Complexity classification of local Hamiltonian problems

Ashley Montanaro

Department of Computer Science, University of Bristol, UK

4 February 2014

arXiv:1311.3161

University of BRISTOL

Joint work with Toby Cubitt:

Introduction

Constraint satisfaction problems are ubiquitous in computer science.

Introduction

Constraint satisfaction problems are ubiquitous in computer science. Two classic examples:

- The 3-SAT problem: given a boolean formula in conjunctive normal form with at most 3 variables per clause, is there a satisfying assignment to the formula?

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_4)$$

Introduction

Constraint satisfaction problems are ubiquitous in computer science. Two classic examples:

- The 3-SAT problem: given a boolean formula in conjunctive normal form with at most 3 variables per clause, is there a satisfying assignment to the formula?

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_4)$$

- Solving 3-term linear equations: given a system of linear equations over $\mathbb{F}_2$ with at most 3 variables per equation, is there a solution to all the equations?

$$x_1 + x_2 + x_4 = 0, \quad x_2 + x_3 = 1, \quad x_1 + x_4 = 0$$

Introduction

Constraint satisfaction problems are ubiquitous in computer science. Two classic examples:

- The 3-SAT problem: given a boolean formula in conjunctive normal form with at most 3 variables per clause, is there a satisfying assignment to the formula?

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2) \wedge (x_4)$$

- Solving 3-term linear equations: given a system of linear equations over $\mathbb{F}_2$ with at most 3 variables per equation, is there a solution to all the equations?

$$x_1 + x_2 + x_4 = 0, \quad x_2 + x_3 = 1, \quad x_1 + x_4 = 0$$

The first of these is NP-complete, the second is in P.

General constraint satisfaction problems

A very general way to study these kind of problems is via the framework of the problem $\mathcal{S}$-CSP.

- Let $\mathcal{S}$ be a set of constraints, where a constraint is a boolean function acting on a constant number of bits.
- An instance of $\mathcal{S}$-CSP on n bits is specified by a sequence of constraints picked from $\mathcal{S}$ applied to subsets of the bits.

General constraint satisfaction problems

A very general way to study these kind of problems is via the framework of the problem $\mathcal{S}$-CSP.

- Let $\mathcal{S}$ be a set of constraints, where a constraint is a boolean function acting on a constant number of bits.
- An instance of $\mathcal{S}$-CSP on n bits is specified by a sequence of constraints picked from $\mathcal{S}$ applied to subsets of the bits.
- Our task is to determine whether there exists an assignment to the variables such that all the constraints are satisfied (evaluate to 1).

General constraint satisfaction problems

A very general way to study these kind of problems is via the framework of the problem $\mathcal{S}$-CSP.

- Let $\mathcal{S}$ be a set of constraints, where a constraint is a boolean function acting on a constant number of bits.
- An instance of $\mathcal{S}$-CSP on n bits is specified by a sequence of constraints picked from $\mathcal{S}$ applied to subsets of the bits.
- Our task is to determine whether there exists an assignment to the variables such that all the constraints are satisfied (evaluate to 1).

Theorem [Schaefer '78]

$\mathcal{S}$-CSP is either in P or NP-complete. Further, which of these is the case can be determined easily for a given $\mathcal{S}$.

Local Hamiltonian problems

A natural quantum generalisation of CSPs is the QMA-complete k-LOCAL HAMILTONIAN problem [Kitaev, Shen and Vyalyi '02].

k-LOCAL HAMILTONIAN

We are given a k-local Hamiltonian $H = \sum_{i=1}^{m} H^{(i)}$ on n qubits, and two numbers $a < b$ such that $b - a \geqslant 1/\operatorname{poly}(n)$. Promised that the smallest eigenvalue of H is either at most a, or at least b, our task is to determine which of these is the case.

Local Hamiltonian problems

A natural quantum generalisation of CSPs is the QMA-complete k-LOCAL HAMILTONIAN problem [Kitaev, Shen and Vyalyi '02].

k-LOCAL HAMILTONIAN

We are given a k-local Hamiltonian $H = \sum_{i=1}^{m} H^{(i)}$ on n qubits, and two numbers $a < b$ such that $b - a \geqslant 1/\operatorname{poly}(n)$. Promised that the smallest eigenvalue of H is either at most a, or at least b, our task is to determine which of these is the case.

- Essentially equivalent to calculating the ground-state energies of physical systems.

Local Hamiltonian problems

A natural quantum generalisation of CSPs is the QMA-complete k-local Hamiltonian problem [Kitaev, Shen and Vyalyi '02].

k-local Hamiltonian

We are given a k-local Hamiltonian $H = \sum_{i=1}^{m} H^{(i)}$ on n qubits, and two numbers $a < b$ such that $b - a \geqslant 1/\operatorname{poly}(n)$. Promised that the smallest eigenvalue of H is either at most a, or at least b, our task is to determine which of these is the case.

- Essentially equivalent to calculating the ground-state energies of physical systems.
- This connection to physics motivates the study of k-local Hamiltonian with restricted types of interactions.
- The aim: to prove QMA-hardness (or otherwise) of problems of more direct physical interest.

The $\mathcal{S}$-Hamiltonian problem

Let $\mathcal{S}$ be a fixed subset of Hermitian matrices on at most k qubits, for some constant k.

$\mathcal{S}$-Hamiltonian

$\mathcal{S}$-Hamiltonian is the special case of k-local Hamiltonian where the overall Hamiltonian H is specified by a sum of matrices $H^{(i)}$, each of which acts non-trivially on at most k qubits, and whose non-trivial part is proportional to a matrix picked from $\mathcal{S}$.

The $\mathcal{S}$-Hamiltonian problem

Let $\mathcal{S}$ be a fixed subset of Hermitian matrices on at most k qubits, for some constant k.

$\mathcal{S}$-Hamiltonian

$\mathcal{S}$-Hamiltonian is the special case of k-local Hamiltonian where the overall Hamiltonian H is specified by a sum of matrices $H^{(i)}$, each of which acts non-trivially on at most k qubits, and whose non-trivial part is proportional to a matrix picked from $\mathcal{S}$.

We then have the following general question:

Problem

Given $\mathcal{S}$, characterise the computational complexity of $\mathcal{S}$-Hamiltonian.

