Zero-error source-channel coding with entanglement

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CWI

Joint work with

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(+Universitat Autónoma de Barcelona)

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Main Question:

Does entanglement help in zero-error communication between two parties with

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- one-way classical noisy channel

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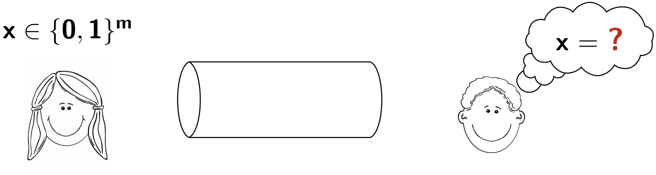
- side information from a dual source

The channel coding problem



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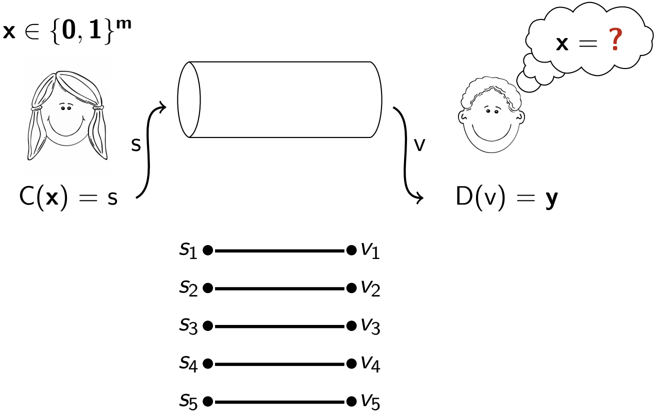

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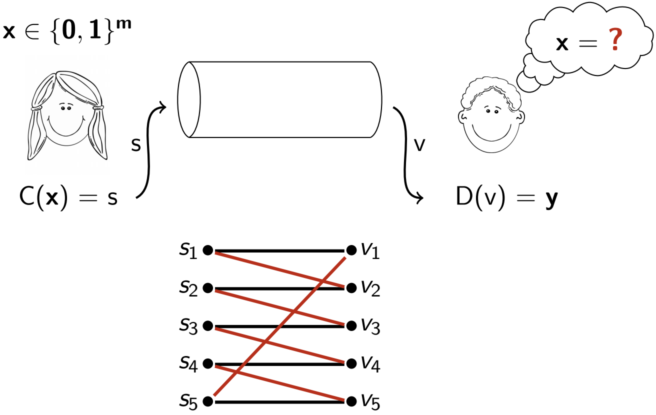

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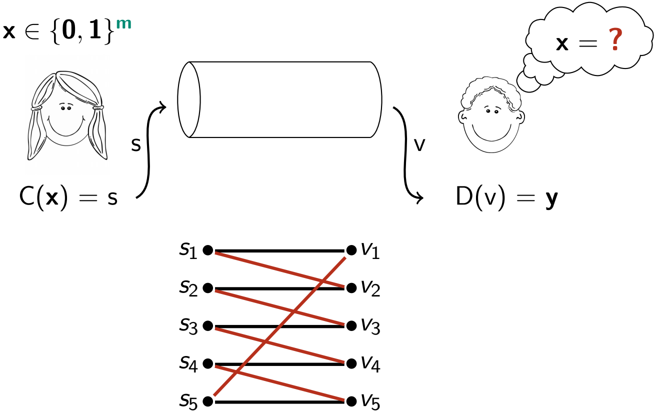
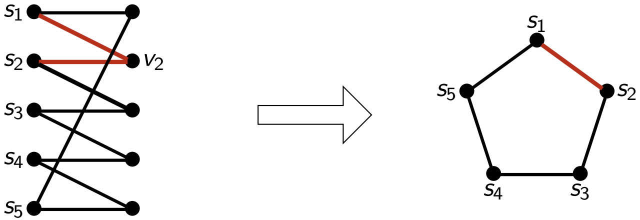


- **Goal:** $\mathbf{y} = \mathbf{x}$ with zero probability of error and maximize m

The confusability graph of a channel

- The confusability graph of a noisy channel has as vertices the channel's inputs, which are adjacent if they are confusable



