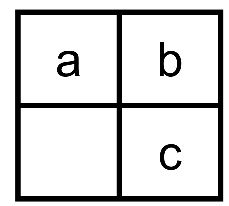

a+b=c (mod 2)

1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
1	1	1	1	0	0	0	0
1	0	0	0	1	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
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⋮ ⋱

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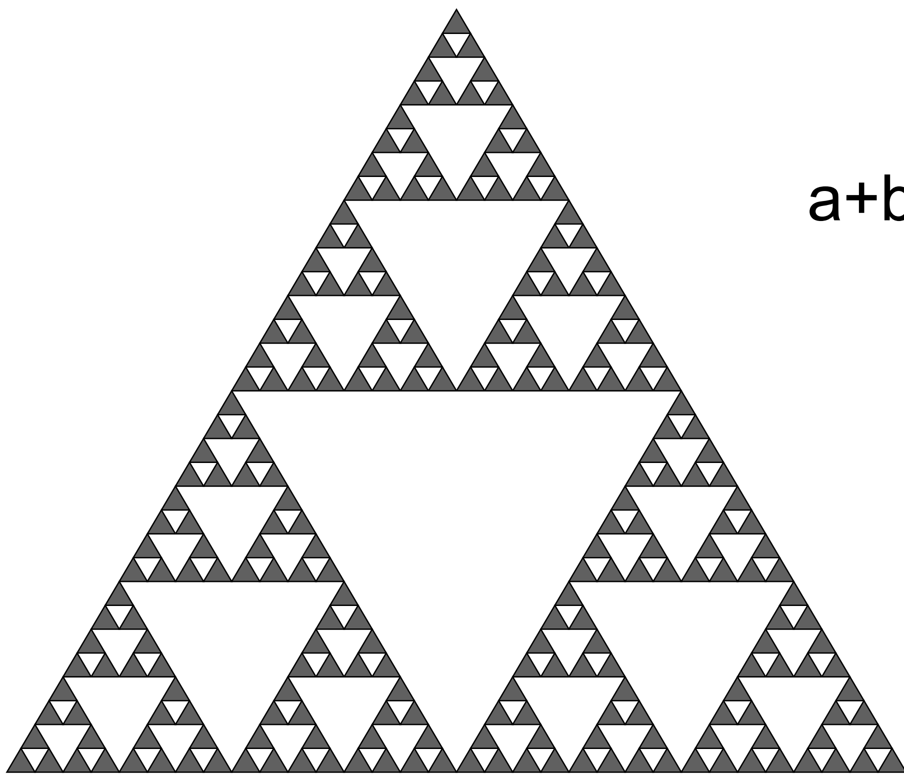


a	b
	c

a+b=c (mod 2)

$\longrightarrow$

1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
1	1	1	1	0	0	0	0
1	0	0	0	1	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1

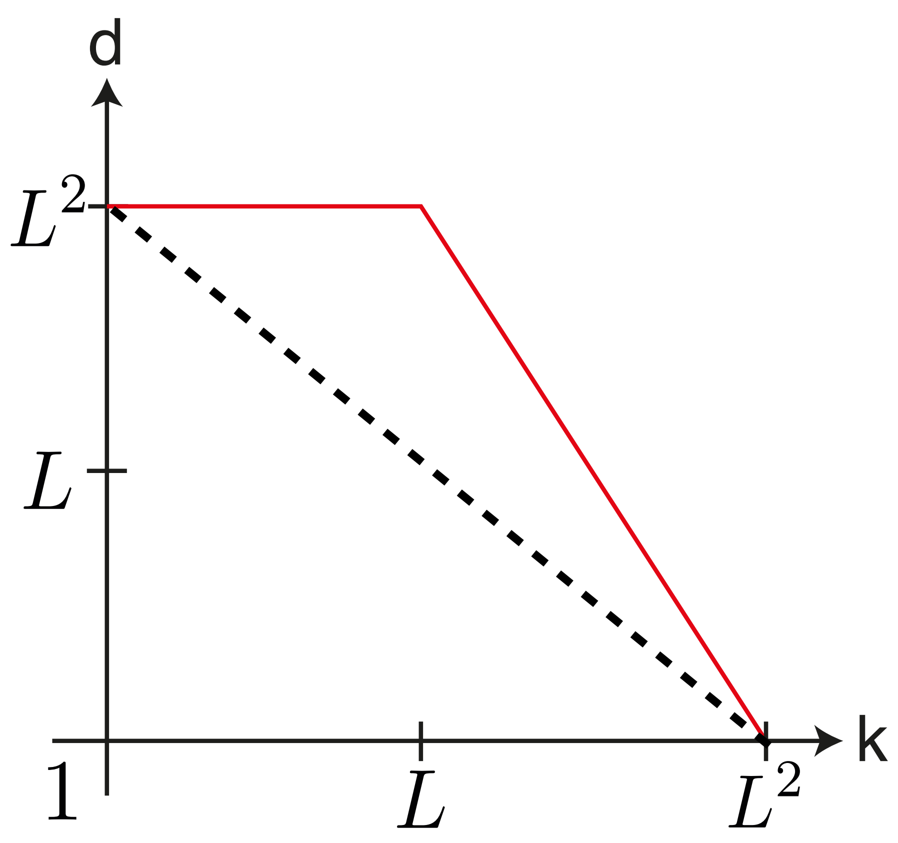
$\vdots$ $\ddots$

Sierpinski triangle as a code

- This system is a good error-correcting code ! (BY 2011)

Fractal dimension !

$$k \sim O(L), \qquad d \sim O(L^{\frac{\log 3}{\log 2}})$$



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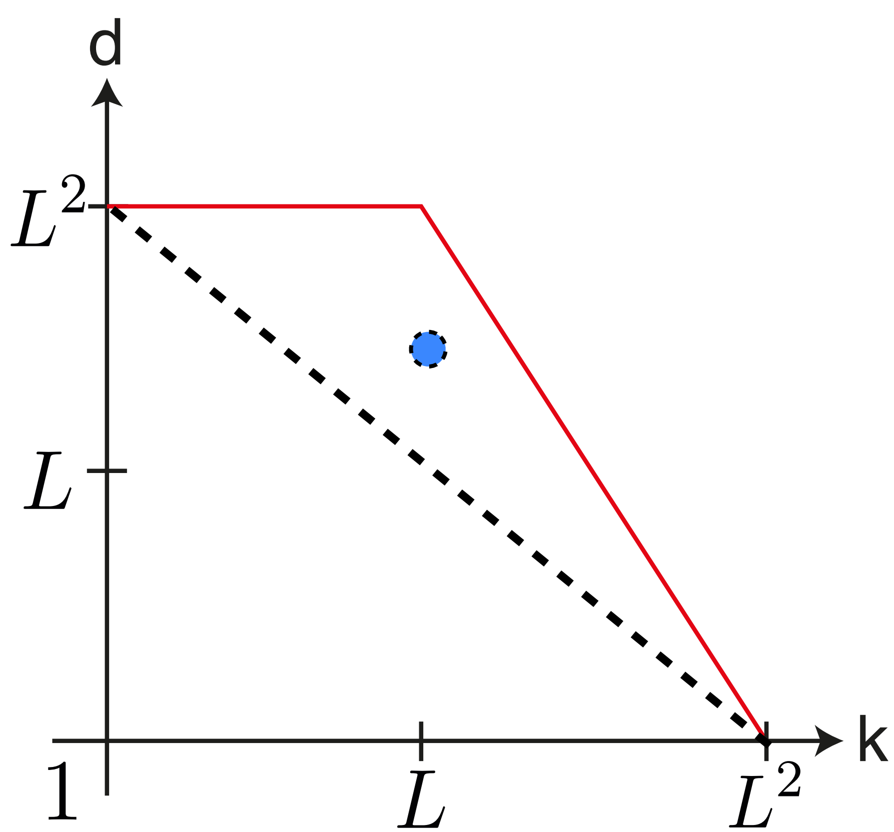

Better than repetition codes !

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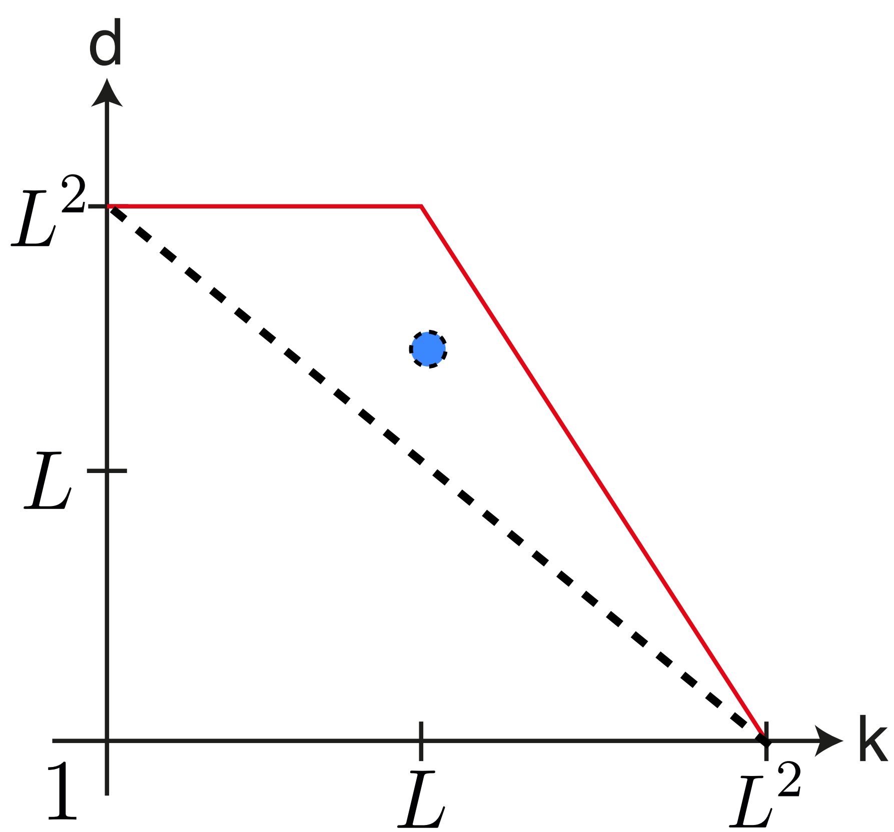
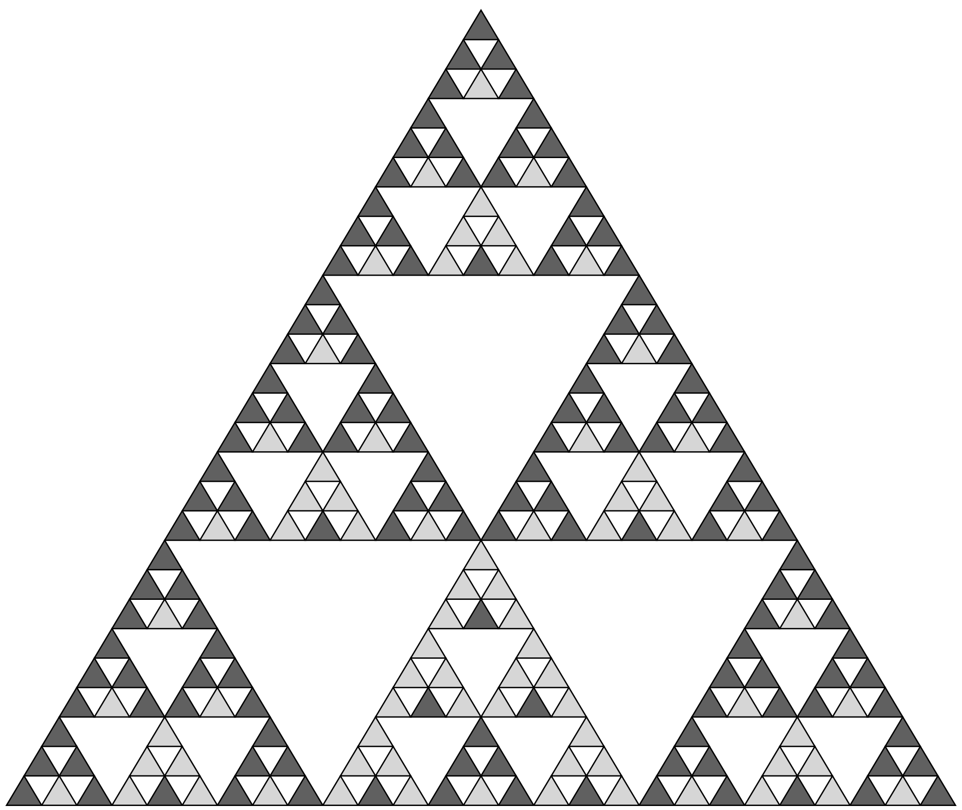


Better than repetition codes !