Some examples

- $\mathcal{S} = \{ZZ\}$: the (general) Ising model (NP-complete):

$$H = \sum_{i<j} \alpha_{ij} Z_i Z_j.$$

Some examples

- $\mathcal{S} = \{ZZ\}$: the (general) Ising model (NP-complete):

$$H = \sum_{i<j} \alpha_{ij} Z_i Z_j.$$

- $\mathcal{S} = \{ZZ, X\}$: the (general) Ising model with transverse magnetic fields (NP-hard, contained within StoqMA):

$$H = \sum_{i<j} \alpha_{ij} Z_i Z_j + \sum_k \beta_k X_k.$$

Some examples

- $\mathcal{S} = \{ZZ\}$: the (general) Ising model (NP-complete):

$$H = \sum_{i<j} \alpha_{ij} Z_i Z_j.$$

- $\mathcal{S} = \{ZZ, X\}$: the (general) Ising model with transverse magnetic fields (NP-hard, contained within StoqMA):

$$H = \sum_{i<j} \alpha_{ij} Z_i Z_j + \sum_k \beta_k X_k.$$

Some QMA-complete cases:

- $\mathcal{S} = \{XX, ZZ, X, Z\}$, $\mathcal{S} = \{XZ, X, Z\}$ [Biamonte and Love '08].
- $\mathcal{S} = \{XX + YY + ZZ, X, Y, Z\}$ [Schuch and Verstraete '09].

Our main result

Let $\mathcal{S}$ be an arbitrary fixed subset of Hermitian matrices on at most 2 qubits.

Theorem

1. If every matrix in $\mathcal{S}$ is 1-local, $\mathcal{S}$-HAMILTONIAN is in P;

Our main result

Let $\mathcal{S}$ be an arbitrary fixed subset of Hermitian matrices on at most 2 qubits.

Theorem

1. If every matrix in $\mathcal{S}$ is 1-local, $\mathcal{S}$-HAMILTONIAN is in P;
2. Otherwise, if there exists $U \in SU(2)$ such that U locally diagonalises $\mathcal{S}$, then $\mathcal{S}$-HAMILTONIAN is NP-complete;

Our main result

Let $\mathcal{S}$ be an arbitrary fixed subset of Hermitian matrices on at most 2 qubits.

Theorem

1. If every matrix in $\mathcal{S}$ is 1-local, $\mathcal{S}$-HAMILTONIAN is in P;
2. Otherwise, if there exists $U \in SU(2)$ such that U locally diagonalises $\mathcal{S}$, then $\mathcal{S}$-HAMILTONIAN is NP-complete;
3. Otherwise, if there exists $U \in SU(2)$ such that, for each 2-qubit matrix $H_i \in \mathcal{S}$, $U^{\otimes 2} H_i (U^\dagger)^{\otimes 2} = \alpha_i Z^{\otimes 2} + A_i I + I B_i$, where $\alpha_i \in \mathbb{R}$ and A_i, B_i are arbitrary 2×2 Hermitian matrices, then $\mathcal{S}$-HAMILTONIAN is polytime-equivalent to the Ising model with transverse fields;

Our main result

Let $\mathcal{S}$ be an arbitrary fixed subset of Hermitian matrices on at most 2 qubits.

Theorem

1. If every matrix in $\mathcal{S}$ is 1-local, $\mathcal{S}$-Hamiltonian is in P;
2. Otherwise, if there exists $U \in SU(2)$ such that U locally diagonalises $\mathcal{S}$, then $\mathcal{S}$-Hamiltonian is NP-complete;
3. Otherwise, if there exists $U \in SU(2)$ such that, for each 2-qubit matrix $H_i \in \mathcal{S}$, $U^{\otimes 2} H_i (U^\dagger)^{\otimes 2} = \alpha_i Z^{\otimes 2} + A_i I + I B_i$, where $\alpha_i \in \mathbb{R}$ and A_i, B_i are arbitrary 2×2 Hermitian matrices, then $\mathcal{S}$-Hamiltonian is polytime-equivalent to the Ising model with transverse fields;
4. Otherwise, $\mathcal{S}$-Hamiltonian is QMA-complete.

Notes and corollaries

The second case is stated in terms of "local diagonalisation":

- We say that $U \in SU(2)$ locally diagonalises a $2^k \times 2^k$ matrix M if $U^{\otimes k} M (U^\dagger)^{\otimes k}$ is diagonal.

Notes and corollaries

The second case is stated in terms of "local diagonalisation":

- We say that $U \in SU(2)$ locally diagonalises a $2^k \times 2^k$ matrix M if $U^{\otimes k} M (U^\dagger)^{\otimes k}$ is diagonal.
- We say that U locally diagonalises $\mathcal{S}$ if U locally diagonalises M for all $M \in \mathcal{S}$.

Notes and corollaries

The second case is stated in terms of "local diagonalisation":

- We say that $U \in SU(2)$ locally diagonalises a $2^k \times 2^k$ matrix M if $U^{\otimes k} M (U^\dagger)^{\otimes k}$ is diagonal.
- We say that U locally diagonalises $\mathcal{S}$ if U locally diagonalises M for all $M \in \mathcal{S}$.
- Note that matrices in $\mathcal{S}$ may be of different sizes.

Notes and corollaries

The second case is stated in terms of "local diagonalisation":

- We say that $U \in SU(2)$ locally diagonalises a $2^k \times 2^k$ matrix M if $U^{\otimes k} M (U^\dagger)^{\otimes k}$ is diagonal.
- We say that U locally diagonalises $\mathcal{S}$ if U locally diagonalises M for all $M \in \mathcal{S}$.
- Note that matrices in $\mathcal{S}$ may be of different sizes.

As corollaries of our main result, we have that:

- The (general) Heisenberg model is QMA-complete ($\mathcal{S} = \{XX + YY + ZZ\}$)
- The (general) XY model is QMA-complete ($\mathcal{S} = \{XX + YY\}$)

…as well as many other cases.

Notes and corollaries

The second case is stated in terms of "local diagonalisation":

- We say that $U \in SU(2)$ locally diagonalises a $2^k \times 2^k$ matrix M if $U^{\otimes k} M (U^\dagger)^{\otimes k}$ is diagonal.
- We say that U locally diagonalises $\mathcal{S}$ if U locally diagonalises M for all $M \in \mathcal{S}$.
- Note that matrices in $\mathcal{S}$ may be of different sizes.

As corollaries of our main result, we have that:

- The (general) Heisenberg model is QMA-complete ($\mathcal{S} = \{XX + YY + ZZ\}$)
- The (general) XY model is QMA-complete ($\mathcal{S} = \{XX + YY\}$)

. . . as well as many other cases. We can think of this result as a quantum analogue of Schaefer's dichotomy theorem.

