Infinite Randomness Expansion and Amplification with a Constant Number of Devices

Matthew Coudron, Henry Yuen

MIT EECS

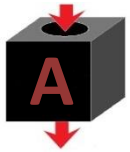
arXiv 1310.6755

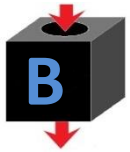
Randomness Expansion

- Roger Colbeck – PhD Thesis, 2006

S = Input Seed

Randomness Expansion





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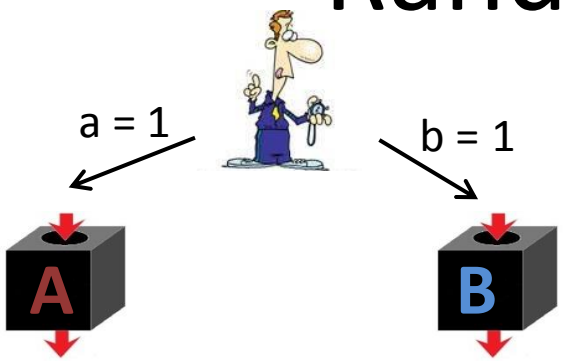

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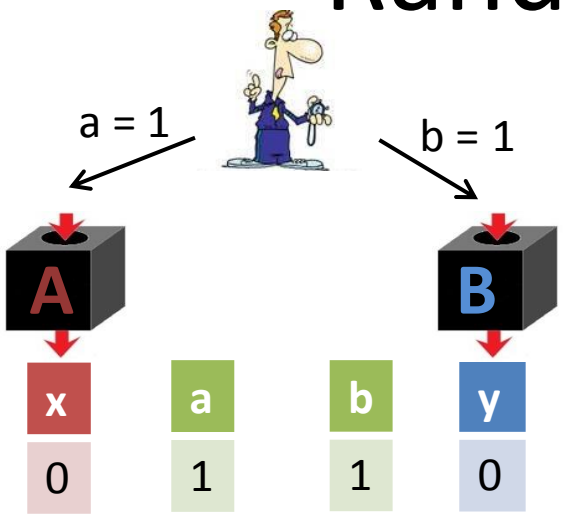

- Roger Colbeck – PhD Thesis, 2006
- Serial rounds

  “Query and Response”

S = Input Seed

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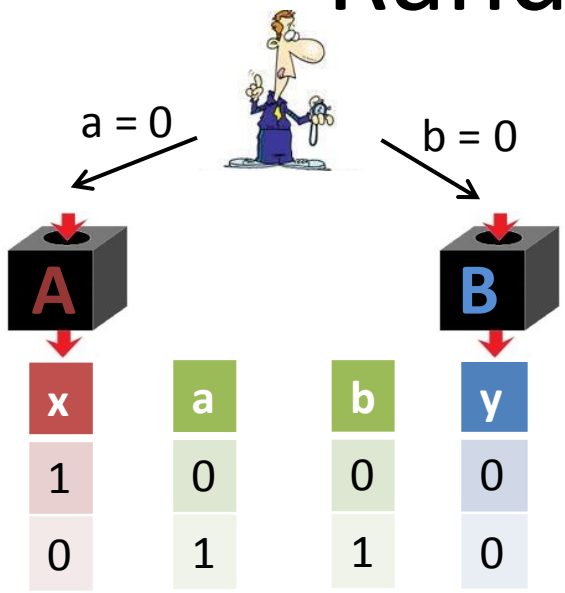

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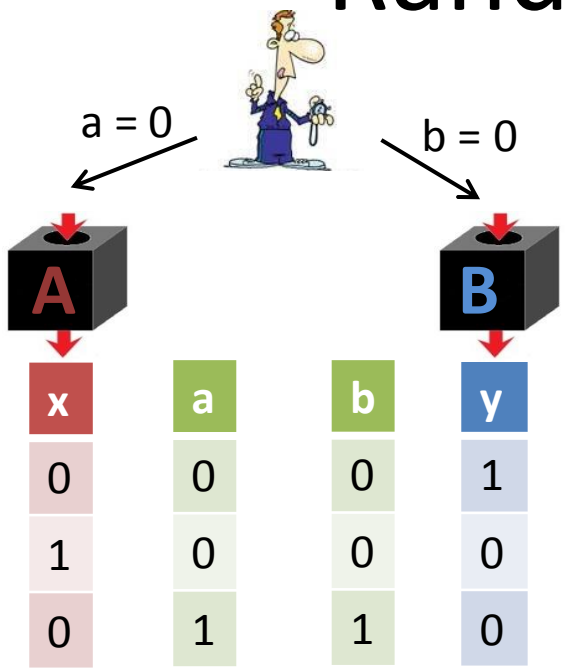

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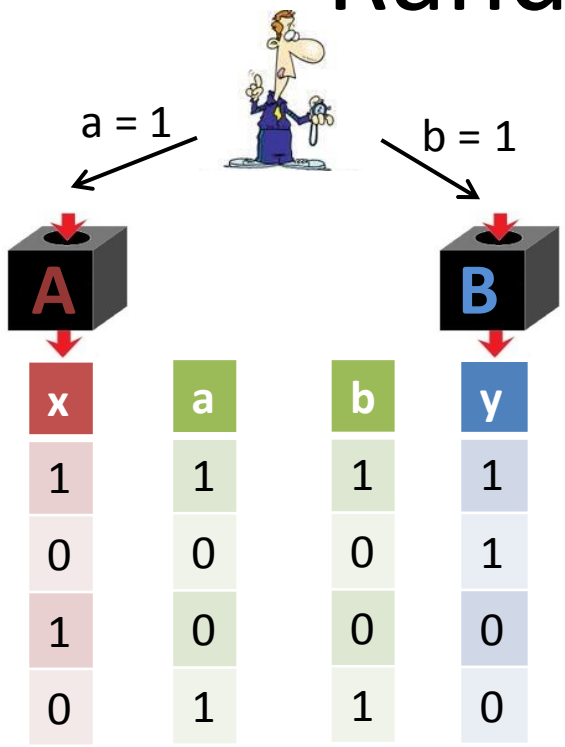