Still below a theoretical limit ...

The Sierpinski triangle (generalized)

- Fractal geometry with self-similar properties



Larger !

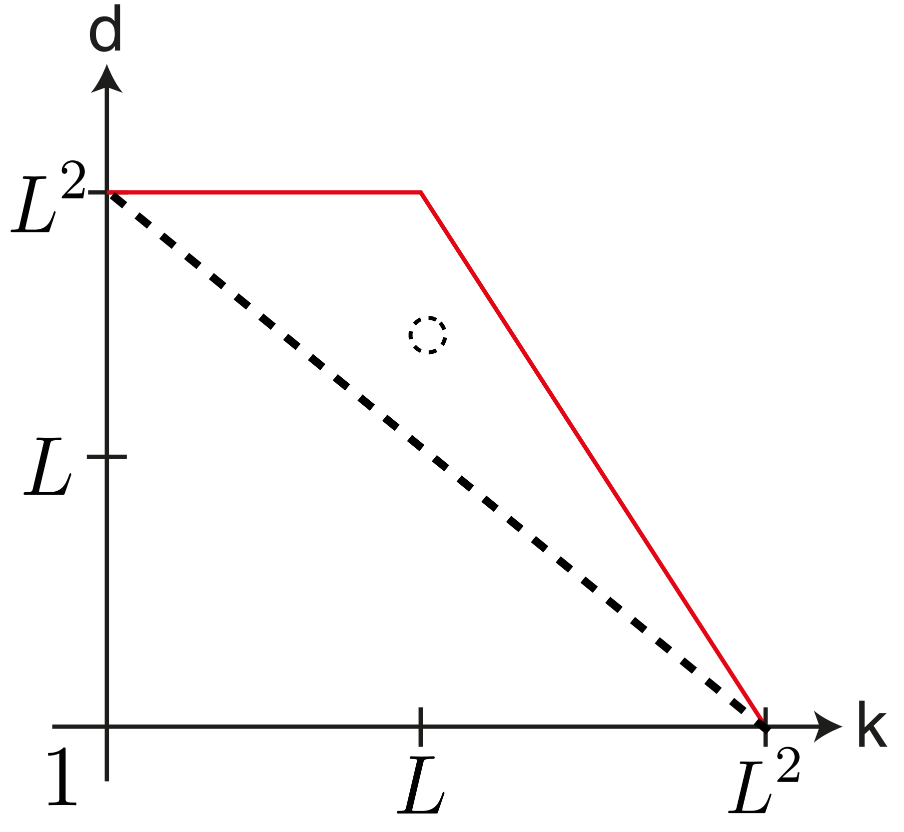
Fractal dimension $\frac{\log 6}{\log 3} \sim 1.631$

Generalized Sierpinski triangle as a code

- This fractal code has ...

larger !

$$k \sim O(L) \qquad d \sim O(L^{\frac{\log 6}{\log 3}})$$



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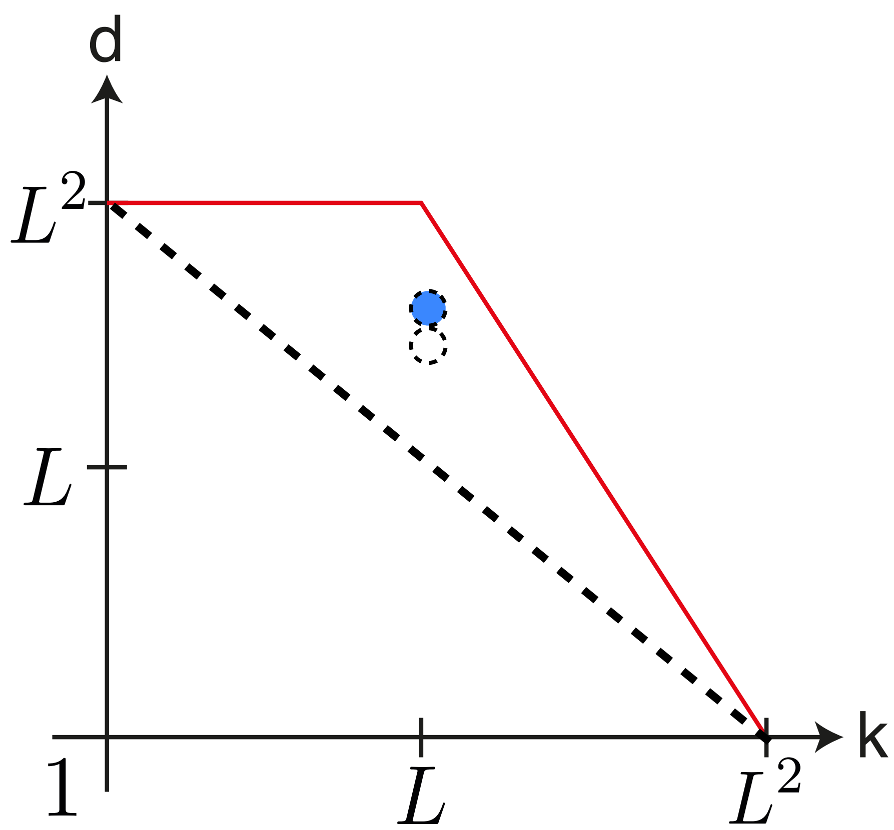


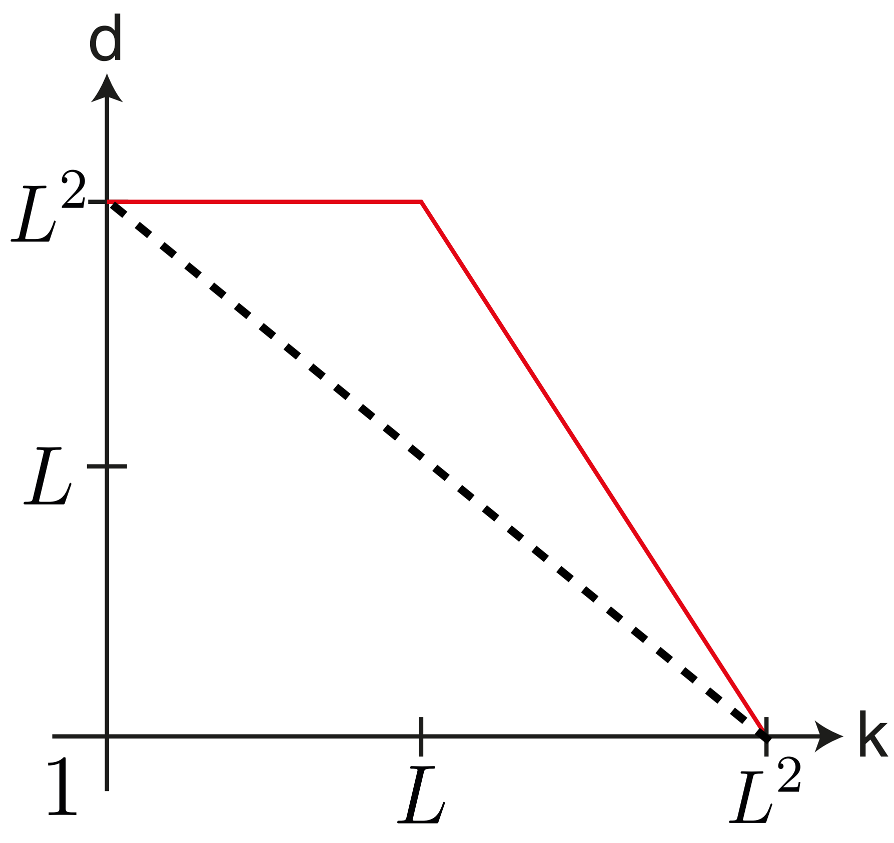
Slightly better than a previous fractal code !

Asymptotic saturation (D=2)

- Sierpinski triangle with p-dim spins (BY 2011)

Fractal dimension

$$\frac{\log(\frac{p(p+1)}{2})}{\log p}$$

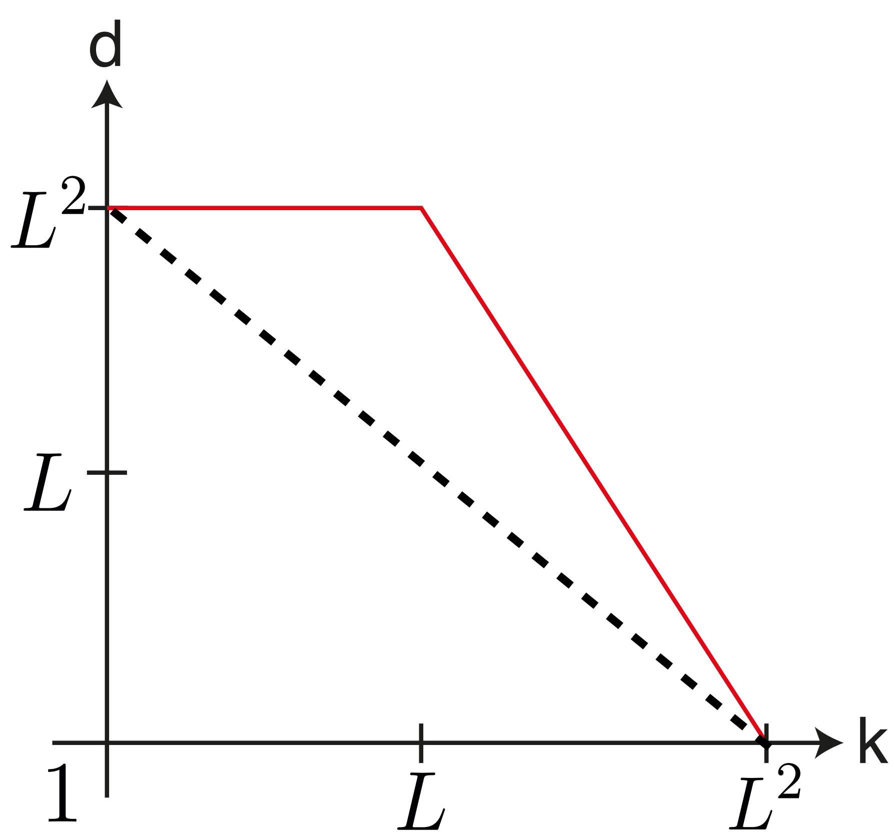


Asymptotic saturation (D=2)