Remarks on this result

- We assume that, given a set of interactions $\mathcal{S}$, we are allowed to produce an overall Hamiltonian by applying each interaction $M \in \mathcal{S}$ scaled by an arbitrary real weight, which can be either positive or negative.

Remarks on this result

- We assume that, given a set of interactions $\mathcal{S}$, we are allowed to produce an overall Hamiltonian by applying each interaction $M \in \mathcal{S}$ scaled by an arbitrary real weight, which can be either positive or negative.
- We assume that we are allowed to apply the interactions in $\mathcal{S}$ across any choice of subsets of the qubits. That is, the interaction pattern is not constrained by any spatial locality, planarity or symmetry considerations.

Remarks on this result

- We assume that, given a set of interactions $\mathcal{S}$, we are allowed to produce an overall Hamiltonian by applying each interaction $M \in \mathcal{S}$ scaled by an arbitrary real weight, which can be either positive or negative.

- We assume that we are allowed to apply the interactions in $\mathcal{S}$ across any choice of subsets of the qubits. That is, the interaction pattern is not constrained by any spatial locality, planarity or symmetry considerations.

- Some of the interactions in $\mathcal{S}$ could be non-symmetric under permutation of the qubits on which they act. We assume that we are allowed to apply such interactions to any permutation of the qubits.

Remarks on this result

- We assume that, given a set of interactions $\mathcal{S}$, we are allowed to produce an overall Hamiltonian by applying each interaction $M \in \mathcal{S}$ scaled by an arbitrary real weight, which can be either positive or negative.

- We assume that we are allowed to apply the interactions in $\mathcal{S}$ across any choice of subsets of the qubits. That is, the interaction pattern is not constrained by any spatial locality, planarity or symmetry considerations.

- Some of the interactions in $\mathcal{S}$ could be non-symmetric under permutation of the qubits on which they act. We assume that we are allowed to apply such interactions to any permutation of the qubits.

- We can assume without loss of generality that the identity matrix $I \in \mathcal{S}$ (we can add an arbitrary "energy shift").

Proof techniques

The basic idea behind the proof of the QMA-hardness part is to use reductions.

> "To prove QMA-hardness of $\mathcal{A}$-Hamiltonian, approximately simulate some other set of interactions $\mathcal{B}$, where $\mathcal{B}$-Hamiltonian is QMA-hard."

Proof techniques

The basic idea behind the proof of the QMA-hardness part is to use reductions.

> "To prove QMA-hardness of $\mathcal{A}$-Hamiltonian, approximately simulate some other set of interactions $\mathcal{B}$, where $\mathcal{B}$-Hamiltonian is QMA-hard."

- Given two Hamiltonians H and V, we form $\widetilde{H} = V + \Delta H$, where Δ is a large parameter.
- Then $\widetilde{H}_{<\Delta/2}$, the low-energy part of $\widetilde{H}$, is effectively the same as V_-, the projection of V onto the lowest-energy eigenspace of H.

Proof techniques

The basic idea behind the proof of the QMA-hardness part is to use reductions.

> "To prove QMA-hardness of $\mathcal{A}$-Hamiltonian, approximately simulate some other set of interactions $\mathcal{B}$, where $\mathcal{B}$-HAMILTONIAN is QMA-hard."

- Given two Hamiltonians H and V, we form $\widetilde{H} = V + \Delta H$, where Δ is a large parameter.
- Then $\widetilde{H}_{<\Delta/2}$, the low-energy part of $\widetilde{H}$, is effectively the same as V_{-}, the projection of V onto the lowest-energy eigenspace of H.

Projection Lemma (informal, based on [Oliveira-Terhal '08])

If $\Delta = \delta \|V\|^2$, then

$$\|\widetilde{H}_{<\Delta/2} - V_{-}\| = O(1/\delta).$$

Example: the Heisenberg model

The case $\mathcal{S} = \{XX + YY + ZZ\}$ illustrates a difficulty with this idea. Let

$$H = \sum_{i<j} \alpha_{ij}(X_i X_j + Y_i Y_j + Z_i Z_j).$$

Example: the Heisenberg model

The case $\mathcal{S} = \{XX + YY + ZZ\}$ illustrates a difficulty with this idea. Let

$$H = \sum_{i<j} \alpha_{ij}(X_i X_j + Y_i Y_j + Z_i Z_j).$$

- $XX + YY + ZZ = 2F - I$ is invariant under conjugation by $U^{\otimes 2}$ for all $U \in SU(2)$ (where F is the swap operator).

Example: the Heisenberg model

The case $\mathcal{S} = \{XX + YY + ZZ\}$ illustrates a difficulty with this idea. Let

$$H = \sum_{i<j} \alpha_{ij}(X_i X_j + Y_i Y_j + Z_i Z_j).$$

- $XX + YY + ZZ = 2F - I$ is invariant under conjugation by $U^{\otimes 2}$ for all $U \in SU(2)$ (where F is the swap operator).
- So the eigenspaces of H are all invariant under conjugation by $U^{\otimes n}$!

This means that we cannot hope to implement an arbitrary Hamiltonian using only this interaction.

Example: the Heisenberg model

The case $\mathcal{S} = \{XX + YY + ZZ\}$ illustrates a difficulty with this idea. Let

$$H = \sum_{i<j} \alpha_{ij}(X_i X_j + Y_i Y_j + Z_i Z_j).$$

- $XX + YY + ZZ = 2F - I$ is invariant under conjugation by $U^{\otimes 2}$ for all $U \in SU(2)$ (where F is the swap operator).
- So the eigenspaces of H are all invariant under conjugation by $U^{\otimes n}$!

This means that we cannot hope to implement an arbitrary Hamiltonian using only this interaction.

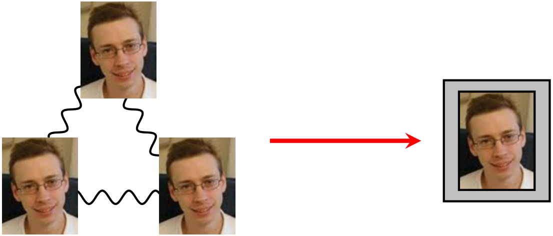
Just as with classical CSPs, the way round this is to use encodings.

Example: the Heisenberg model

- We would like to find a gadget that encodes qubits, and lets us encode operations across qubits.