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  “Query and Response”

S = Input Seed

Randomness Expansion

a = 1

b = 0

A

B

x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
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O = Output

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Test

O = Output

- Roger Colbeck – PhD Thesis, 2006
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Test

Pass

O = Output

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  “Query and Response”

O has high Min-Entropy
(high “randomness”)

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Randomness Expansion

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Test

Fail

Abort Protocol

Pass

O = Output

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VV Protocol [Vazirani, Vidick '11]

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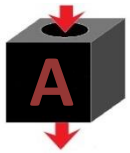

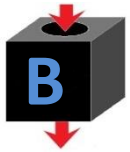

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O = Output

- Exponential expansion

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- Exponential expansion

n bit seed -> O has $2^{\sqrt[3]{n}}$ Min Entropy

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$$\rho_{SDE} \approx \rho_U \otimes \rho_{DE} \Rightarrow \rho_{OE} \approx \rho_U \otimes \rho_E$$

Questions

1) What is the greatest possible rate of randomness expansion? Exponential? Higher?

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Our Result:

Infinite randomness expansion with 8 devices.
(We can also do 6)

Input Seed

Compose?

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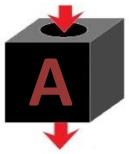

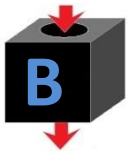

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O

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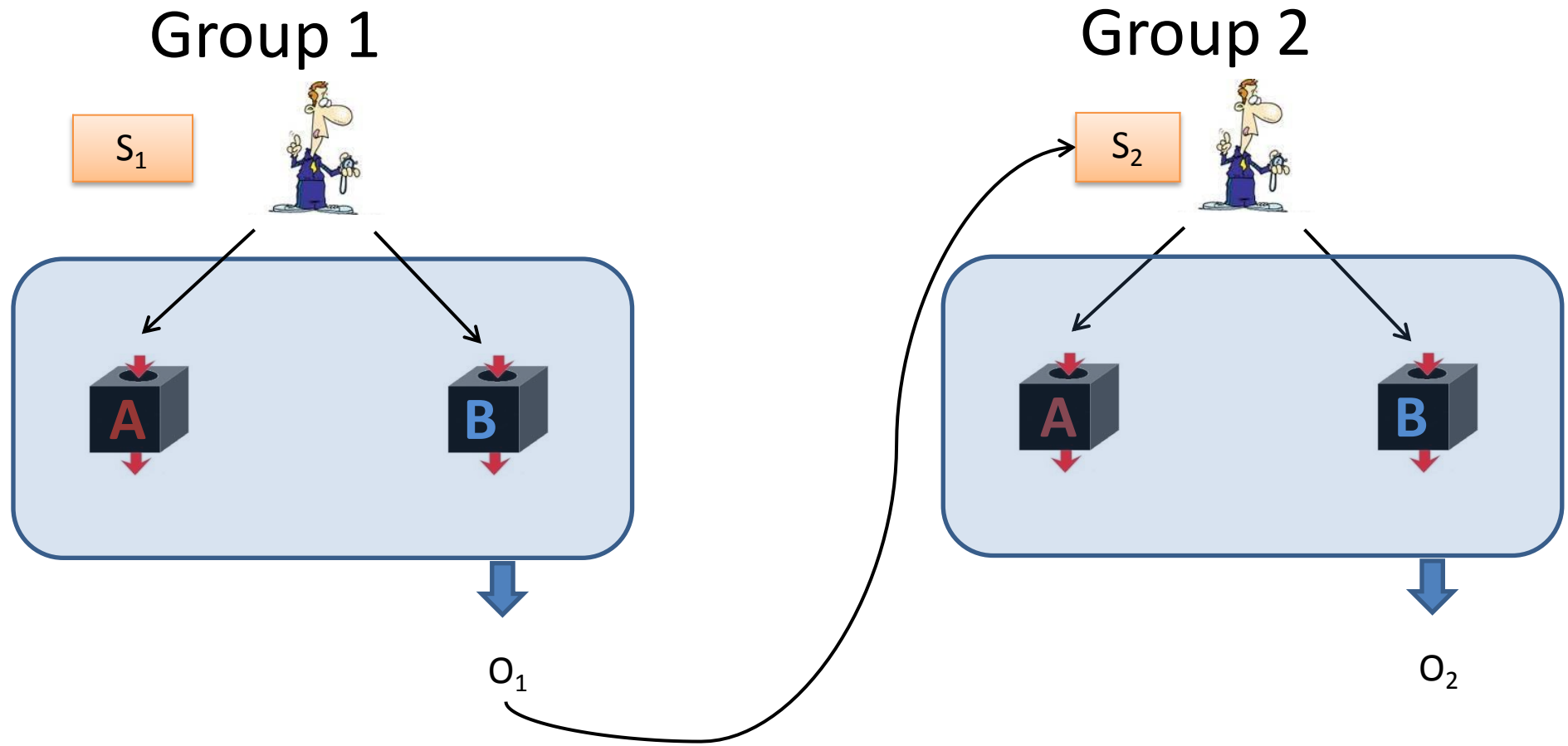