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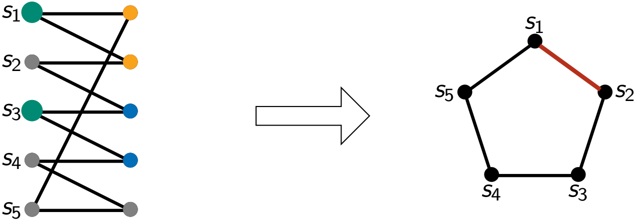

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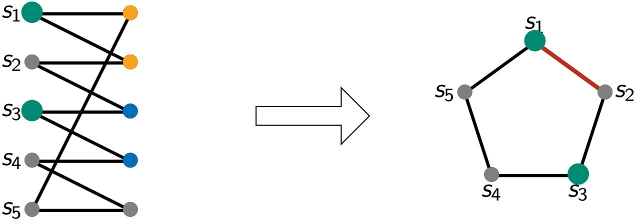

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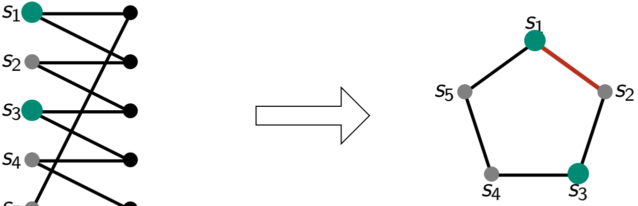

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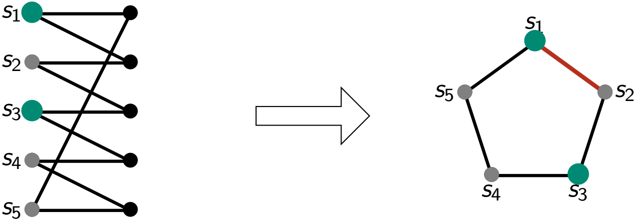

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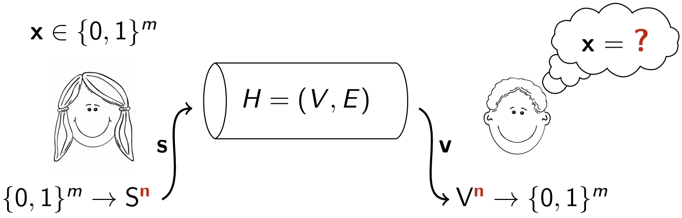
- $\alpha(H)$ is the maximum number of distinct messages Alice can send to Bob

Block coding and the Shannon capacity



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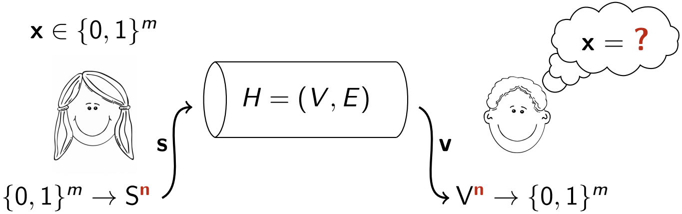

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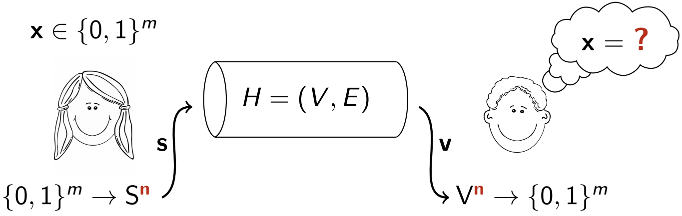