- Sierpinski triangle with p-dim spins (BY 2011)

Fractal dimension

$$\frac{\log(\frac{p(p+1)}{2})}{\log p} \to 2 \qquad \text{for} \quad p \to \infty.$$



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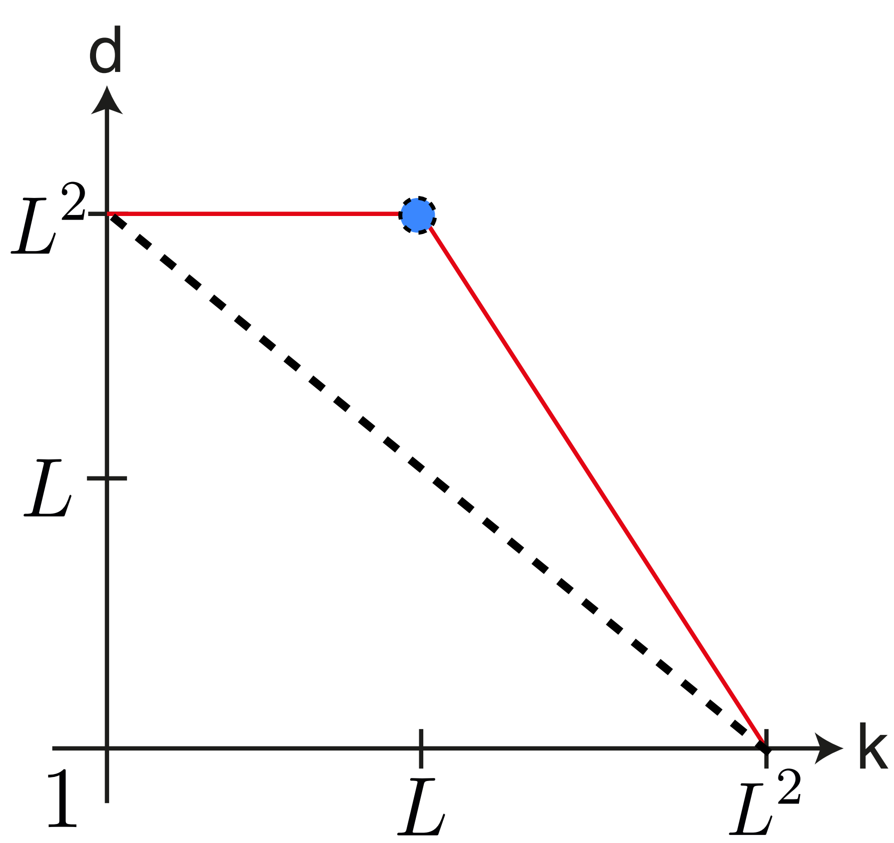

$$k \sim O(L), \qquad d \sim O(L^{2-\epsilon})$$

Asymptotically saturate the bound !

Asymptotic saturation (D>2)

- Higher-dimensions ?

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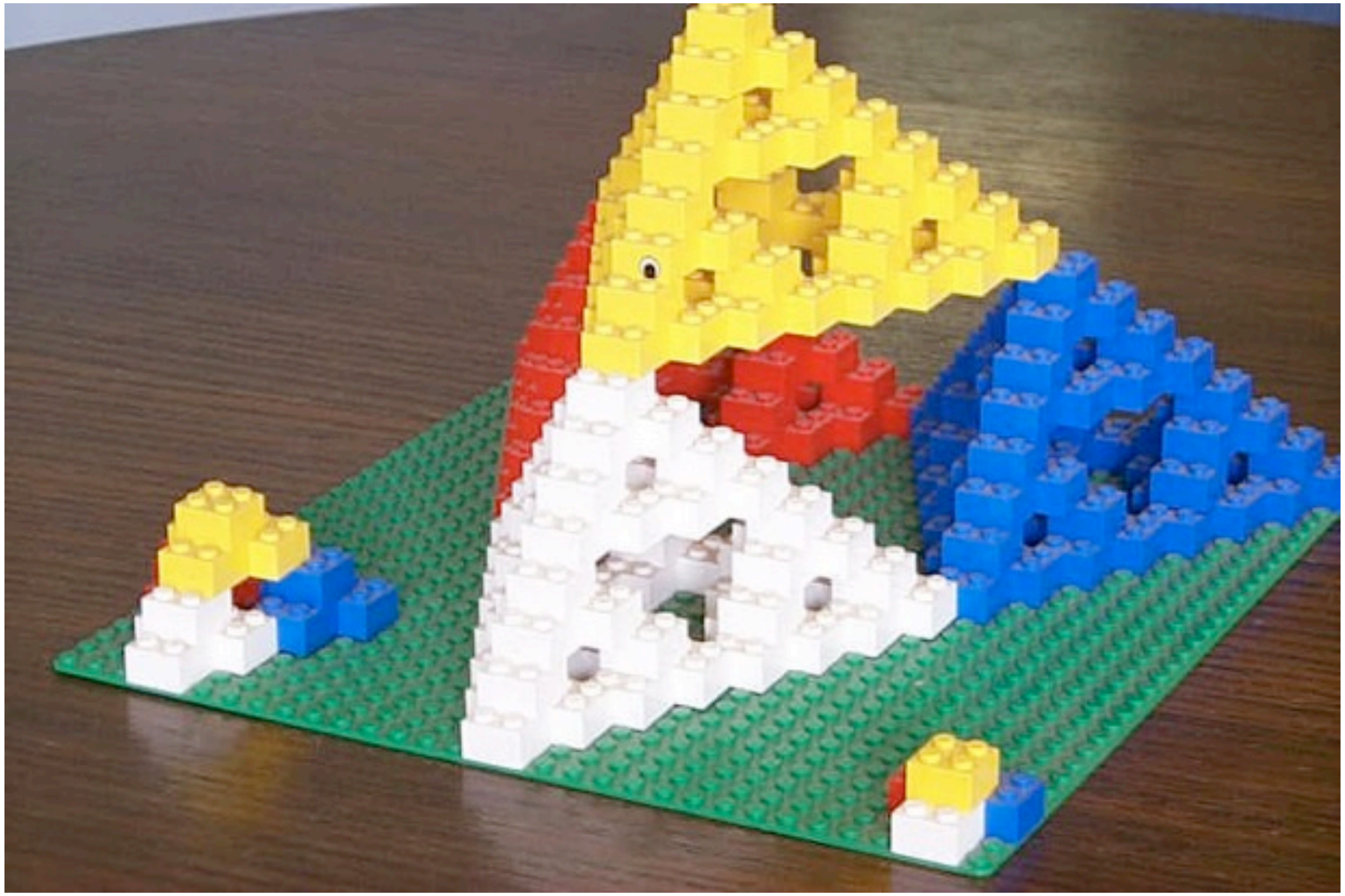

Fractal dimension

$$\frac{\log\left(\frac{p(p+1)\cdots(p+D-1)}{D!}\right)}{\log(p)}$$

Asymptotic saturation (D>2)

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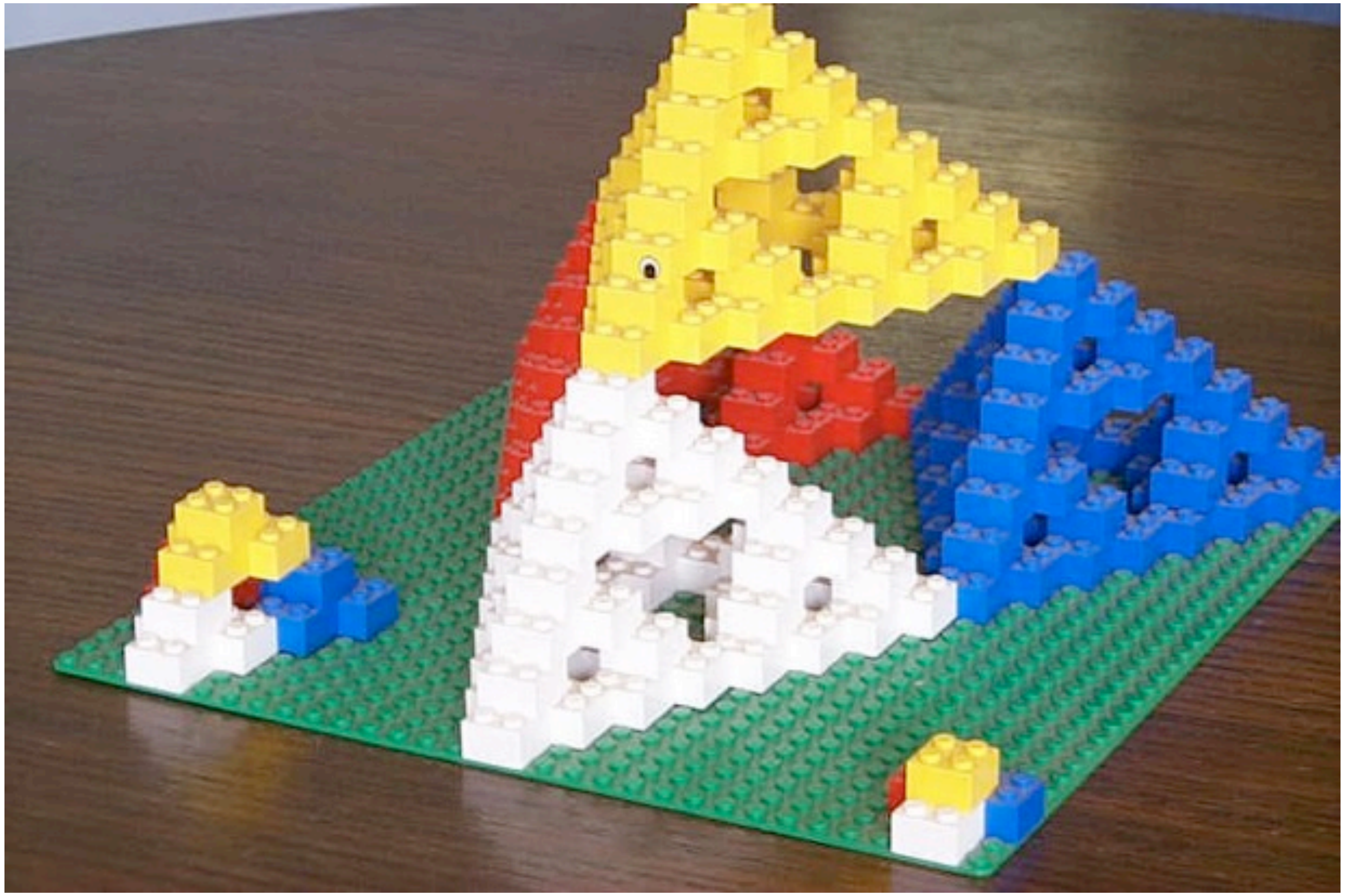