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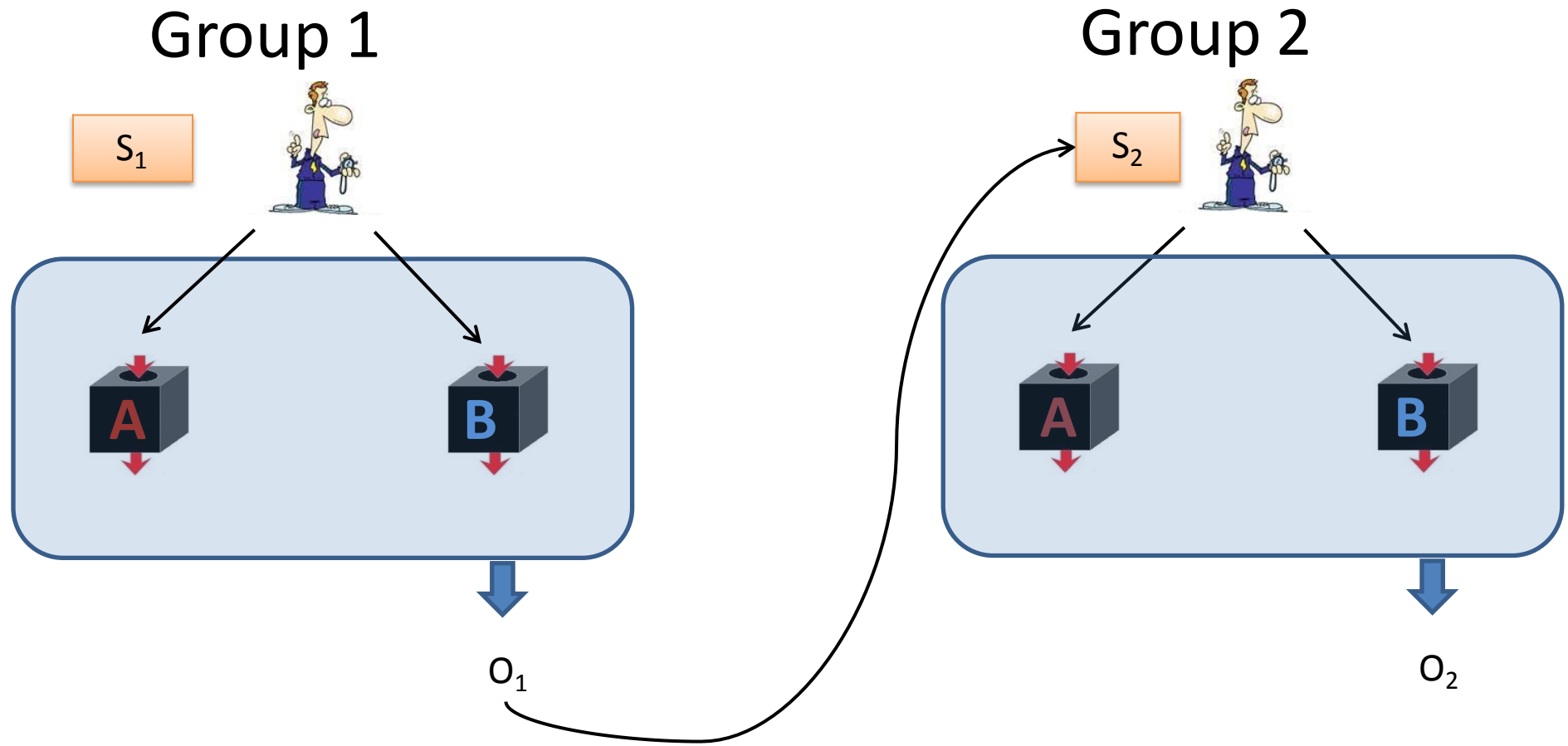
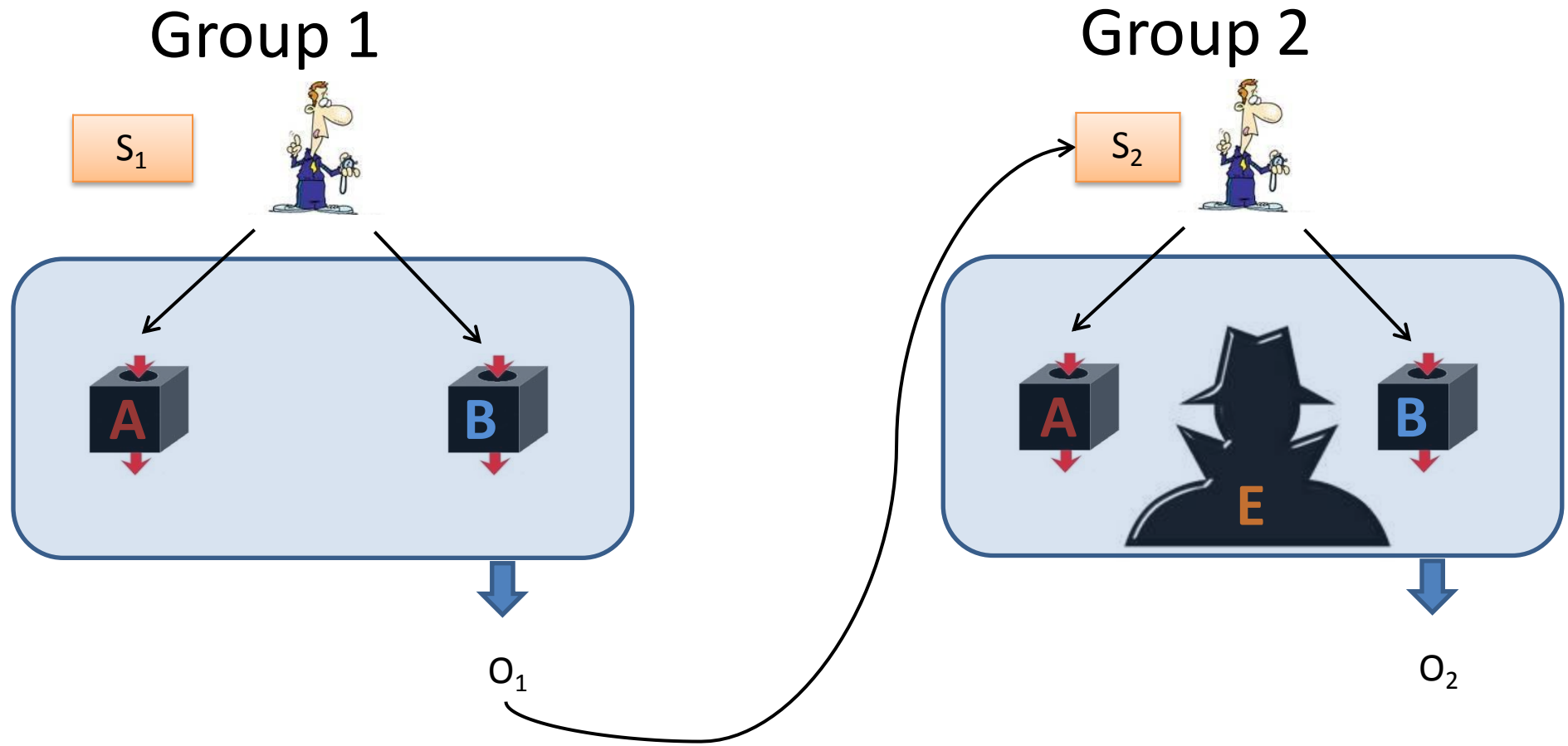


Secure Against Eavesdropper

Compose?