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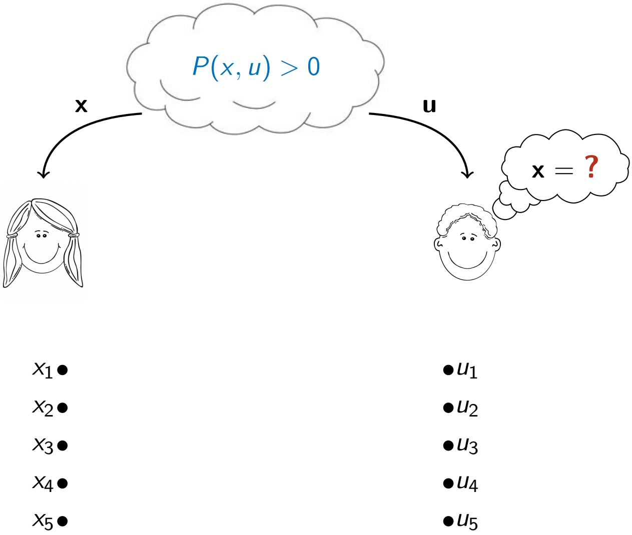
- The Shannon capacity lead to many interesting developments in combinatorics: e.g., perfect graphs, semidefinite optimization

The source coding problem



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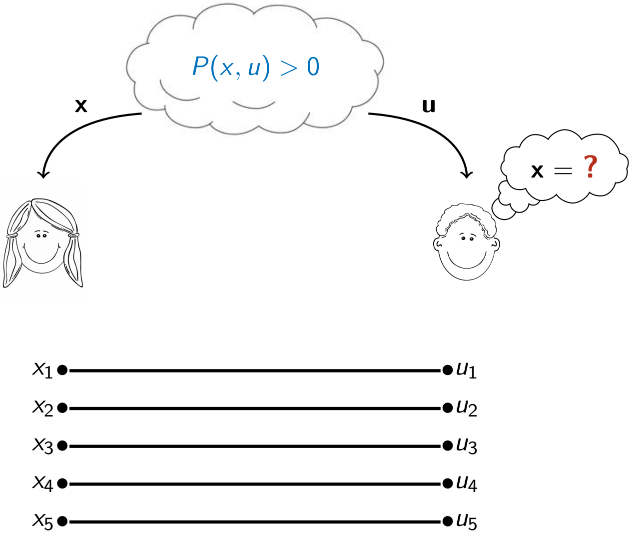

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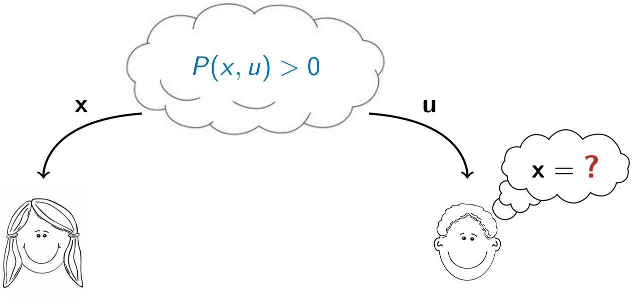

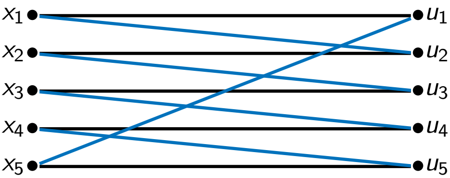

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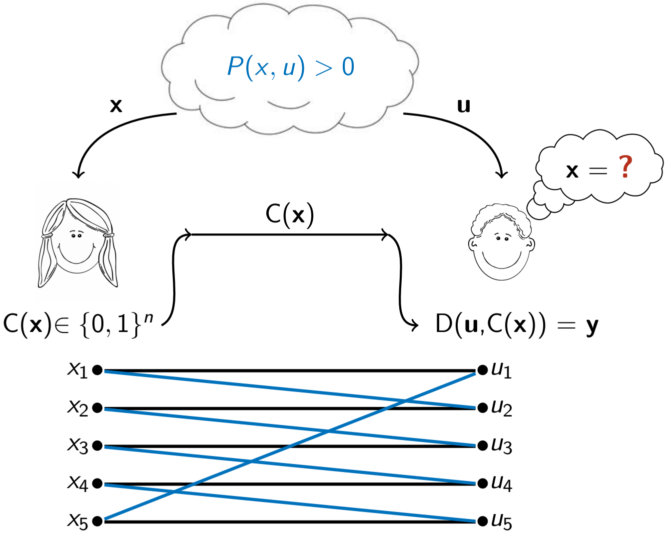