Fractal dimension

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Asymptotic saturation (D>2)

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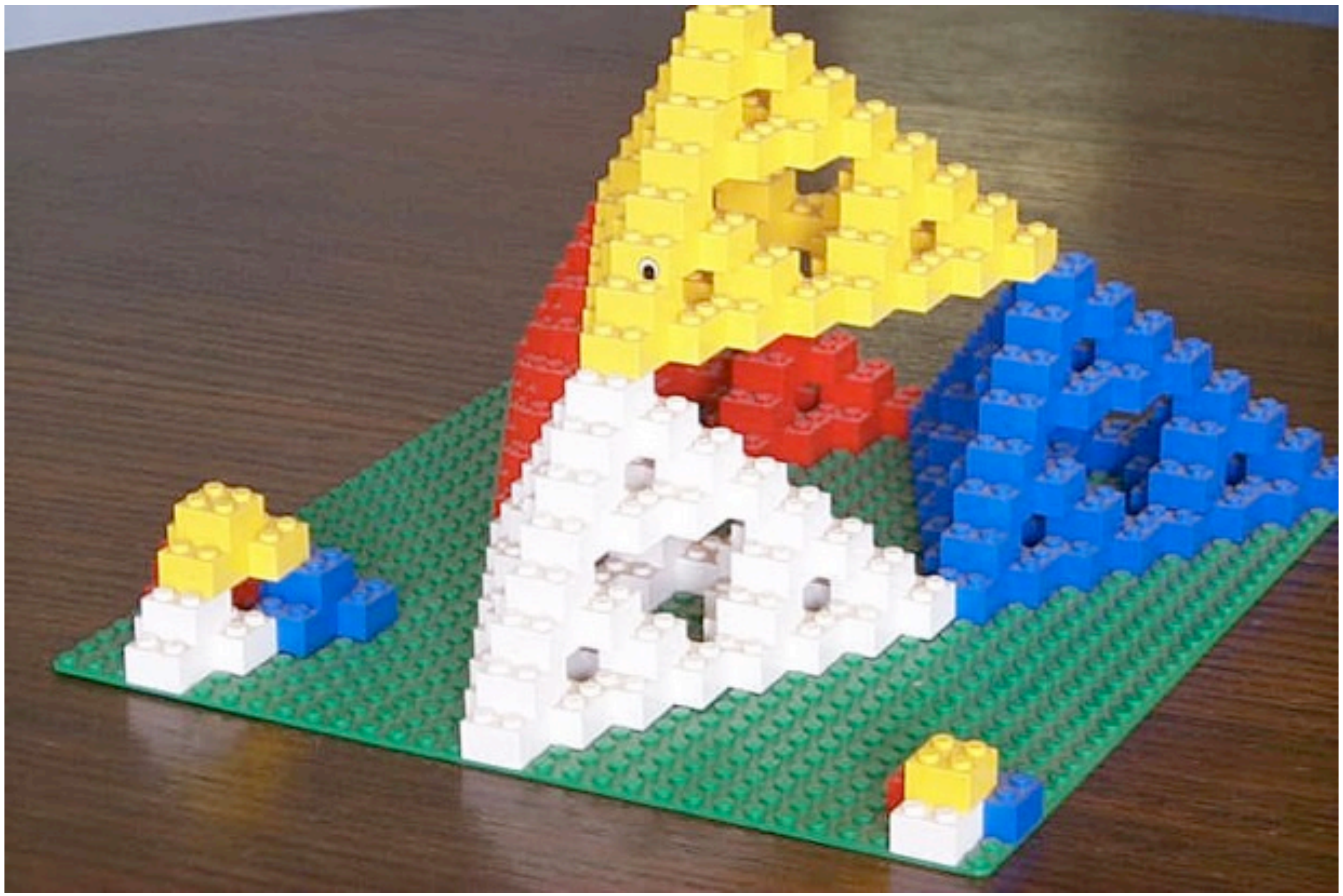


Fractal dimension

$$\frac{\log\left(\frac{p(p+1)\cdots(p+D-1)}{D!}\right)}{\log(p)} \xrightarrow[p\to\infty]{} D$$

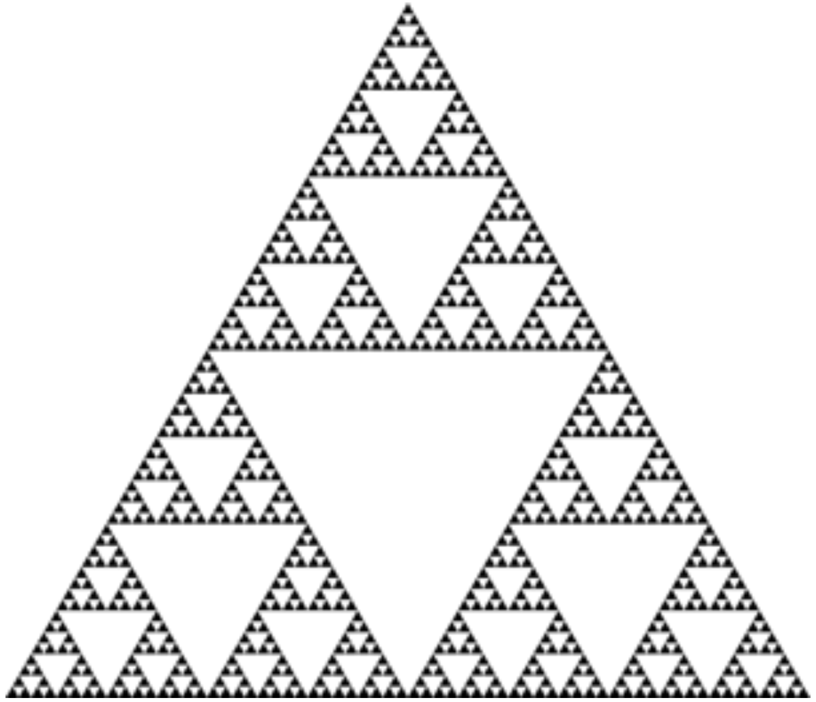
Fractal codes saturate the bound for D > 2 too !

$$k \sim O(L^{D-1}), \qquad d \sim O(L^{D-\epsilon}),$$

(classical) local code bound

$$kd^{1/D} \leq O(n)$$

saturation ←



Quantum generalizations ?

(classical) local code bound

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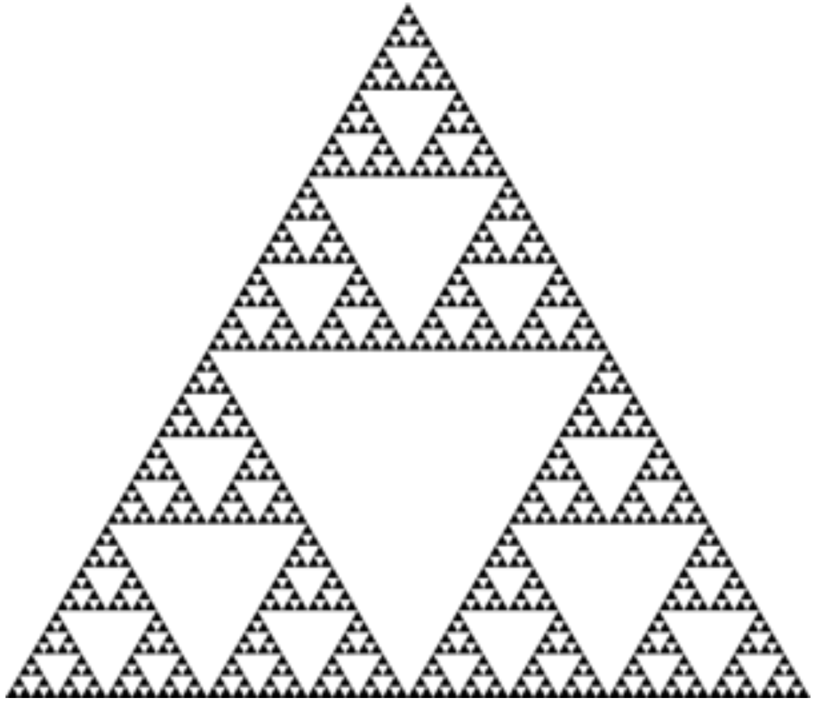

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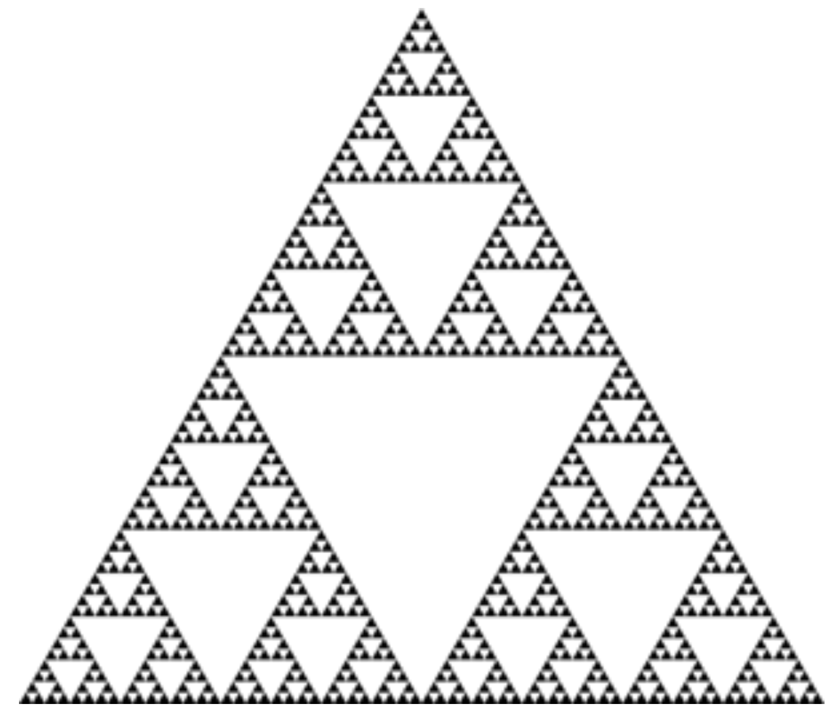
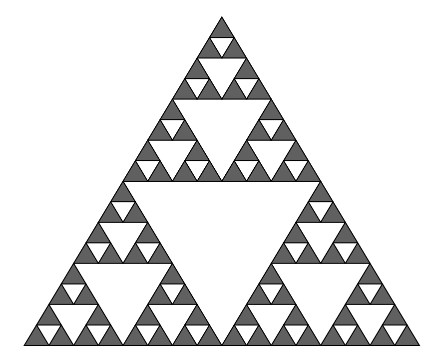
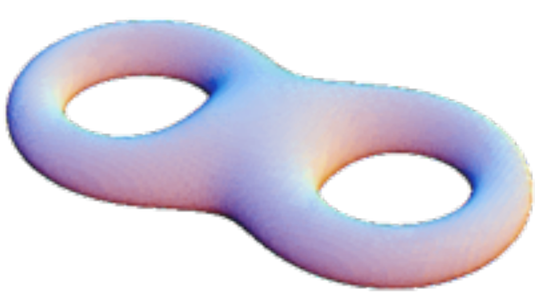


Physical properties ?