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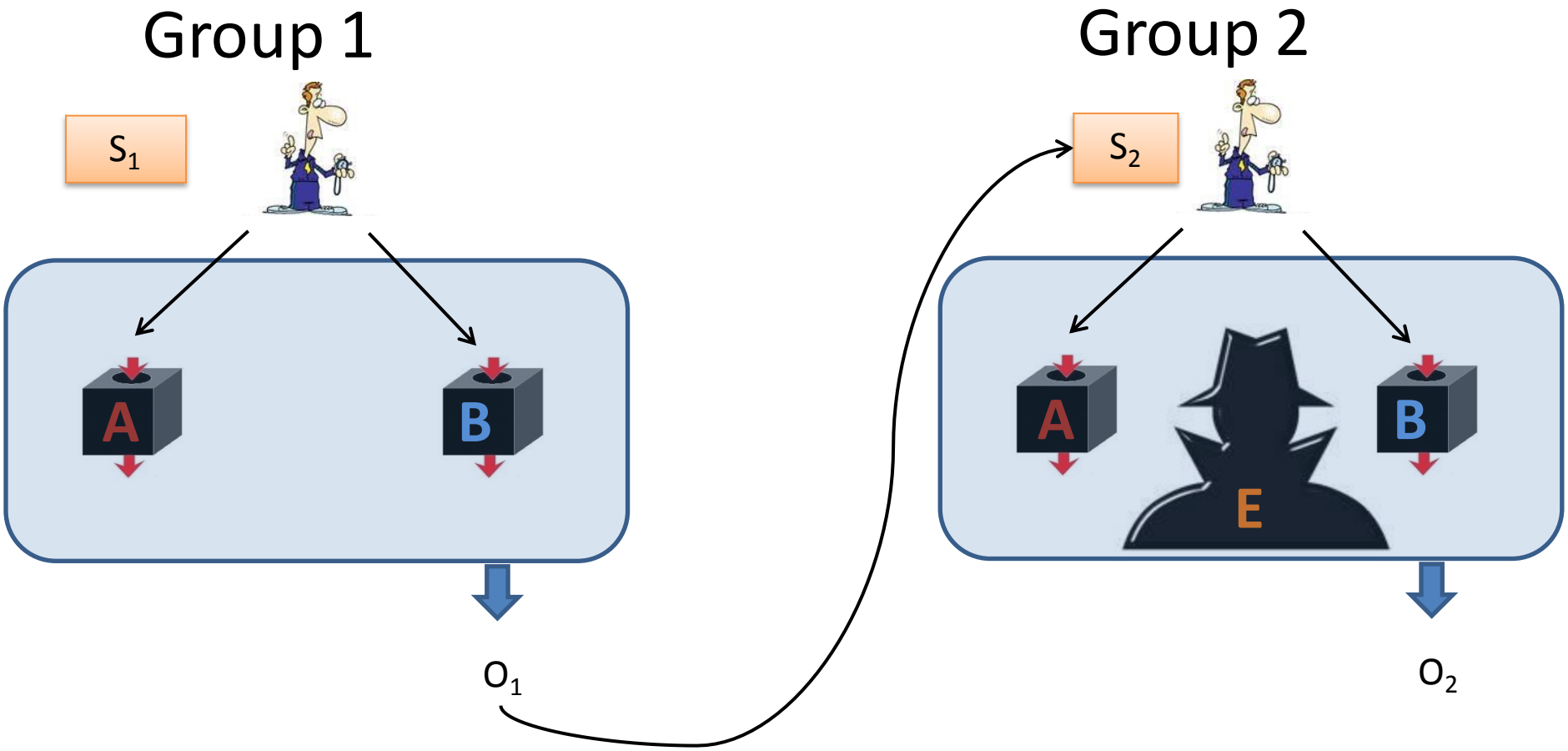
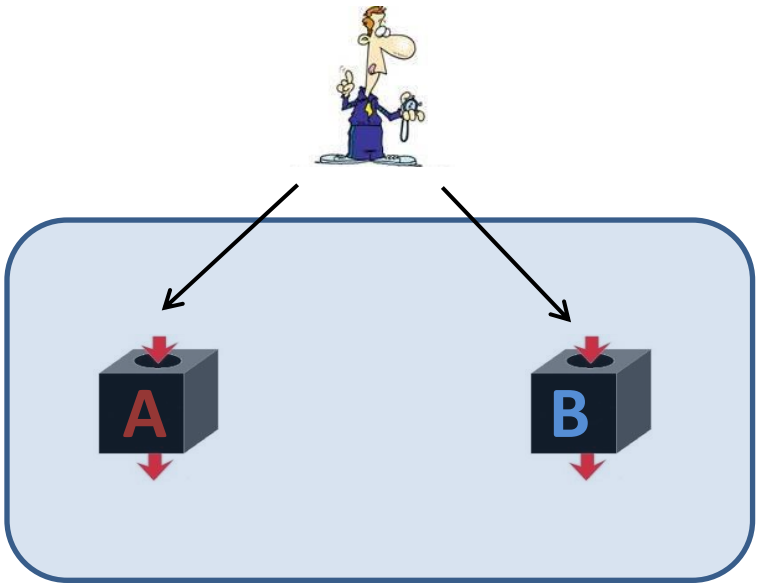