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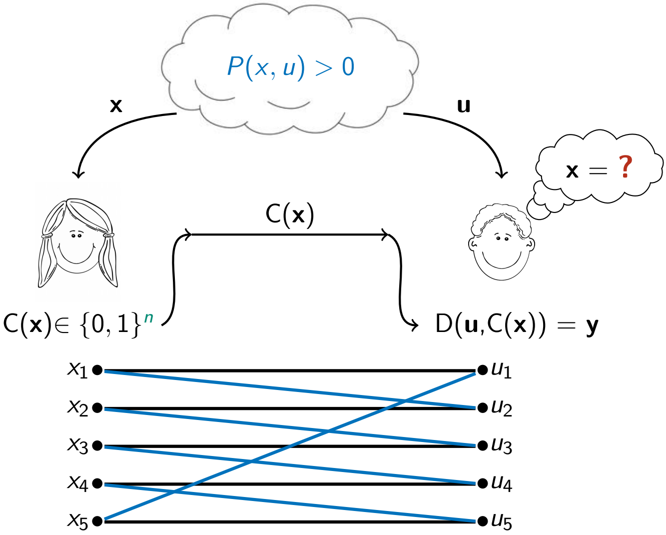
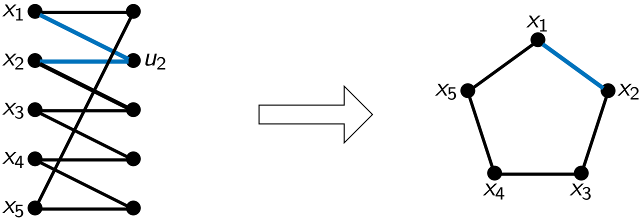


- **Goal:** $\mathbf{y} = \mathbf{x}$ with zero probability of error and minimize n

The characteristic graph of a dual source

- The characteristic graph of a dual source has as vertices Alice's inputs, which are adjacent if they are confusable to Bob



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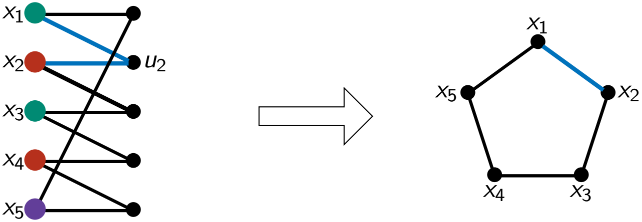

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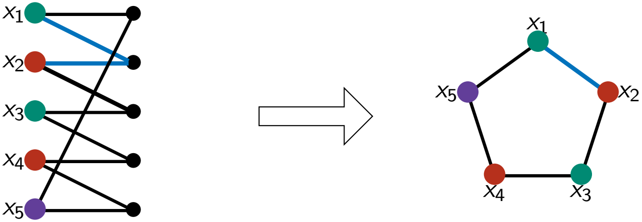

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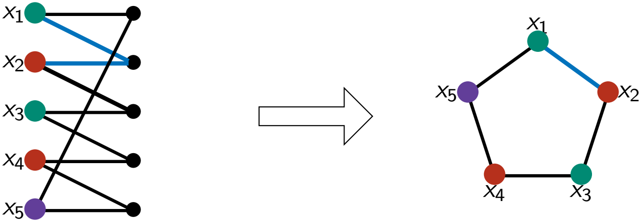

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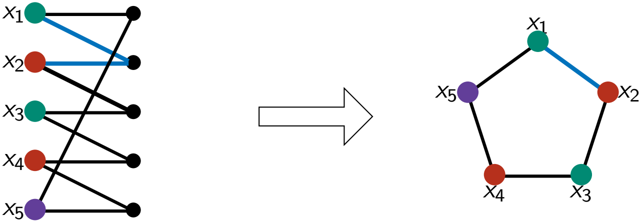

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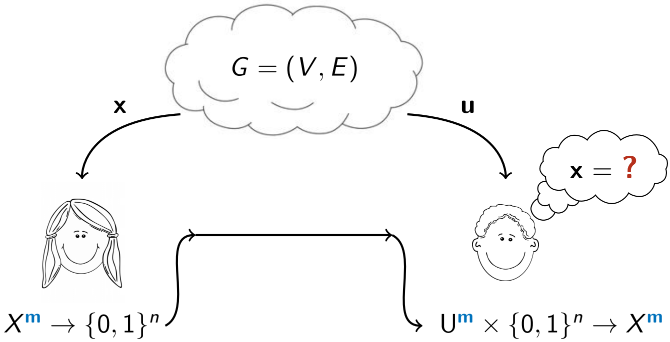
- $\chi(G)$ is the minimum size of the message set that Alice must use