Question:

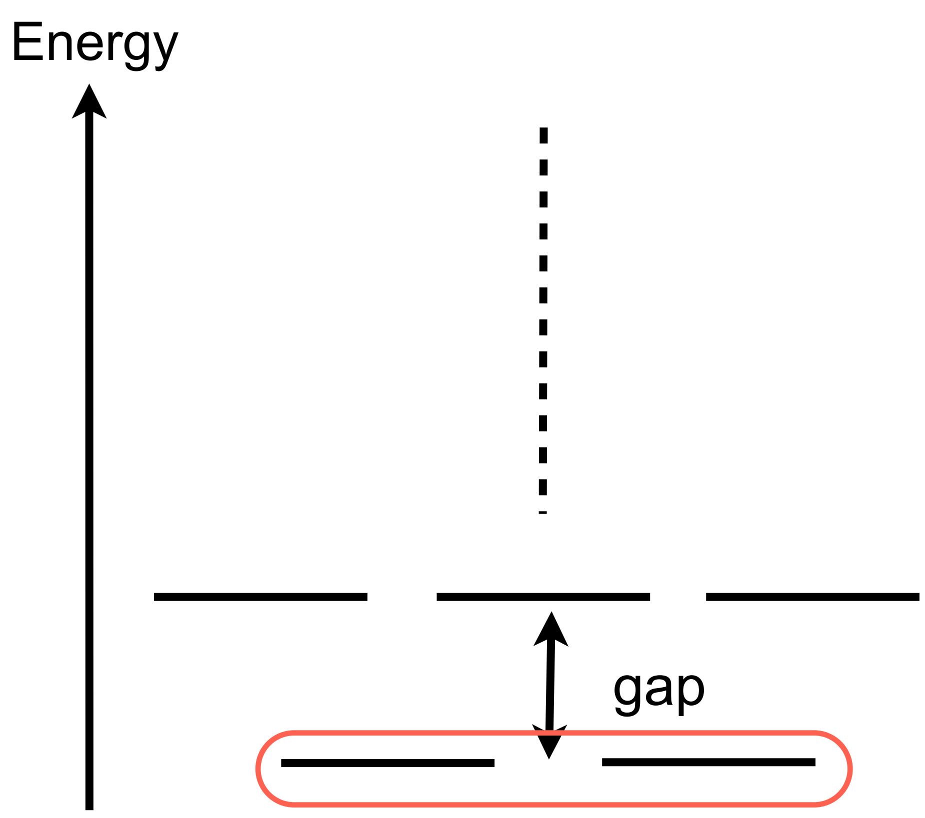
- Is Topological Quantum Field Theory a universal theory of topological order?





What is topological order ?

- [Def] Ground State Properties are stable against any types of small local perturbations.



* for non-chiral topological order

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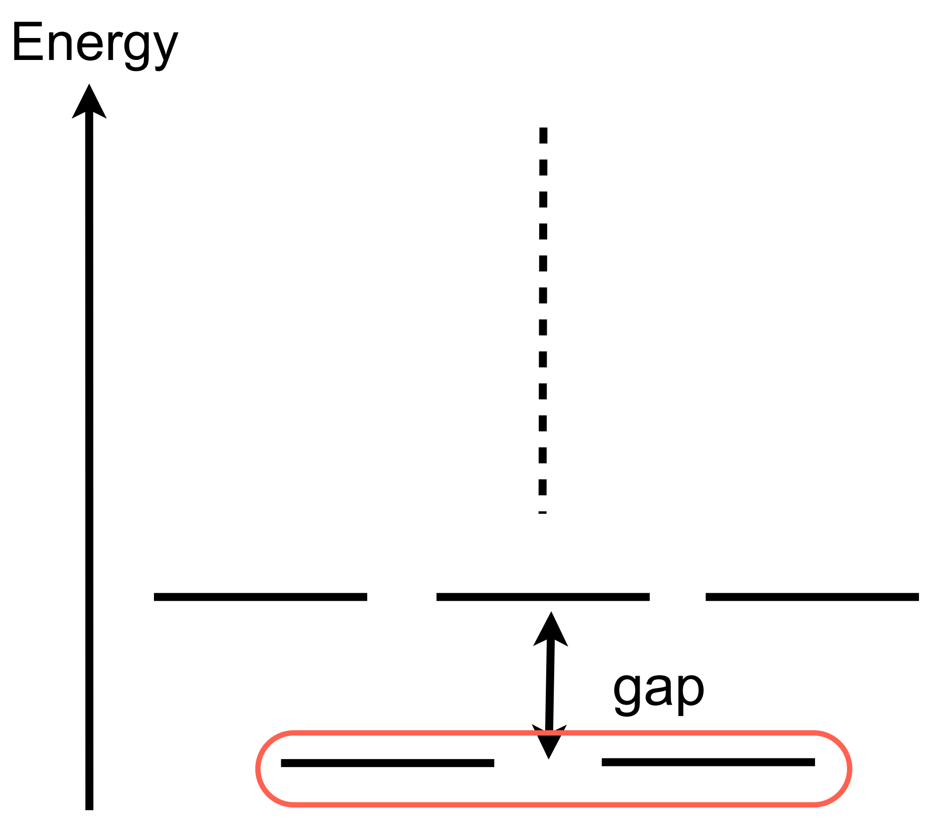
→ The system is a *quantum error-correcting code.

(no local order parameter)



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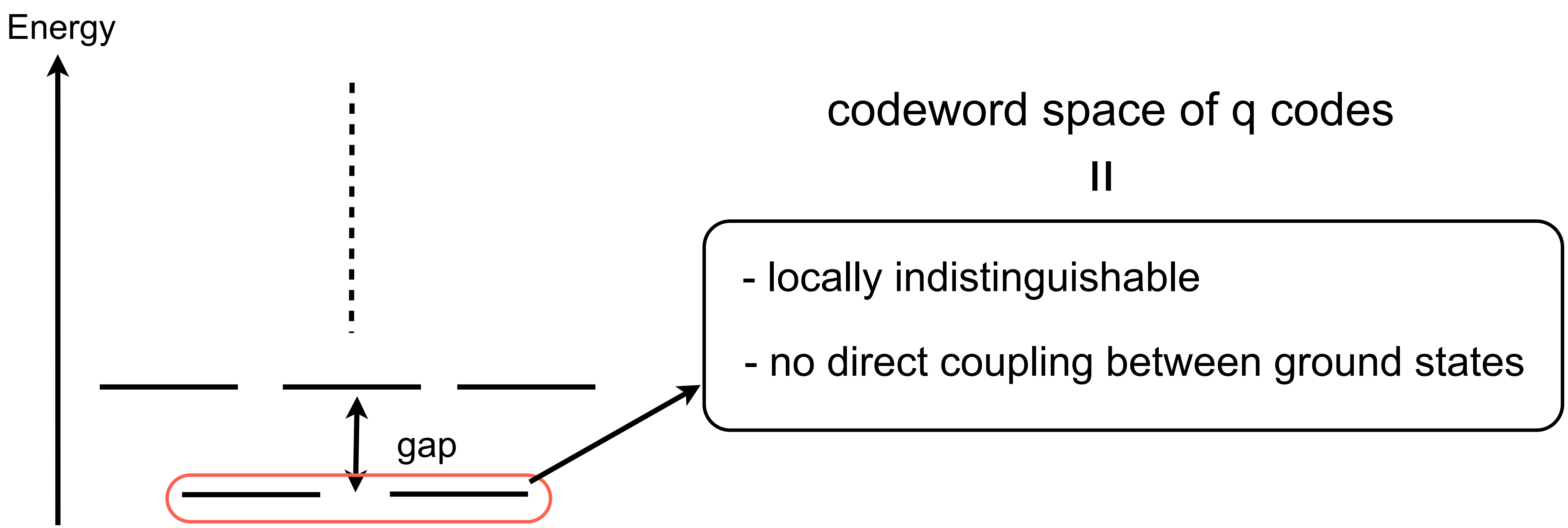

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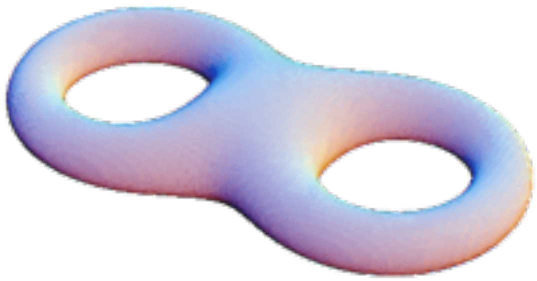
Topological order and TQFT (Wen 90)

Fractional Quantum Hall effect

(a) The number of ground states depend only on the number of genus.

(b) Ground state properties are stable against small perturbations



Topological Quantum Field Theory

(a) Invariant under diffeomorphism (continuous deformations)

(b) Local deformation of metric leaves the theory invariant.

Topological order and TQFT (Wen 90)

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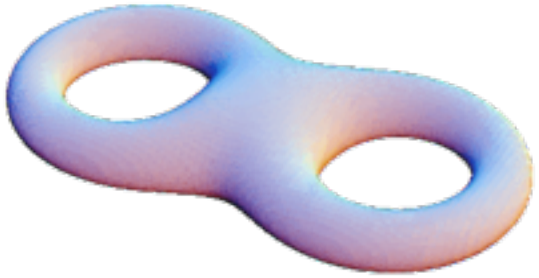
(b) Ground state properties are stable against small perturbations ← definition of topological order



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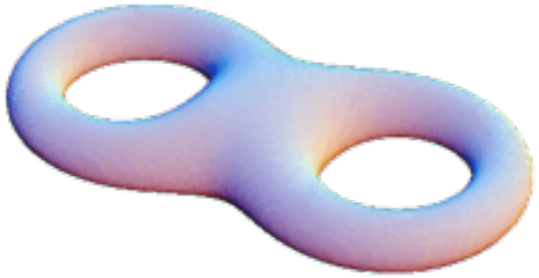
Topological order and TQFT (Wen 90)

Fractional Quantum Hall effect

(a) The number of ground states depend only on the number of genus. ← extra property !