Secure Against Eavesdropper

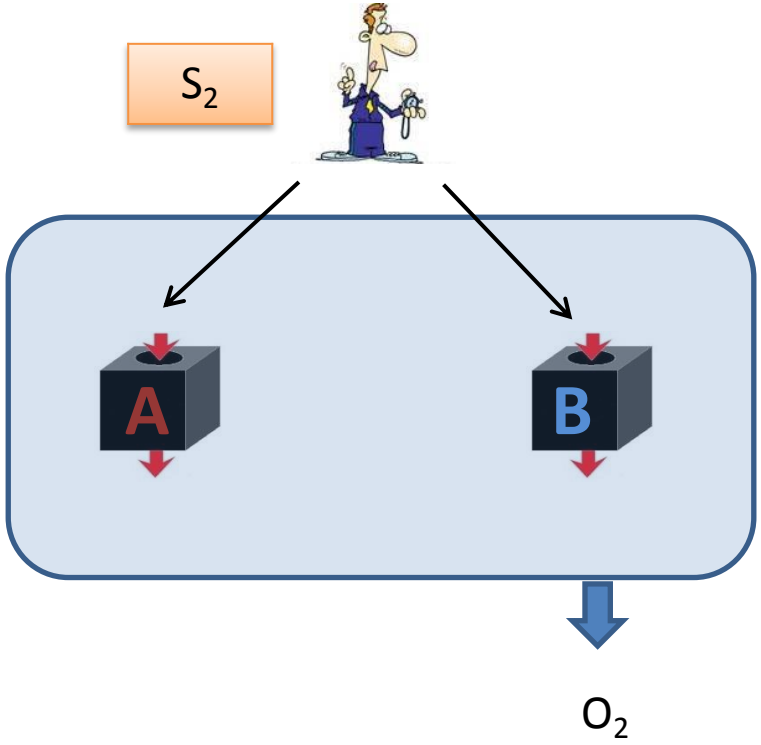
$$\rho_{S_1 G_1 G_2} \approx \rho_U \otimes \rho_{G_1 G_2} \Rightarrow \rho_{O_1 G_2} \approx \rho_U \otimes \rho_{G_2}$$

Alternate?

Group 1



Group 2



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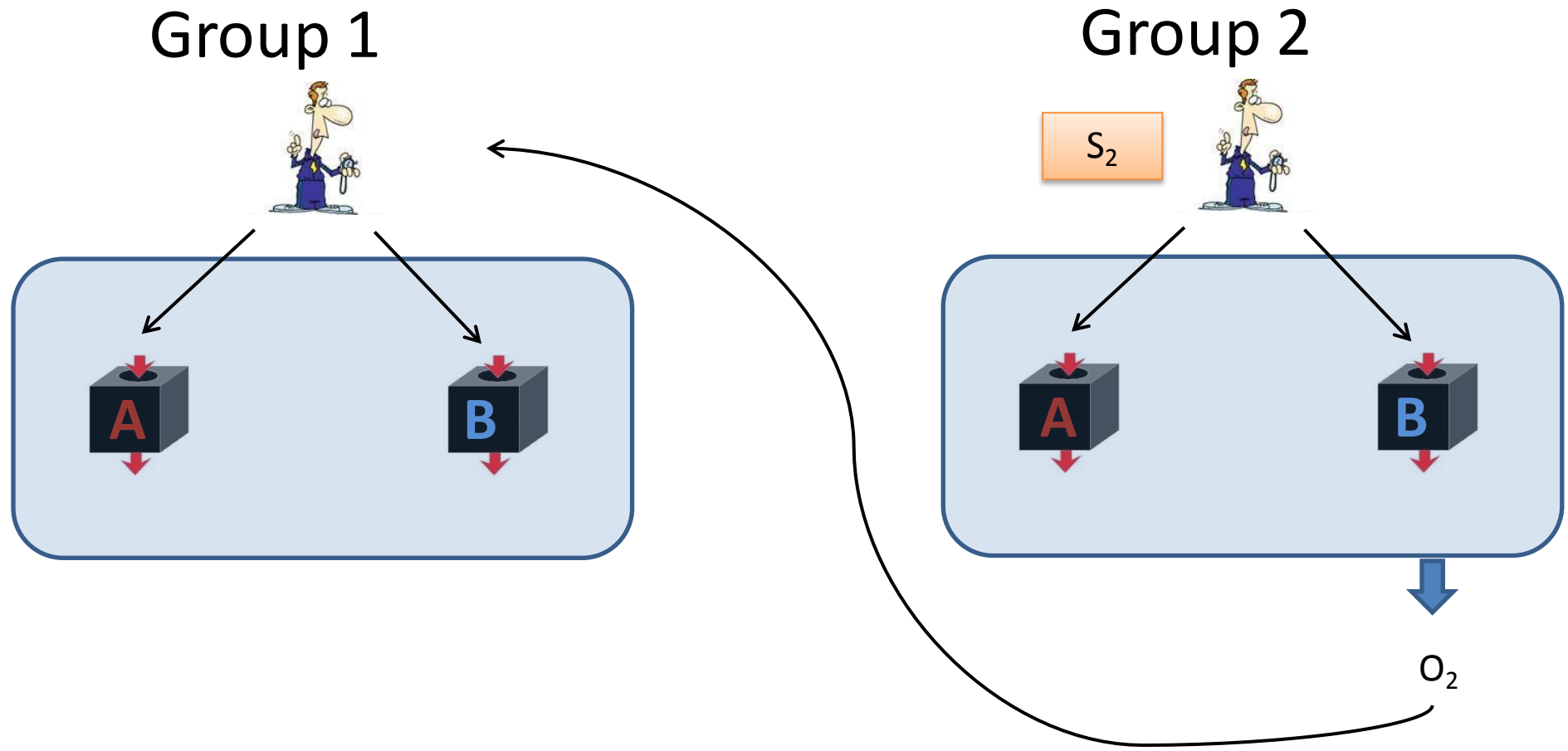