Block coding and the Witsenhausen rate



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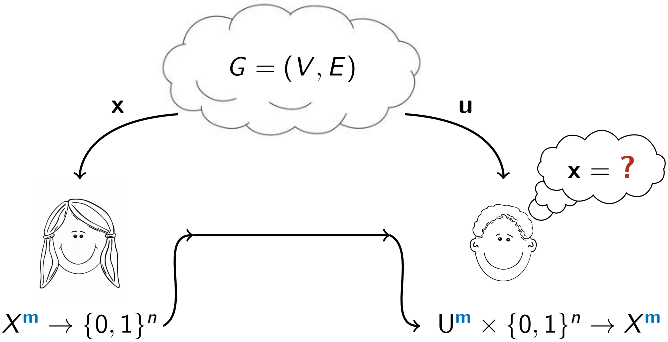

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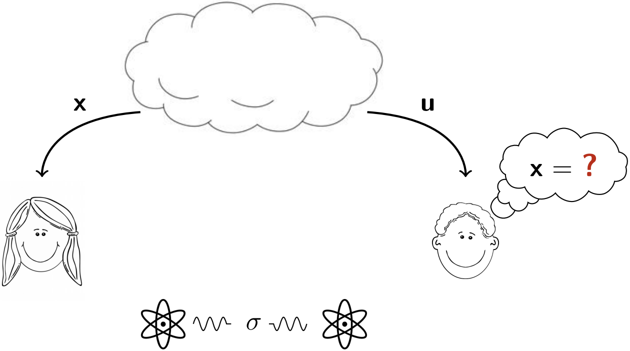
- The **Witsenhausen rate** of G

$$R(G) = \lim_{m \to \infty} \frac{1}{m} \log \chi(G^{\boxtimes m})$$

gives the minimum average number of bits needed per source input

The source coding problem with entanglement



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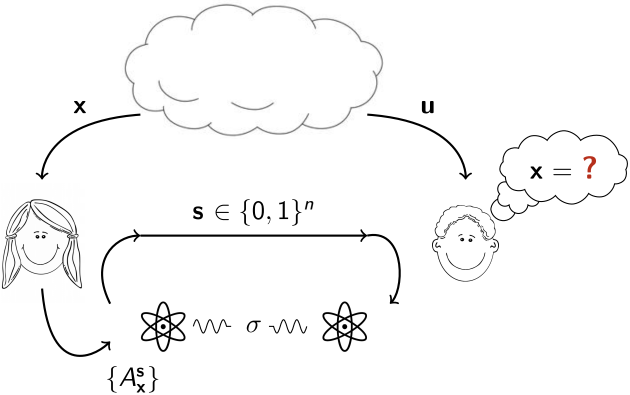

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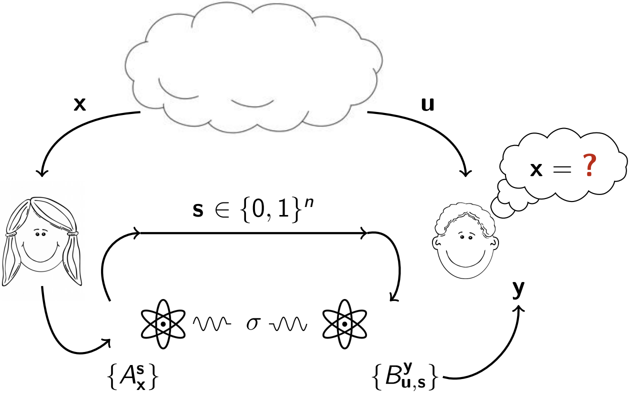