(b) Ground state properties are stable against small perturbations ← definition of topological order



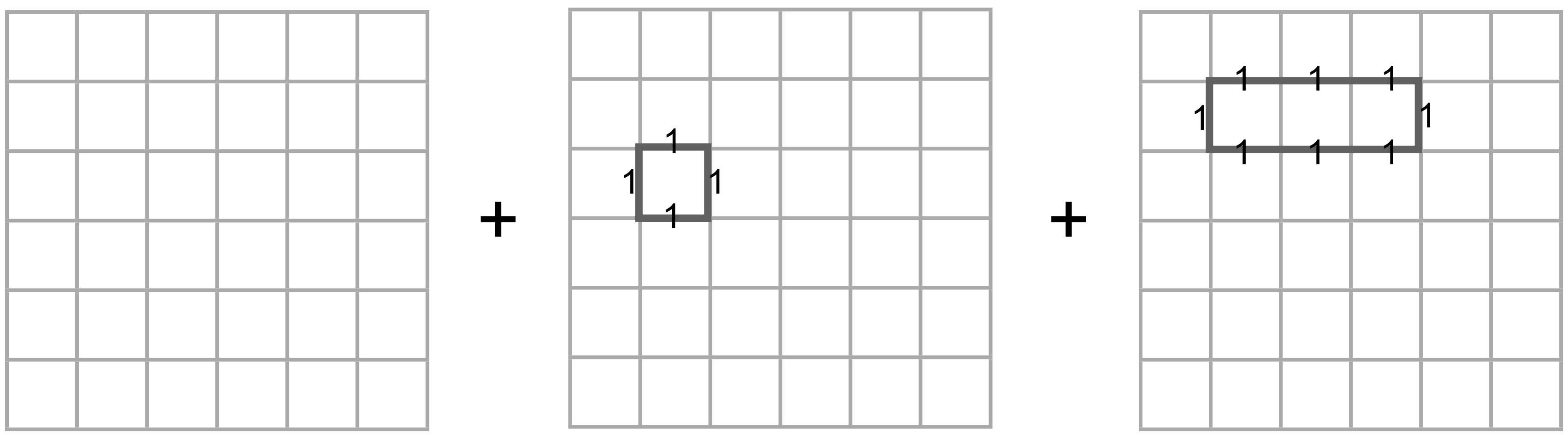
Topological Quantum Field Theory

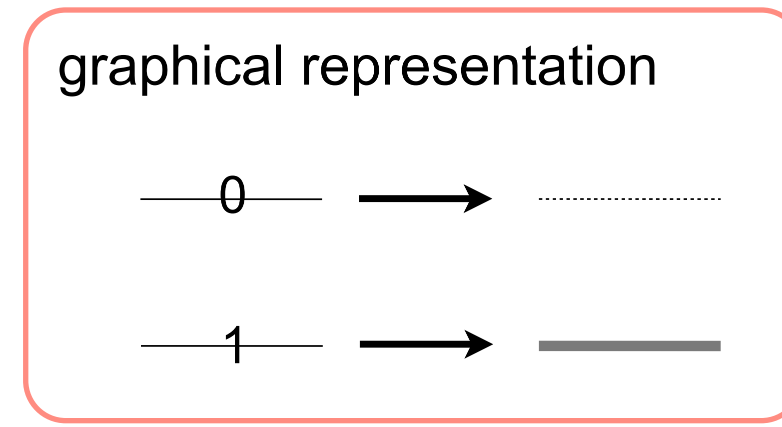
(a) Invariant under diffeomorphism (continuous deformations)

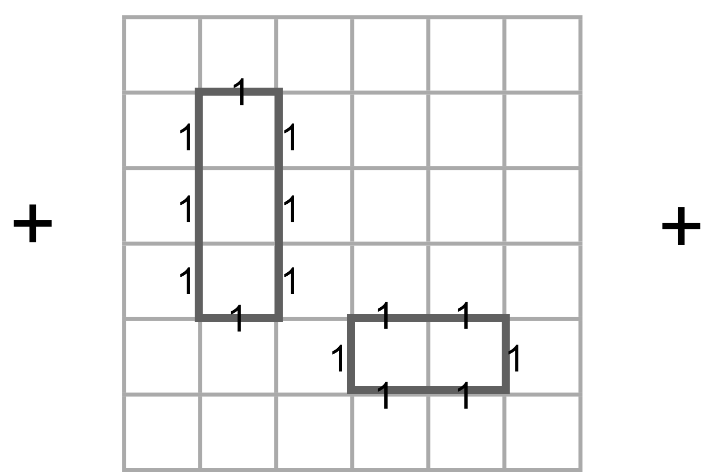
(b) Local deformation of metric leaves the theory invariant.

The Toric code : loop condensation

A ground state is a condensation of fluctuating loops (Z_2)







a superposition of loops of different sizes and shapes

⟶ TQFT

Wavefunction is invariant under diffeomorphism

⟶ Fixed-point under RG transformations !

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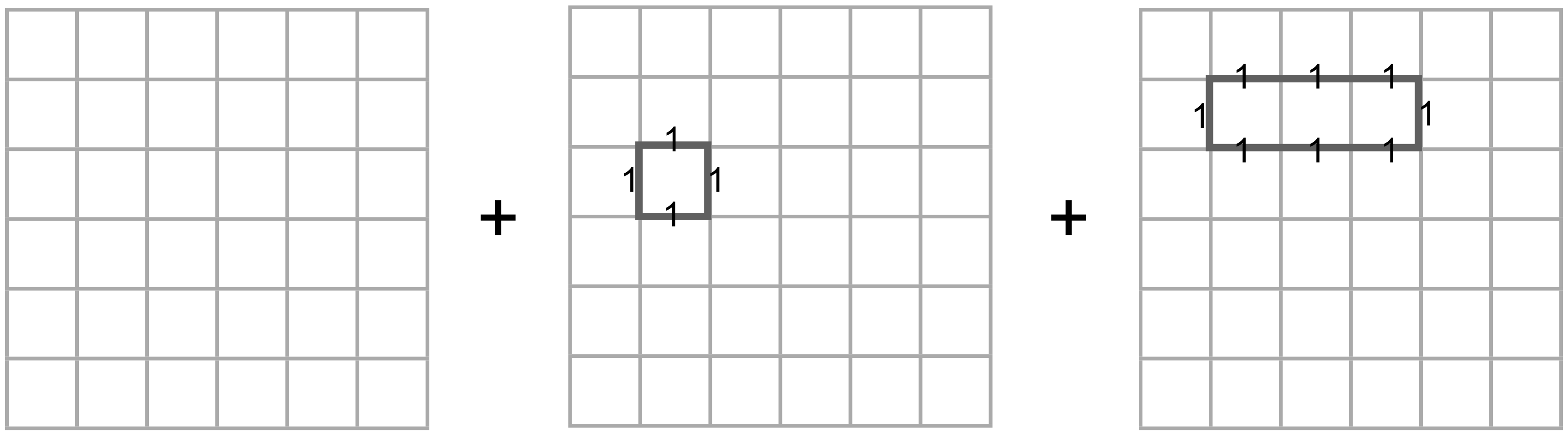

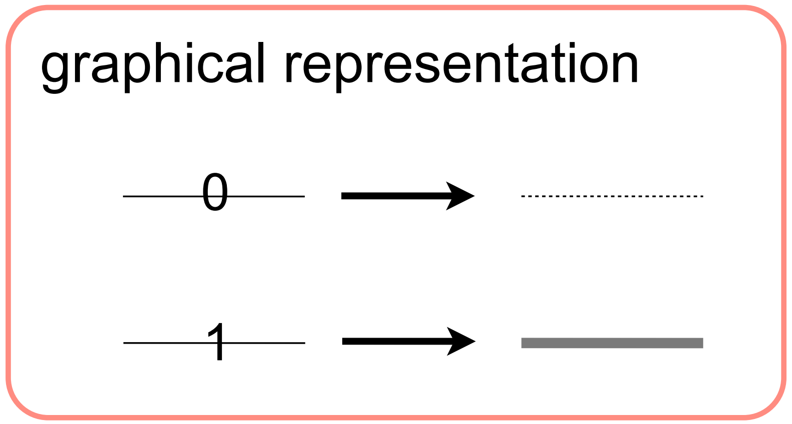

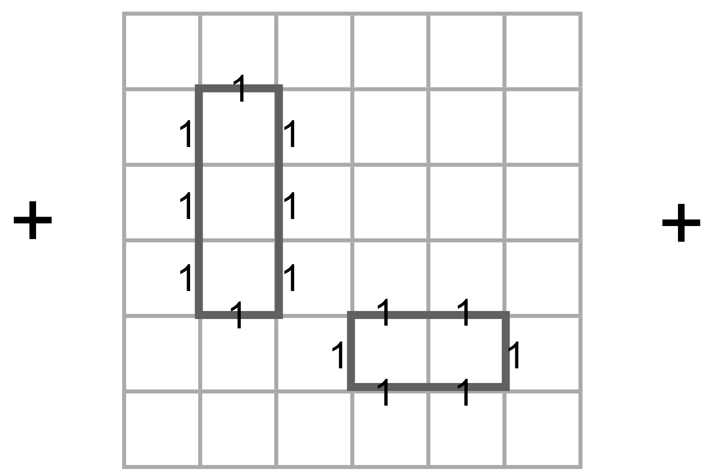

a superposition of loops of different sizes and shapes

→ TQFT

Levin-Wen model

=Turaev-Viro TQFT

Wavefunction is invariant under diffeomorphism

→ Fixed-point under RG transformations !

2dim topological order is within TQFT ?

Theorem (BY 2010)

Consider a 2D stabilizer Hamiltonian with translation symmetries.

If it is topologically ordered, then it is equivalent to a single or multiple copies of the Toric code.

This seems to imply....

In 2D, topological order = TQFT

2dim topological order is within TQFT ?

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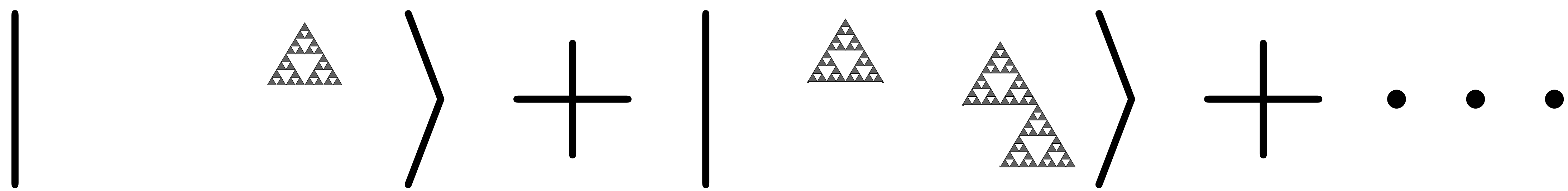
then, D>2 ??