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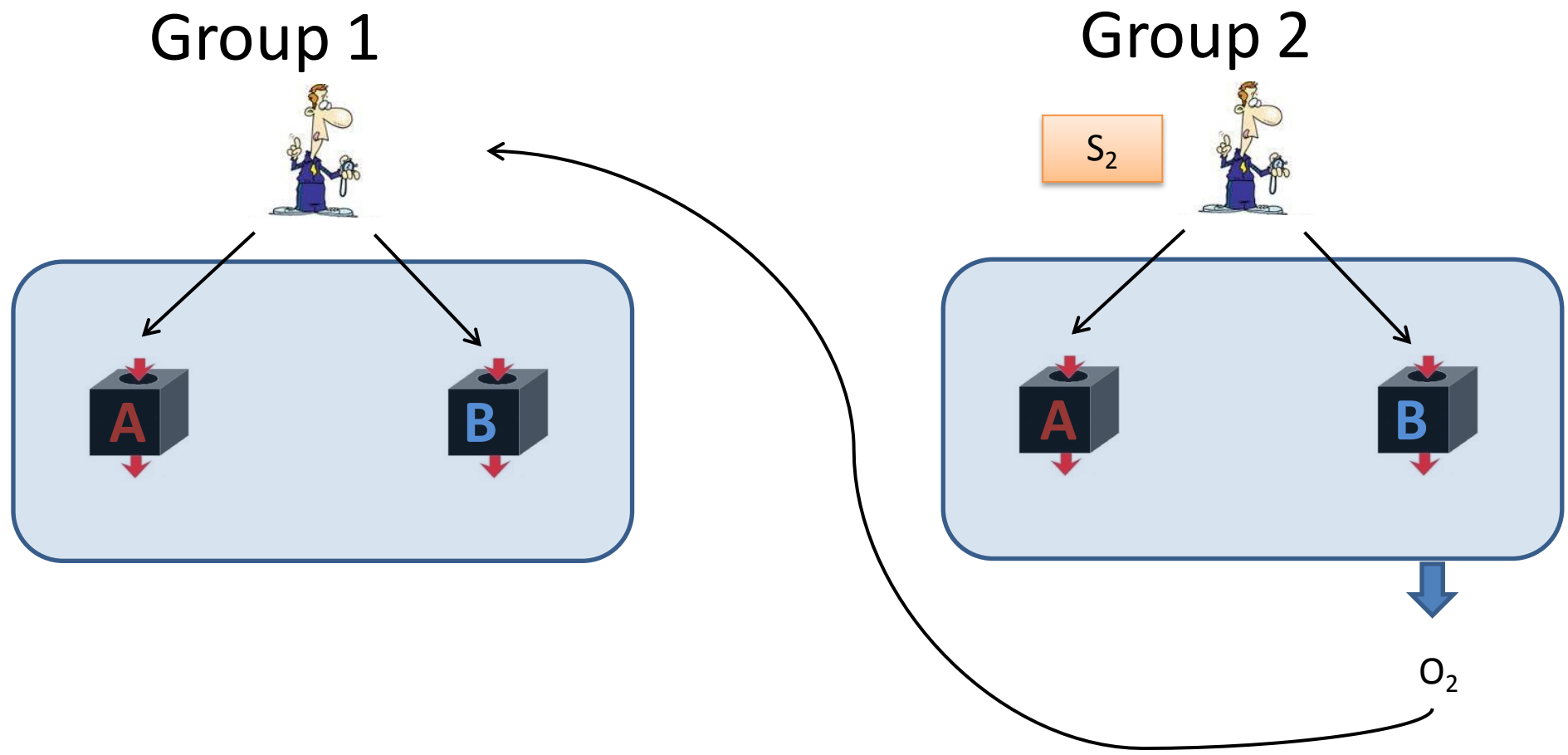

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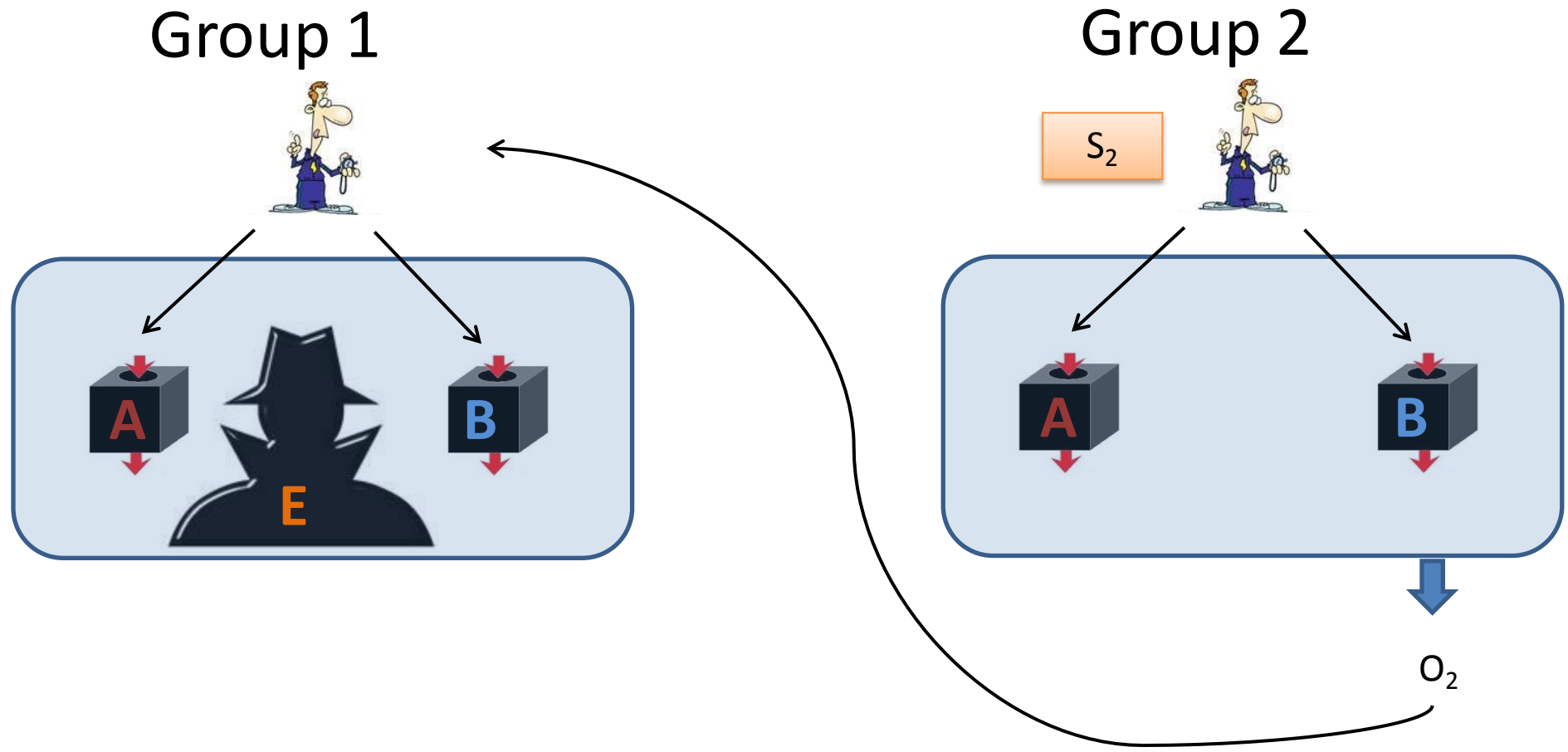