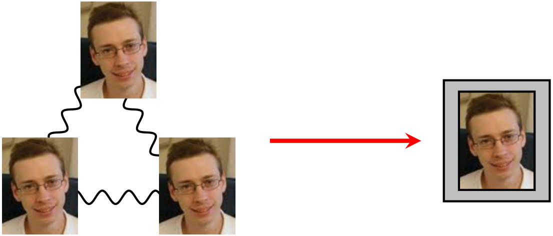
Example: the Heisenberg model

- We would like to find a gadget that encodes qubits, and lets us encode operations across qubits.
- We try to encode a logical qubit within a triangle of 3 physical qubits:



Example: the Heisenberg model

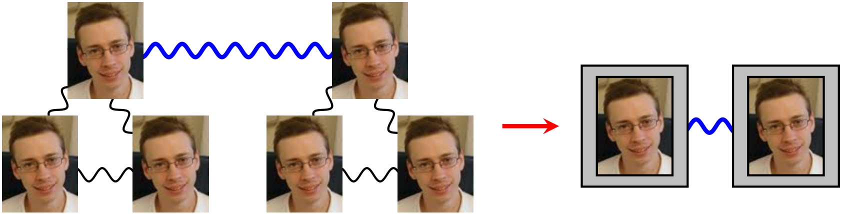
- We would like to find a gadget that encodes qubits, and lets us encode operations across qubits.
- We try to encode a logical qubit within a triangle of 3 physical qubits:



- This is inspired by previous work on universality of the exchange interaction [Kempe et al. '00].
- We can find a 4-dimensional subspace of the 3 qubits such that, within this subspace, we can make logical $Z \otimes I$ and $X \otimes I$ operators.

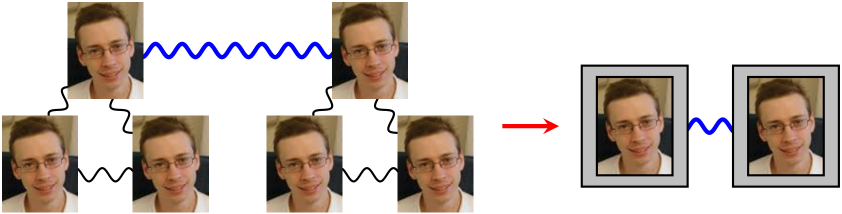
Example: the Heisenberg model

We would now like to make pairwise interactions across logical qubits.



Example: the Heisenberg model

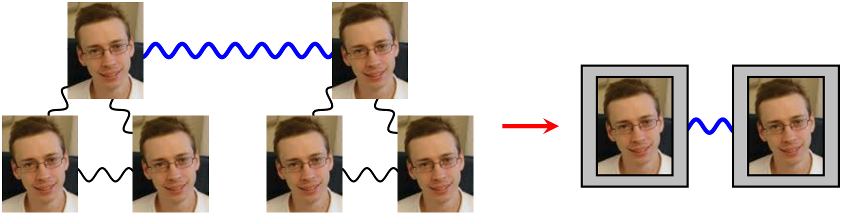
We would now like to make pairwise interactions across logical qubits.



- This can almost be done by applying Heisenberg interactions across different choices of physical qubits.

Example: the Heisenberg model

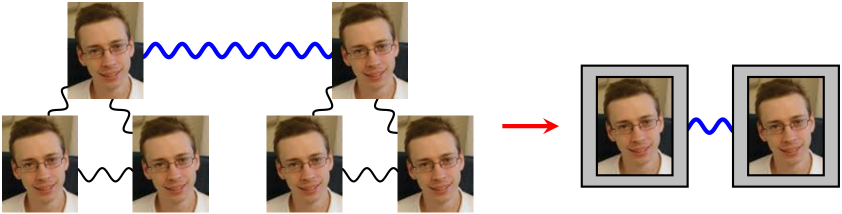
We would now like to make pairwise interactions across logical qubits.



- This can almost be done by applying Heisenberg interactions across different choices of physical qubits.
- Let the logical qubits in the first (resp. second) triangle be labelled (1,2) (resp. (3,4)).

Example: the Heisenberg model

We would now like to make pairwise interactions across logical qubits.



- This can almost be done by applying Heisenberg interactions across different choices of physical qubits.
- Let the logical qubits in the first (resp. second) triangle be labelled (1,2) (resp. (3,4)).
- By applying suitable linear combinations across qubits, we can effectively make

$$X_1 X_3 (2F - I)_{24}, \quad Z_1 Z_3 (2F - I)_{24}, \quad I_1 I_3 (2F - I)_{24}.$$

Example: the Heisenberg model

So, using Heisenberg interactions alone, we can implement an arbitrary (logical) Hamiltonian of the form

$$H = \sum_{k=1}^{n} (\alpha_k X_k + \beta_k Z_k) I_{k'} + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j)(2F - I)_{i'j'},$$

where we identify the i'th logical qubit pair with indices (i, i').

Example: the Heisenberg model

So, using Heisenberg interactions alone, we can implement an arbitrary (logical) Hamiltonian of the form

$$H = \sum_{k=1}^{n} (\alpha_k X_k + \beta_k Z_k) I_{k'} + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j)(2F - I)_{i'j'},$$

where we identify the i'th logical qubit pair with indices (i, i').

- We would like to remove the $(2F - I)$ operators.

Example: the Heisenberg model

So, using Heisenberg interactions alone, we can implement an arbitrary (logical) Hamiltonian of the form

$$H = \sum_{k=1}^{n} (\alpha_k X_k + \beta_k Z_k) I_{k'} + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j)(2F - I)_{i'j'},$$

where we identify the i'th logical qubit pair with indices (i, i').

- We would like to remove the $(2F - I)$ operators.
- To do this, we force the primed qubits to be in some state by very strong $F_{i'j'}$ interactions: we add the (logical) term

$$G = \Delta \sum_{i<j} w_{ij} F_{i'j'}$$

where w_{ij} are some weights and Δ is very large.

Example: the Heisenberg model

So, using Heisenberg interactions alone, we can implement an arbitrary (logical) Hamiltonian of the form

$$H = \sum_{k=1}^{n} (\alpha_k X_k + \beta_k Z_k) I_{k'} + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j)(2F - I)_{i'j'},$$

where we identify the i'th logical qubit pair with indices (i, i').

- We would like to remove the $(2F - I)$ operators.
- To do this, we force the primed qubits to be in some state by very strong $F_{i'j'}$ interactions: we add the (logical) term

$$G = \Delta \sum_{i<j} w_{ij} F_{i'j'}$$

  where w_{ij} are some weights and Δ is very large.
- We can do this by making $I_i I_j (2F - I)_{i'j'}$ as on last slide.