Main result

- Quantum fractal codes (D>2) are topologically ordered, but <u>beyond</u> TQFT.



condensation of fractal objects

Quantum code from classical codes

- A framework to construct a quantum code from a pair of (cyclic) classical codes (BY 2013).

a pair of (D-1)-dim classical codes

D-dim quantum code

model A

model B

→ quantum model (A,B)

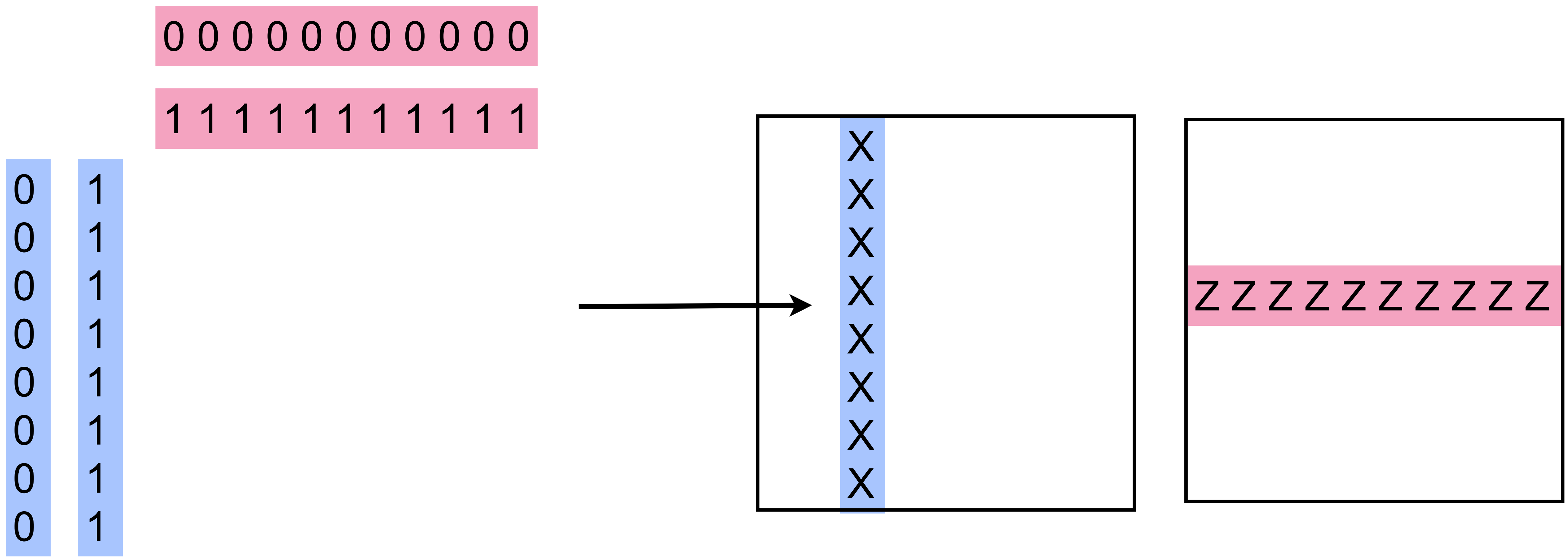
Geometric shapes of condensed objects look like model A and model B.

Quantum code from classical codes

1dim repetition

1dim repetition

→ 2dim Toric



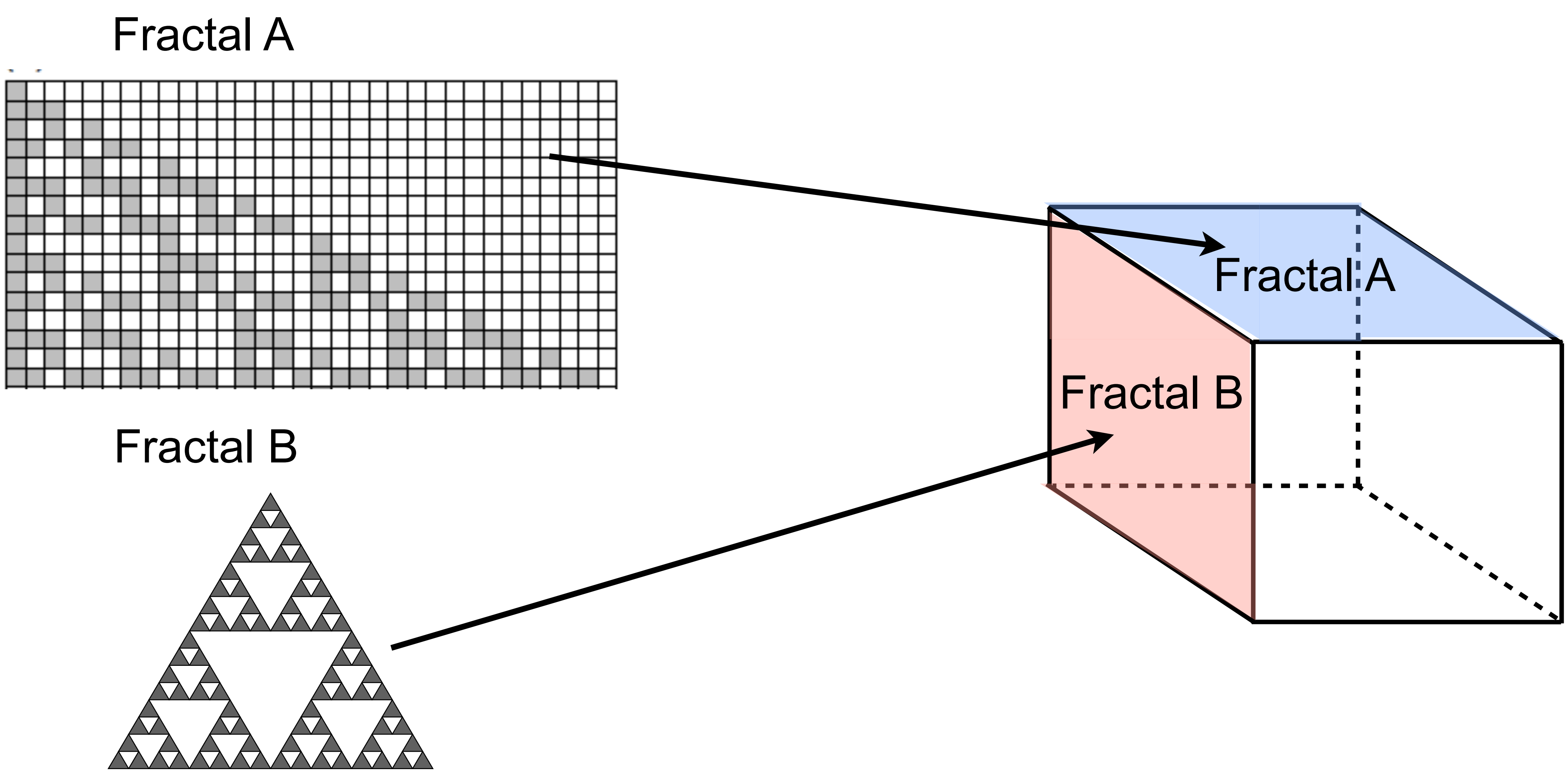
1dim repetition

2dim Toric

Quantum code from classical codes

2dim fractal
2dim fractal → 3dim quantum fractal

eg)



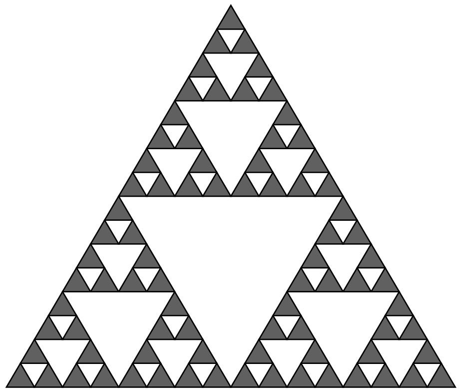
Why are they beyond TQFT?

(a) A large (diverging) number of ground states: $k \sim L$

TQFT has O(1) ground states by its definition...

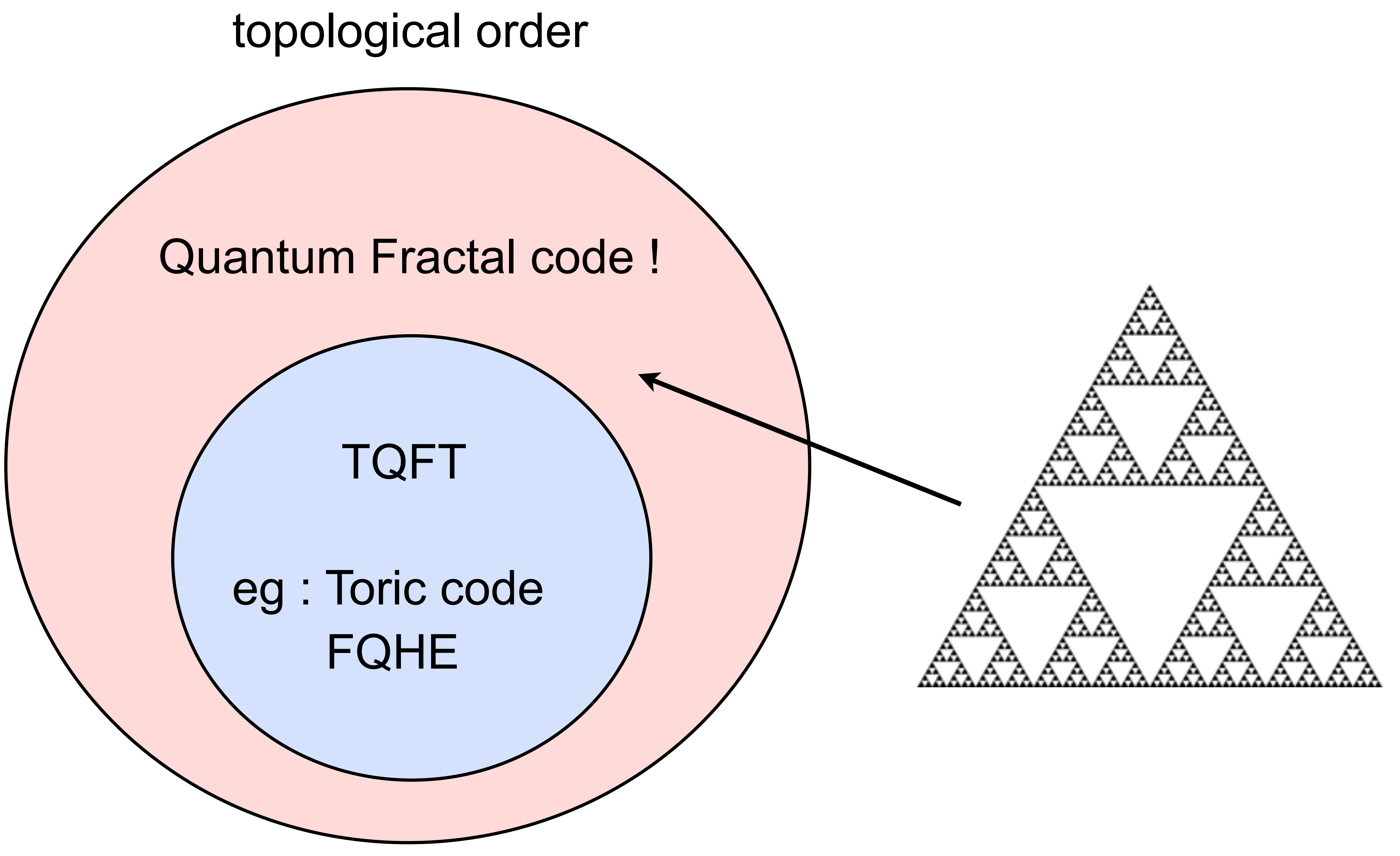
(b) Fractal objects are not continuously deformable.