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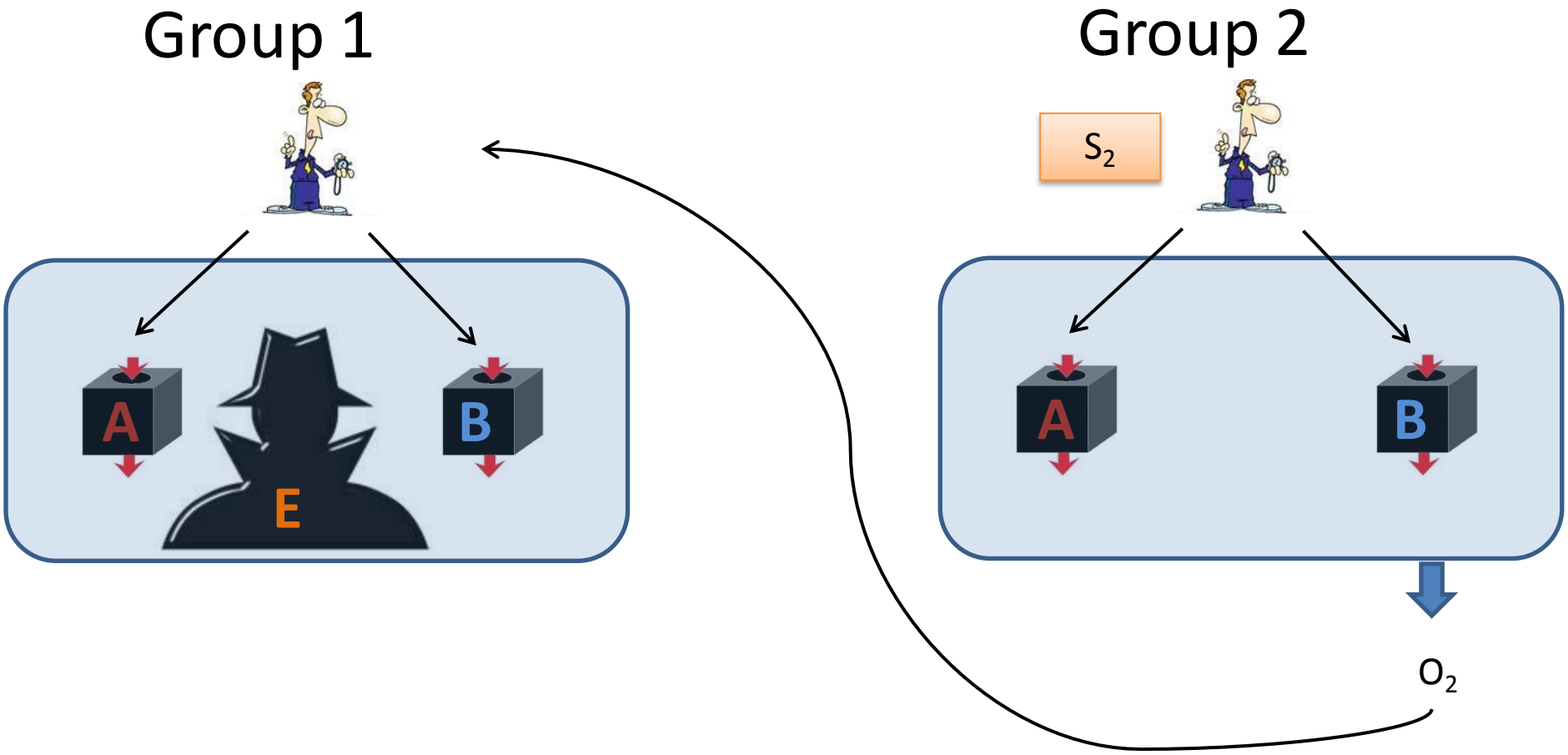