Example: the Heisenberg model

If the ground state $|\psi\rangle$ of G is non-degenerate, the primed qubits will all be effectively projected onto the ground state, and H will become (up to a small additive error)

$$\widetilde{H} = \sum_{k=1}^{n} \alpha_k X_k + \beta_k Z_k + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j) \langle\psi|(2F - I)_{i'j'}|\psi\rangle.$$

Example: the Heisenberg model

If the ground state $|\psi\rangle$ of G is non-degenerate, the primed qubits will all be effectively projected onto the ground state, and H will become (up to a small additive error)

$$\widetilde{H} = \sum_{k=1}^{n} \alpha_k X_k + \beta_k Z_k + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j) \langle\psi|(2F - I)_{i'j'}|\psi\rangle.$$

- So we need to find a G such that the ground state is non-degenerate and $\langle\psi|(2F - I)_{i'j'}|\psi\rangle \neq 0$ for all i, j (and also these quantities should be easily computable).

Example: the Heisenberg model

If the ground state $|\psi\rangle$ of G is non-degenerate, the primed qubits will all be effectively projected onto the ground state, and H will become (up to a small additive error)

$$\widetilde{H} = \sum_{k=1}^{n} \alpha_k X_k + \beta_k Z_k + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j) \langle\psi|(2F - I)_{i'j'}|\psi\rangle.$$

- So we need to find a G such that the ground state is non-degenerate and $\langle\psi|(2F - I)_{i'j'}|\psi\rangle \neq 0$ for all i, j (and also these quantities should be easily computable).
- Not so easy! This corresponds to an exactly solvable special case of the Heisenberg model, and not many of these are known.

Example: the Heisenberg model

If the ground state $|\psi\rangle$ of G is non-degenerate, the primed qubits will all be effectively projected onto the ground state, and H will become (up to a small additive error)

$$\widetilde{H} = \sum_{k=1}^{n} \alpha_k X_k + \beta_k Z_k + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j) \langle\psi|(2F - I)_{i'j'}|\psi\rangle.$$

- So we need to find a G such that the ground state is non-degenerate and $\langle\psi|(2F - I)_{i'j'}|\psi\rangle \neq 0$ for all i, j (and also these quantities should be easily computable).
- Not so easy! This corresponds to an exactly solvable special case of the Heisenberg model, and not many of these are known.
- Luckily for us, the Lieb-Mattis model [Lieb and Mattis '62] has precisely the properties we need.

The normal form

We've dealt with the Heisenberg model… what about everything else?

The normal form

We've dealt with the Heisenberg model… what about everything else? We can simplify things using a very similar normal form to one identified by [Dür et al. '01, Bennett et al. '02]:

Lemma

Let H be a 2-qubit interaction which is symmetric under swapping qubits. Then there exists $U \in SU(2)$ such that the 2-local part of $U^{\otimes 2} H (U^\dagger)^{\otimes 2}$ is of the form

$$\alpha XX + \beta YY + \gamma ZZ.$$

The normal form

We've dealt with the Heisenberg model…what about everything else? We can simplify things using a very similar normal form to one identified by [Dür et al. '01, Bennett et al. '02]:

Lemma

Let H be a 2-qubit interaction which is symmetric under swapping qubits. Then there exists $U \in SU(2)$ such that the 2-local part of $U^{\otimes 2} H (U^\dagger)^{\otimes 2}$ is of the form

$$\alpha XX + \beta YY + \gamma ZZ.$$

Why is this useful? If we conjugate each term by $U^{\otimes 2}$ in a 2-local Hamiltonian with only H interactions, it doesn't change the eigenvalues:

$$\sum_{i \neq j} \alpha_{ij} (U^{\otimes 2} H (U^\dagger)^{\otimes 2})_{ij} = U^{\otimes n} \left(\sum_{i \neq j} \alpha_{ij} H_{ij} \right) (U^\dagger)^{\otimes n}.$$

The other QMA-complete cases

This normal form drastically reduces the number of interactions we have to consider to a few special cases:

- The XY model $\mathcal{S} = \{XX + YY\}$ uses similar techniques to the Heisenberg model, but the gadgets are a bit simpler.
- For $\mathcal{S} = \{XX + \alpha YY + \beta ZZ\}$, we can reduce from the XY model.
- We also need to deal with the antisymmetric case $\mathcal{S} = \{XZ - ZX\}$.
- For interactions with 1-local terms, using gadgets we can effectively delete the 1-local parts.

The other QMA-complete cases

This normal form drastically reduces the number of interactions we have to consider to a few special cases:

- The XY model $\mathcal{S} = \{XX + YY\}$ uses similar techniques to the Heisenberg model, but the gadgets are a bit simpler.
- For $\mathcal{S} = \{XX + \alpha YY + \beta ZZ\}$, we can reduce from the XY model.
- We also need to deal with the antisymmetric case $\mathcal{S} = \{XZ - ZX\}$.
- For interactions with 1-local terms, using gadgets we can effectively delete the 1-local parts.

Finding and verifying each of the gadgets required was somewhat painful and required the use of a computer algebra package.

Conclusions and open problems

We have (almost) completely characterised the complexity of 2-local qubit Hamiltonians.

Despite this, our work is only just beginning…

Conclusions and open problems

We have (almost) completely characterised the complexity of 2-local qubit Hamiltonians.

Despite this, our work is only just beginning…

- What about k-qubit interactions for $k > 2$? We also have a complete characterisation here in the special case where we assume that we are allowed access to arbitrary local terms (i.e. $\{X, Y, Z\} \subseteq \mathcal{S}$).

Conclusions and open problems

We have (almost) completely characterised the complexity of 2-local qubit Hamiltonians.

Despite this, our work is only just beginning...

- What about k-qubit interactions for $k > 2$? We also have a complete characterisation here in the special case where we assume that we are allowed access to arbitrary local terms (i.e. $\{X, Y, Z\} \subseteq \mathcal{S}$).

- What about local dimension $d > 2$? Classically, the complexity of d-ary CSPs is still unresolved.

More open problems

- What about restrictions on the interaction pattern or weights? e.g. 1-dimensional systems, 2-D lattices, the antiferromagnetic Heisenberg model etc.

More open problems

- What about restrictions on the interaction pattern or weights? e.g. 1-dimensional systems, 2-D lattices, the antiferromagnetic Heisenberg model etc.