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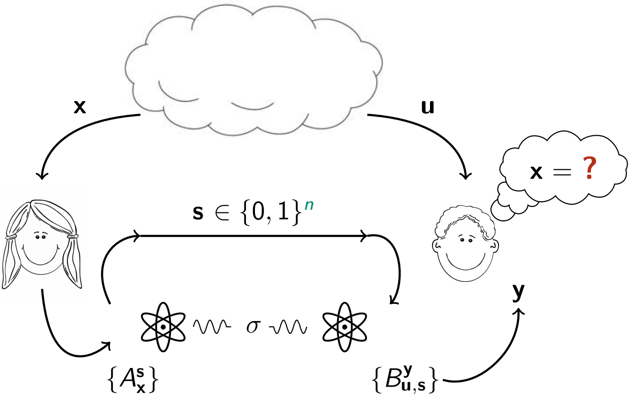

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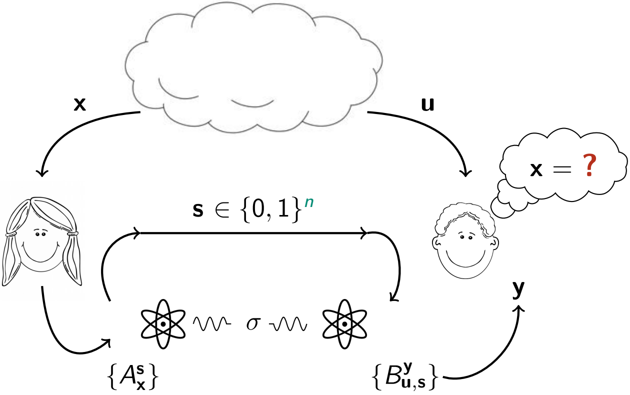

- As for the classical case, the goal is to have $\mathbf{y} = \mathbf{x}$ with zero probability of error and minimize n
- The entangled Witsenhausen rate $R^\star$ depends on the characteristic graph G and can be defined by simple constraints on $\sigma, \{A^{\mathbf{s}}_{\mathbf{x}}\}$

Entangled Shannon capacity

The entangled Shannon capacity $c^\star$ of a graph H is defined as

$$c^\star(H) = \lim_{n\to\infty} \frac{1}{n} \log \alpha^\star(H^{\boxtimes n})$$

- [Cubitt et al. '10] introduced $\alpha^\star, c^\star$ and showed that entanglement can increase that number of possible messages that can be sent with one use of the channel (i.e. $\alpha < \alpha^\star$)
- [Leung et al. '12] and [Briët et al. '12] showed that entanglement can increase the capacity of a channel (i.e. $c < c^\star$ by a constant factor)

Lovász ϑ number

$$\vartheta(G) = \min \quad \lambda \in \mathbb{R}$$
$$\text{such that} \quad \exists \text{ PSD matrix } Z \in \mathbb{R}^{V \times V}$$
$$Z(u,u) = \lambda - 1 \text{ for all } u \in V$$
$$Z(u,v) = -1 \text{ for all } \{u,v\} \notin E$$

- Introduced by Lovász ['79] to compute $c(C_5)$
- ϑ can be computed efficiently (up to any approximation)
- $c(G) \leq \log \vartheta(G) \leq R(\overline{G})$ ([Lovász '79] and [Nayak et al. '06])
- [Beigi '10] and [Duan et al. '13] proved $c^\star(H) \leq \log \vartheta(H)$

ϑ bound on the entangled Witsenhausen rate

Theorem

$$\log \vartheta(G) \leq R^{\star}(\overline{G})$$

- Thus

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Separation between the classical and entangled Witsenhausen rate

Theorem

There exists an infinite family of graphs H_k such that

$$\frac{R^\star(H_k)}{R(H_k)} \quad \leq \quad O\Big(\frac{\log k}{k}\Big).$$

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- The quarter orthogonality graph H_k is a subgraph of the orthogonality graph induced by the vertices $x \in \{\pm 1\}^k$ with $x_1 = +1$ and an even number of -1's

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Let $k = 4p^\ell - 1$ where p is an odd prime and $\ell \in \mathbb{N}$, then

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- The lower bound on $R(H_k)$ is derived from a technique that upper bounds $c(H_k)$. It is obtained using an instance of the linear algebra method due to [Alon '98] with a construction of certain low-degree polynomials over finite field due to [Barrington et al. '94]