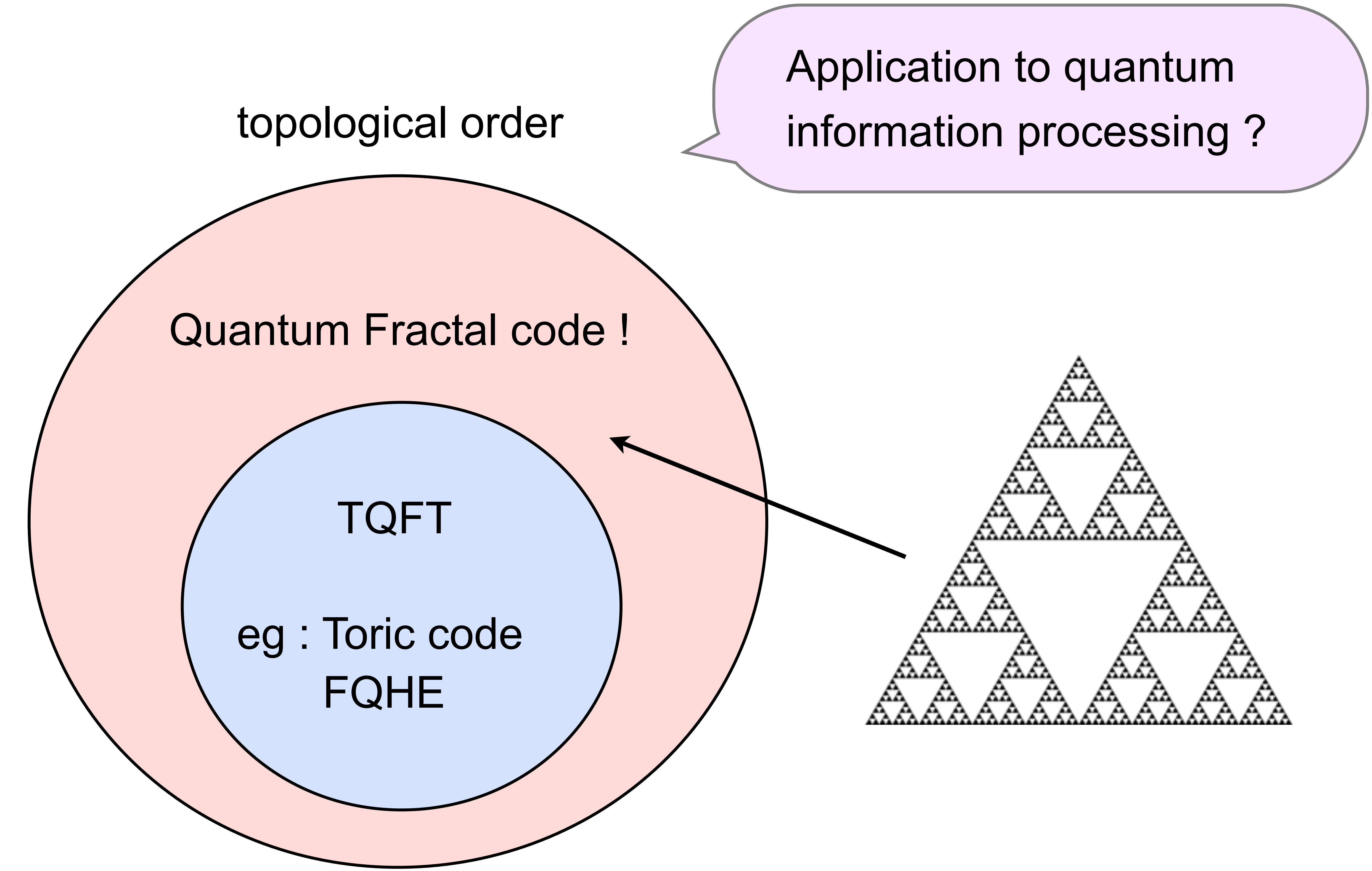
Only discrete scale symmetries.



(c) Ground states correspond to limit cycles of (Kadanoff-type) real-space RG transformations with imaginary scaling dimensions.

*Details can be found in Phys. Rev. B 88, 125122


topological order
Quantum Fractal code !
TQFT
eg : Toric code
FQHE



Application to quantum
information processing ?
topological order
Quantum Fractal code !
TQFT
eg : Toric code
FQHE

Application 1: (Marginally) Self-Correcting Quantum Memory

Does self-correcting quantum memory exist in 3d ?

- Cubic code (Haah 2011) "marginally" self-correcting with $T_C=0$
  - No string-like logical operator.
  - Log(L) energy barrier

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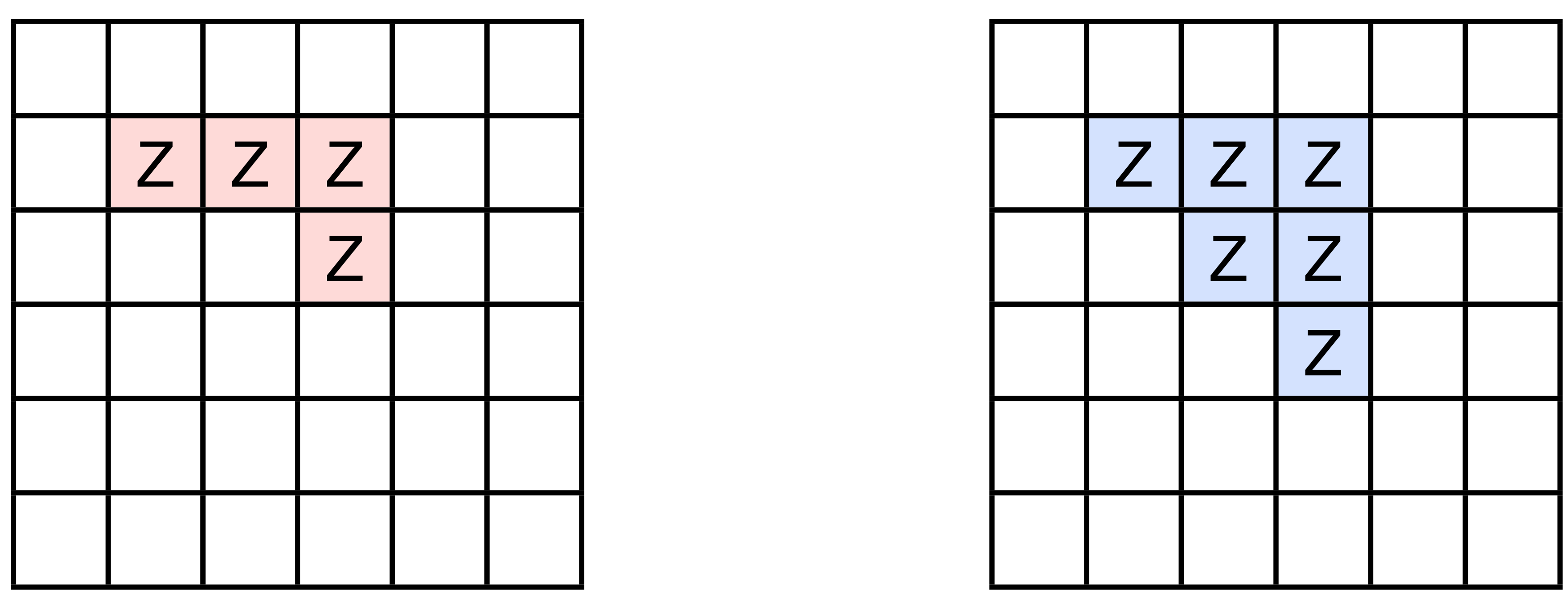


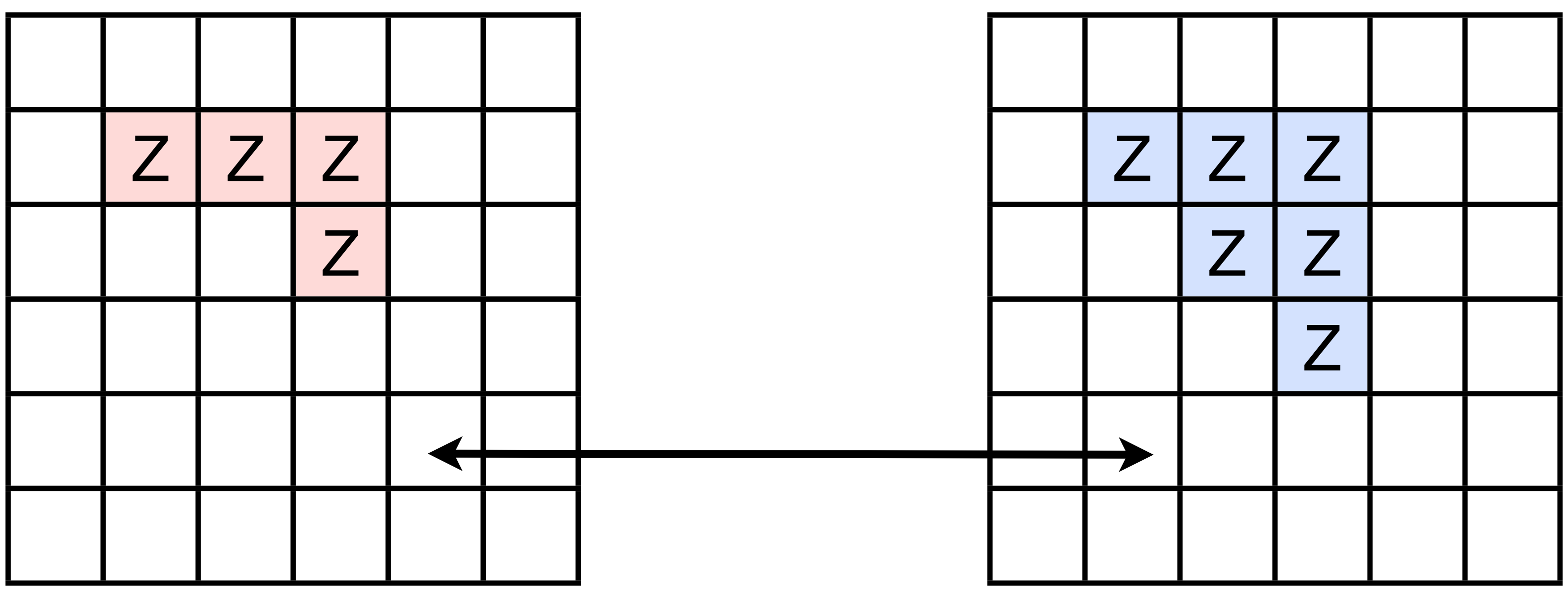
model A model B

Application 1: (Marginally) Self-Correcting Quantum Memory

Does self-correcting quantum memory exist in 3d ?

Theorem (BY2013)

The model is free from string-like logical operators if and only if model A and model B are “algebraically different”.



model A

model B

different fractals

Application 2: (asymptotically) good quantum LDPC code

Quantum local code bound (Bravyi et al 2009)

$$kd^{\frac{2}{D-1}} \leq O(n).$$

Conjecture (BY2013)

Quantum fractal codes asymptotically saturate the bound with

$$k \sim O(L^{D-2}) \qquad d \underset{?}{\sim} O(L^{D-1-\epsilon})$$

- $(D-1-\epsilon)$ - dim logical operators exist.

Application 2: (asymptotically) good quantum LDPC code

- Asymptotically good quantum LDPC code?

infinite dimensional limit ($D \to \infty$)

$$k \sim O(n^{1-\epsilon}) \qquad d \underset{?}{\sim} O(n^{1-\epsilon})$$

current best quantum LDPC code

$$k \sim O(n) \qquad d \sim O(n^{0.5})$$

Classical and quantum fractal code

; topological order beyond TQFT and asymptotically good quantum LDPC code

[1] Beni Yoshida, Annals of Physics 338, 134 (2013)

[2] Beni Yoshida, Phys. Rev. B 88, 125122 (2013)

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