- See very recent independent work (previous talk!) proving QMA-hardness for $\mathcal{S} = \{XX + YY, Z\}$ when weights of $XX + YY$ terms are positive and weights of Z terms are negative [Childs, Gosset and Webb '13]...

More open problems

- What about restrictions on the interaction pattern or weights? e.g. 1-dimensional systems, 2-D lattices, the antiferromagnetic Heisenberg model etc.

- See very recent independent work (previous talk!) proving QMA-hardness for $\mathcal{S} = \{XX + YY, Z\}$ when weights of $XX + YY$ terms are positive and weights of Z terms are negative [Childs, Gosset and Webb '13]. . .

- What about quantum k-SAT?

More open problems

- What about restrictions on the interaction pattern or weights? e.g. 1-dimensional systems, 2-D lattices, the antiferromagnetic Heisenberg model etc.

- See very recent independent work (previous talk!) proving QMA-hardness for $\mathcal{S} = \{XX + YY, Z\}$ when weights of $XX + YY$ terms are positive and weights of Z terms are negative [Childs, Gosset and Webb '13]…

- What about quantum k-SAT?

- Finally, what is the complexity of the transverse Ising model? Our intuition: at least MA-hard…for now, we encapsulate it as a new complexity class TIM.

Thanks!

arXiv:1311.3161

Allowing local terms

One variant of this framework is to allow arbitrary local terms ("magnetic fields").

$\mathcal{S}$-Hamiltonian with local terms

$\mathcal{S}$-Hamiltonian with local terms is the special case of $\mathcal{S}$-Hamiltonian where $\mathcal{S}$ is assumed to contain X, Y, Z.

- This is equivalent to $\mathcal{S}$ containing all 1-local interactions.

Allowing local terms

One variant of this framework is to allow arbitrary local terms ("magnetic fields").

$\mathcal{S}$-Hamiltonian with local terms

$\mathcal{S}$-Hamiltonian with local terms is the special case of $\mathcal{S}$-Hamiltonian where $\mathcal{S}$ is assumed to contain X, Y, Z.

- This is equivalent to $\mathcal{S}$ containing all 1-local interactions.
- For any $\mathcal{S}$, $\mathcal{S}$-Hamiltonian with local terms is at least as hard as $\mathcal{S}$-Hamiltonian.

Allowing local terms

One variant of this framework is to allow arbitrary local terms ("magnetic fields").

$\mathcal{S}$-Hamiltonian with local terms

$\mathcal{S}$-Hamiltonian with local terms is the special case of $\mathcal{S}$-Hamiltonian where $\mathcal{S}$ is assumed to contain X, Y, Z.

- This is equivalent to $\mathcal{S}$ containing all 1-local interactions.
- For any $\mathcal{S}$, $\mathcal{S}$-Hamiltonian with local terms is at least as hard as $\mathcal{S}$-Hamiltonian.

It is known that $\mathcal{S}$-Hamiltonian with local terms is QMA-complete when:

- $\mathcal{S} = \{XX + YY + ZZ\}$ [Schuch and Verstraete '09]
- $\mathcal{S} = \{XX, ZZ\}$ or $\mathcal{S} = \{XZ\}$ [Biamonte and Love '08]

The case with local terms

Let $\mathcal{S}$ be a fixed subset of Hermitian matrices on at most k qubits, for some constant k.

Theorem

Let $\mathcal{S}'$ be the subset formed by removing all 1-local terms from each element of $\mathcal{S}$, and then deleting all 0-local matrices. Then:

1. If $\mathcal{S}'$ is empty, $\mathcal{S}$-HAMILTONIAN WITH LOCAL TERMS is in P;

The case with local terms

Let $\mathcal{S}$ be a fixed subset of Hermitian matrices on at most k qubits, for some constant k.

Theorem

Let $\mathcal{S}'$ be the subset formed by removing all 1-local terms from each element of $\mathcal{S}$, and then deleting all 0-local matrices. Then:

1. If $\mathcal{S}'$ is empty, $\mathcal{S}$-Hamiltonian with local terms is in P;
2. Otherwise, if there exists $U \in SU(2)$ such that U locally diagonalises $\mathcal{S}'$, then $\mathcal{S}$-Hamiltonian with local terms is poly-time equivalent to the transverse Ising model;

The case with local terms

Let $\mathcal{S}$ be a fixed subset of Hermitian matrices on at most k qubits, for some constant k.

Theorem

Let $\mathcal{S}'$ be the subset formed by removing all 1-local terms from each element of $\mathcal{S}$, and then deleting all 0-local matrices. Then:

1. If $\mathcal{S}'$ is empty, $\mathcal{S}$-Hamiltonian with local terms is in P;
2. Otherwise, if there exists $U \in SU(2)$ such that U locally diagonalises $\mathcal{S}'$, then $\mathcal{S}$-Hamiltonian with local terms is poly-time equivalent to the transverse Ising model;
3. Otherwise, $\mathcal{S}$-Hamiltonian with local terms is QMA-complete.

The idea

The basic idea:

> "To prove QMA-hardness of $\mathcal{A}$-Hamiltonian, approximately simulate some other set of interactions $\mathcal{B}$, where $\mathcal{B}$-HAMILTONIAN is QMA-hard."

The idea

The basic idea:

> "To prove QMA-hardness of $\mathcal{A}$-Hamiltonian, approximately simulate some other set of interactions $\mathcal{B}$, where $\mathcal{B}$-HAMILTONIAN is QMA-hard."

- To do this, we use two kinds of reductions, both based on perturbation theory.

The idea

The basic idea:

> "To prove QMA-hardness of $\mathcal{A}$-Hamiltonian, approximately simulate some other set of interactions $\mathcal{B}$, where $\mathcal{B}$-HAMILTONIAN is QMA-hard."

- To do this, we use two kinds of reductions, both based on perturbation theory.
- The first-order perturbative gadgets we use are based on ideas going back to [Oliveira and Terhal '08] and [Schuch and Verstraete '08].

The idea

The basic idea:

> “To prove QMA-hardness of $\mathcal{A}$-Hamiltonian, approximately simulate some other set of interactions $\mathcal{B}$, where $\mathcal{B}$-Hamiltonian is QMA-hard.”

- To do this, we use two kinds of reductions, both based on perturbation theory.
- The first-order perturbative gadgets we use are based on ideas going back to [Oliveira and Terhal ’08] and [Schuch and Verstraete ’08].
- The basic idea: to implement an effective interaction across two qubits a and c, add a new mediator qubit b interacting with each of a and c, and put a strong 1-local interaction on b.