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- The upper bound on $R^{\star}(H_k)$ relies on the construction of a orthogonal representation of the graph H_k (similar idea as used by [Cameron et al. '07])

Separation between the classical and entangled Shannon capacity

Theorem

Let $k = 4p^\ell - 1$ where p is an odd prime and $\ell \in \mathbb{N}$, then

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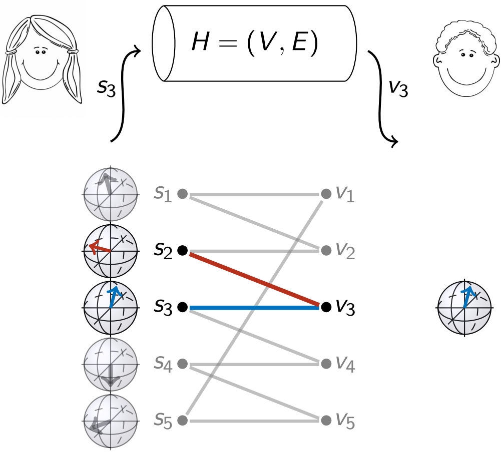
- The lower bound on $c^\star(H_k)$ uses a new method based on the teleportation scheme of [Bennet et al. '93]

Lower bound on entangled Shannon capacity



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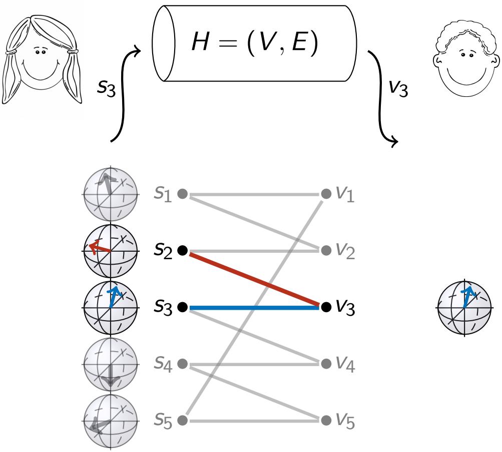

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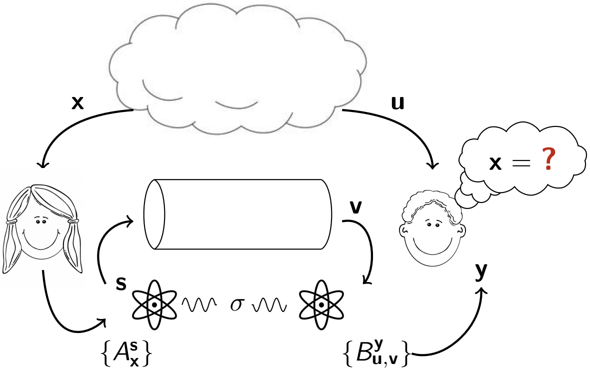
- Send $|V|^t$ messages in $t+1$ steps if $\log \alpha(H) \geq 2t \log \mathrm{orthdim}(H)$

Further results



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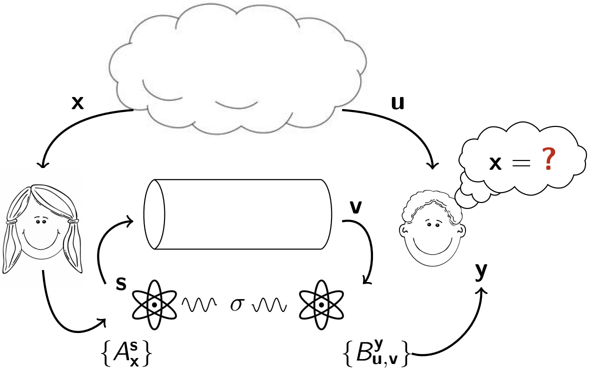

- We study the zero-error source-channel coding problem with entanglement, a generalization of the zero-error channel coding and of the source coding with entanglement
- We present an infinite family of source and channels combinations for which entanglement allows an exponential saving in communication in zero-error source-channel coding

Conclusions and open questions

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Thank you!