Example

Claim (similar to results of [Schuch and Verstraete '08])

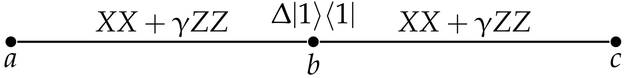
For any $\gamma \neq 0$, $\{XX + \gamma ZZ\}$-Hamiltonian with local terms is QMA-complete.

Example

Claim (similar to results of [Schuch and Verstraete '08])

For any $\gamma \neq 0$, $\{XX + \gamma ZZ\}$-HAMILTONIAN WITH LOCAL TERMS is QMA-complete.

- We use the following perturbative gadget, taking Δ to be a large coefficient:

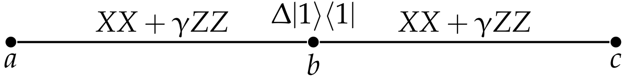


Example

Claim (similar to results of [Schuch and Verstraete '08])

For any $\gamma \neq 0$, $\{XX + \gamma ZZ\}$-Hamiltonian with local terms is QMA-complete.

- We use the following perturbative gadget, taking Δ to be a large coefficient:



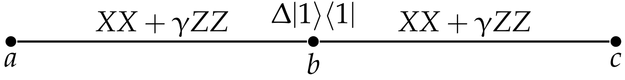
- This forces qubit b to (approximately) be in the state $|0\rangle$.

Example

Claim (similar to results of [Schuch and Verstraete '08])

For any $\gamma \neq 0$, $\{XX + \gamma ZZ\}$-Hamiltonian with local terms is QMA-complete.

- We use the following perturbative gadget, taking Δ to be a large coefficient:



- This forces qubit b to (approximately) be in the state $|0\rangle$.
- It turns out that, up to local and lower-order terms, the effective interaction across the remaining qubits is

$$H_{\text{eff}} \propto X_a X_c.$$

Example

- So, given access to terms of the form $XX + \gamma ZZ$, we can effectively make XX terms. By subtracting from $XX + \gamma ZZ$, we can also make ZZ terms.

Example

- So, given access to terms of the form $XX + \gamma ZZ$, we can effectively make XX terms. By subtracting from $XX + \gamma ZZ$, we can also make ZZ terms.
- The claim follows from the result of [Biamonte and Love '08] that $\{XX, ZZ\}$-Hamiltonian with local terms is QMA-complete.

Example

- So, given access to terms of the form $XX + \gamma ZZ$, we can effectively make XX terms. By subtracting from $XX + \gamma ZZ$, we can also make ZZ terms.

- The claim follows from the result of [Biamonte and Love '08] that $\{XX, ZZ\}$-Hamiltonian with local terms is QMA-complete.

We can similarly show that:

- For any $\beta, \gamma \neq 0$, $\{XX + \beta YY + \gamma ZZ\}$-Hamiltonian with local terms is QMA-complete.
- $\{XZ - ZX\}$-Hamiltonian with local terms is QMA-complete.

Example

- So, given access to terms of the form $XX + \gamma ZZ$, we can effectively make XX terms. By subtracting from $XX + \gamma ZZ$, we can also make ZZ terms.
- The claim follows from the result of [Biamonte and Love '08] that $\{XX, ZZ\}$-Hamiltonian with local terms is QMA-complete.

We can similarly show that:

- For any $\beta, \gamma \neq 0$, $\{XX + \beta YY + \gamma ZZ\}$-Hamiltonian with local terms is QMA-complete.
- $\{XZ - ZX\}$-Hamiltonian with local terms is QMA-complete.

This turns out to be all the cases we need to complete the characterisation of $\mathcal{S}$-Hamiltonian with local terms!

The different cases in the characterisation

To finish off the 2-local special case of $\mathcal{S}$-HAMILTONIAN WITH LOCAL TERMS:

- If the 2-local part of any interaction in $\mathcal{S}$ is locally equivalent to $XX + \beta YY + \gamma ZZ$ or $XZ - ZX$, we have QMA-completeness;

The different cases in the characterisation

To finish off the 2-local special case of $\mathcal{S}$-Hamiltonian with local terms:

- If the 2-local part of any interaction in $\mathcal{S}$ is locally equivalent to $XX + \beta YY + \gamma ZZ$ or $XZ - ZX$, we have QMA-completeness;
- If the 2-local part of all the interactions is locally equivalent to ZZ, using local rotations we can show equivalence to the transverse Ising model;

The different cases in the characterisation

To finish off the 2-local special case of $\mathcal{S}$-Hamiltonian with local terms:

- If the 2-local part of any interaction in $\mathcal{S}$ is locally equivalent to $XX + \beta YY + \gamma ZZ$ or $XZ - ZX$, we have QMA-completeness;
- If the 2-local part of all the interactions is locally equivalent to ZZ, using local rotations we can show equivalence to the transverse Ising model;
- If neither of these is true, we must have one interaction equivalent to XX, another to AA for some $A \neq X$ (exercise!).

The different cases in the characterisation

To finish off the 2-local special case of $\mathcal{S}$-HAMILTONIAN WITH LOCAL TERMS:

- If the 2-local part of any interaction in $\mathcal{S}$ is locally equivalent to $XX + \beta YY + \gamma ZZ$ or $XZ - ZX$, we have QMA-completeness;
- If the 2-local part of all the interactions is locally equivalent to ZZ, using local rotations we can show equivalence to the transverse Ising model;
- If neither of these is true, we must have one interaction equivalent to XX, another to AA for some $A \neq X$ (exercise!).
- So we can make $XX + AA$, which suffices for QMA-completeness.

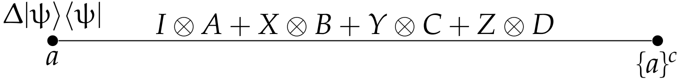
The k-local case for $k > 2$

We can generalise to $\mathcal{S}$-Hamiltonian with local terms when $\mathcal{S}$ contains k-qubit interactions, for any constant $k > 2$.

The k-local case for $k > 2$

We can generalise to $\mathcal{S}$-Hamiltonian with local terms when $\mathcal{S}$ contains k-qubit interactions, for any constant $k > 2$.

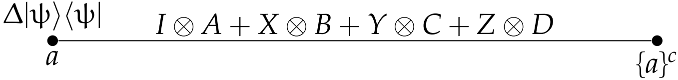
- Basic idea: using local terms, produce effective $(k-1)$-qubit interactions from k-qubit interactions, via the gadget



The k-local case for $k > 2$

We can generalise to $\mathcal{S}$-Hamiltonian with local terms when $\mathcal{S}$ contains k-qubit interactions, for any constant $k > 2$.

- Basic idea: using local terms, produce effective $(k-1)$-qubit interactions from k-qubit interactions, via the gadget

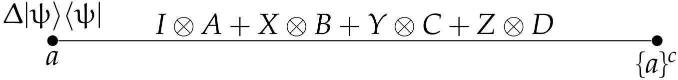


- By letting $|\psi\rangle$ be the eigenvector of X, Y or Z with eigenvalue ± 1, we can produce the effective interactions $A \pm B$, $A \pm C$ and $A \pm D$ (up to a small additive error).

The k-local case for $k > 2$

We can generalise to $\mathcal{S}$-Hamiltonian with local terms when $\mathcal{S}$ contains k-qubit interactions, for any constant $k > 2$.

- Basic idea: using local terms, produce effective $(k-1)$-qubit interactions from k-qubit interactions, via the gadget

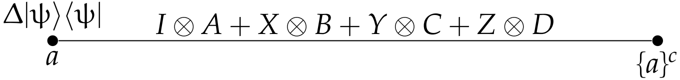


- By letting $|\psi\rangle$ be the eigenvector of X, Y or Z with eigenvalue ± 1, we can produce the effective interactions $A \pm B$, $A \pm C$ and $A \pm D$ (up to a small additive error).
- By adding/subtracting these matrices we can make each of $\{A, B, C, D\}$.

The k-local case for $k > 2$

We can generalise to $\mathcal{S}$-Hamiltonian with local terms when $\mathcal{S}$ contains k-qubit interactions, for any constant $k > 2$.

- Basic idea: using local terms, produce effective $(k-1)$-qubit interactions from k-qubit interactions, via the gadget



- By letting $|\psi\rangle$ be the eigenvector of X, Y or Z with eigenvalue ± 1, we can produce the effective interactions $A \pm B$, $A \pm C$ and $A \pm D$ (up to a small additive error).
- By adding/subtracting these matrices we can make each of $\{A, B, C, D\}$.
- So either $\mathcal{S}$ is QMA-complete, or all 2-local "parts" of each interaction in $\mathcal{S}$ are simultaneously diagonalisable by local unitaries. This case turns out to be in TIM.

$\mathcal{S}$-HAMILTONIAN: The list of lemmas

It suffices to prove QMA-completeness of the following cases:

1. $\{XX + YY + ZZ\}$-HAMILTONIAN;
2. $\{XX + YY\}$-HAMILTONIAN;
3. $\{XZ - ZX\}$-HAMILTONIAN;
4. $\{XX + \beta YY + \gamma ZZ\}$-HAMILTONIAN;
5. $\{XX + \beta YY + \gamma ZZ + AI + IA\}$-HAMILTONIAN;
6. $\{XZ - ZX + AI - IA\}$-HAMILTONIAN.

In the above, β, γ are real numbers such that at least one of β and γ is non-zero, and A is an arbitrary single-qubit Hermitian matrix.

$\mathcal{S}$-Hamiltonian: The list of lemmas

We also need some reductions from cases which are not necessarily QMA-complete:

- $\{ZZ, X, Z\}$-Hamiltonian reduces to $\{ZZ + AI + IA\}$-Hamiltonian;
- $\{ZZ, X, Z\}$-Hamiltonian reduces to $\{ZZ, AI - IA\}$-Hamiltonian.

In the above, A is any single-qubit Hermitian matrix which does not commute with Z.

And the very final case to consider:

- Let $\mathcal{S}$ be a set of diagonal Hermitian matrices on at most 2 qubits. Then, if every matrix in $\mathcal{S}$ is 1-local, $\mathcal{S}$-Hamiltonian is in P. Otherwise, $\mathcal{S}$-Hamiltonian is NP-complete.

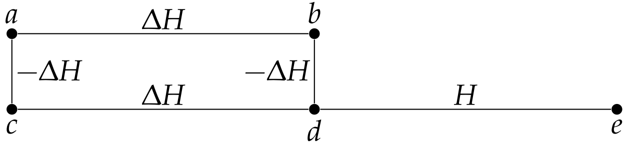
Example gadget for cases with 1-local terms

Let $H := XX + \beta YY + \gamma ZZ + AI + IA$, where β or γ is non-zero.

Lemma

$\{H\}$-HAMILTONIAN is QMA-complete.

The gadget used looks like:



- The ground state of $G := H_{ab} + H_{cd} - H_{ac} - H_{bd}$ is maximally entangled across the split $(a\text{-}c : d)$.
- So if we project H_{de} onto this state, the effective interaction produced is A on qubit e.
- This allows us to effectively delete the 1-local part of H.

The Lieb-Mattis model

The Lieb-Mattis model describes Hamiltonians of the form

$$H_{LM} = \sum_{i \in A, j \in B} X_i X_j + Y_i Y_j + Z_i Z_j,$$

where A and B are disjoint subsets of qubits.

The Lieb-Mattis model

The Lieb-Mattis model describes Hamiltonians of the form

$$H_{LM} = \sum_{i \in A, j \in B} X_i X_j + Y_i Y_j + Z_i Z_j,$$

where A and B are disjoint subsets of qubits.

Claim **[Lieb and Mattis '62, ...]**

If $|A| = |B| = n$, the ground state $|\phi\rangle$ of H_{LM} is unique. For i and j such that $i, j \in A$ or $i, j \in B$, $\langle\phi|F_{ij}|\phi\rangle = 1$. Otherwise, $\langle\phi|F_{ij}|\phi\rangle = -2/n$.

The Lieb-Mattis model

The Lieb-Mattis model describes Hamiltonians of the form

$$H_{LM} = \sum_{i \in A, j \in B} X_i X_j + Y_i Y_j + Z_i Z_j,$$

where A and B are disjoint subsets of qubits.

Claim [Lieb and Mattis '62, ...]

If $|A| = |B| = n$, the ground state $|\phi\rangle$ of H_{LM} is unique. For i and j such that $i, j \in A$ or $i, j \in B$, $\langle \phi | F_{ij} | \phi \rangle = 1$. Otherwise, $\langle \phi | F_{ij} | \phi \rangle = -2/n$.

Using this claim, we can effectively implement any Hamiltonian of the form

$$\widetilde{H} = \sum_{k=1}^{n} \alpha_k X_k + \beta_k Z_k + \sum_{i<j} \gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j,$$

which suffices for QMA-completeness [Biamonte and Love '08].