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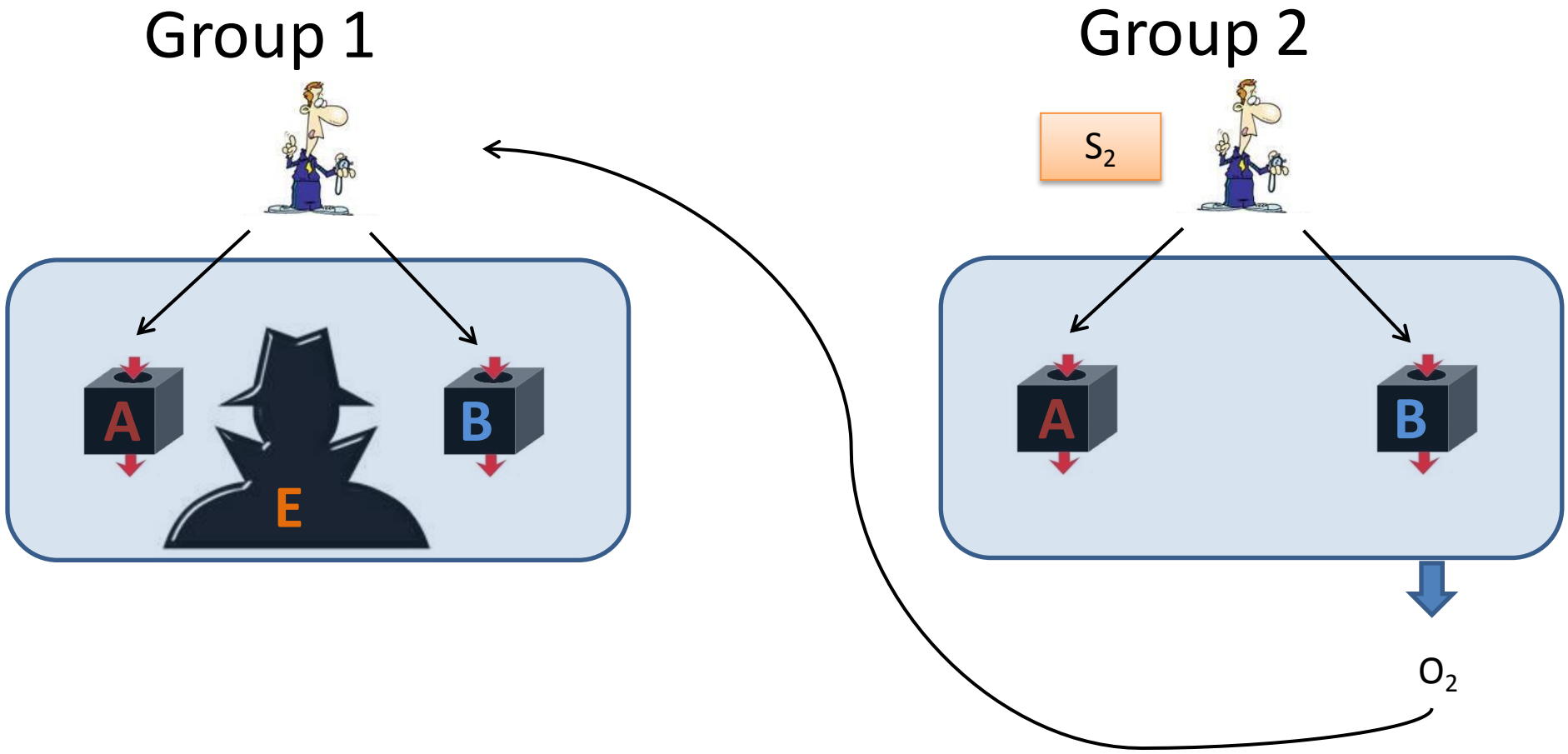

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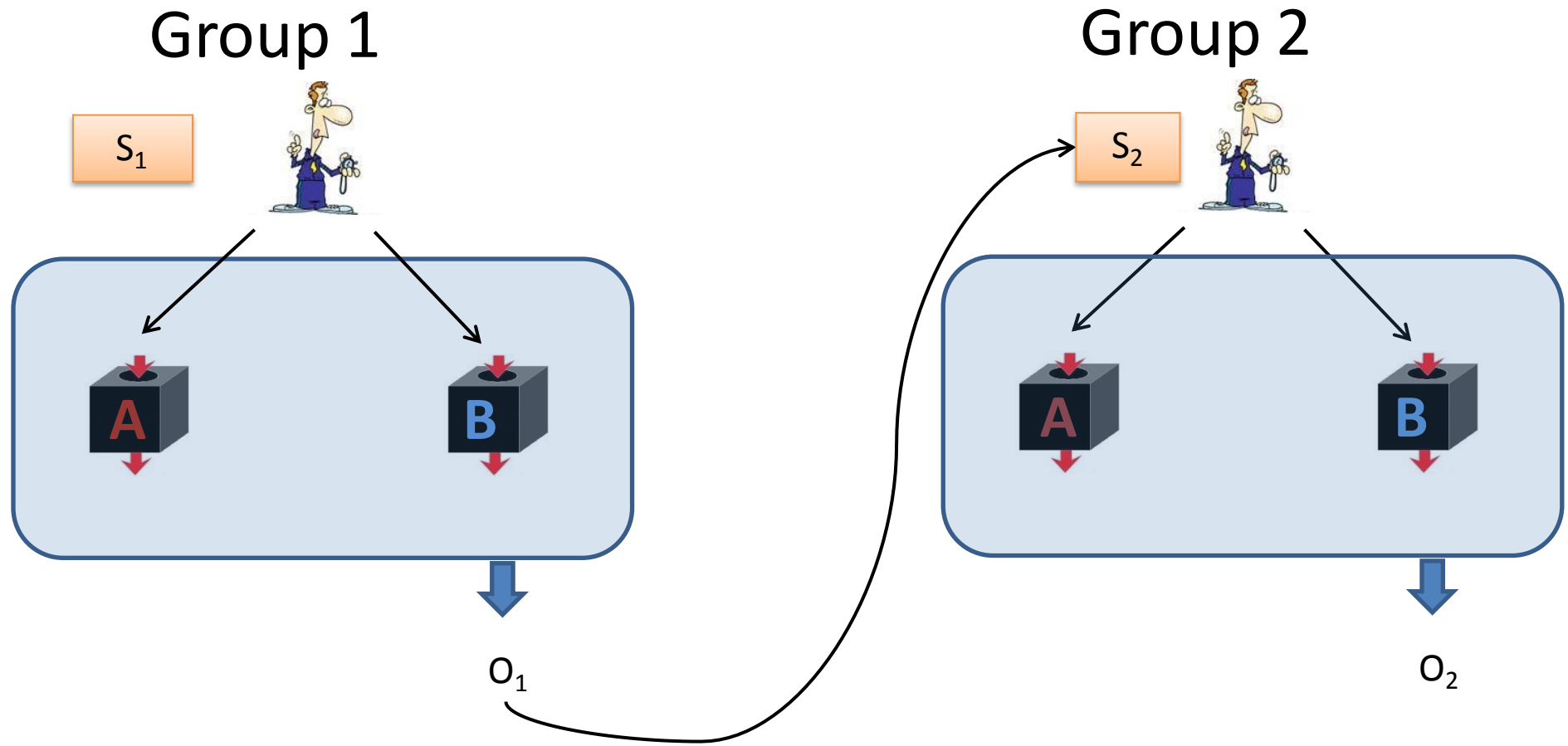

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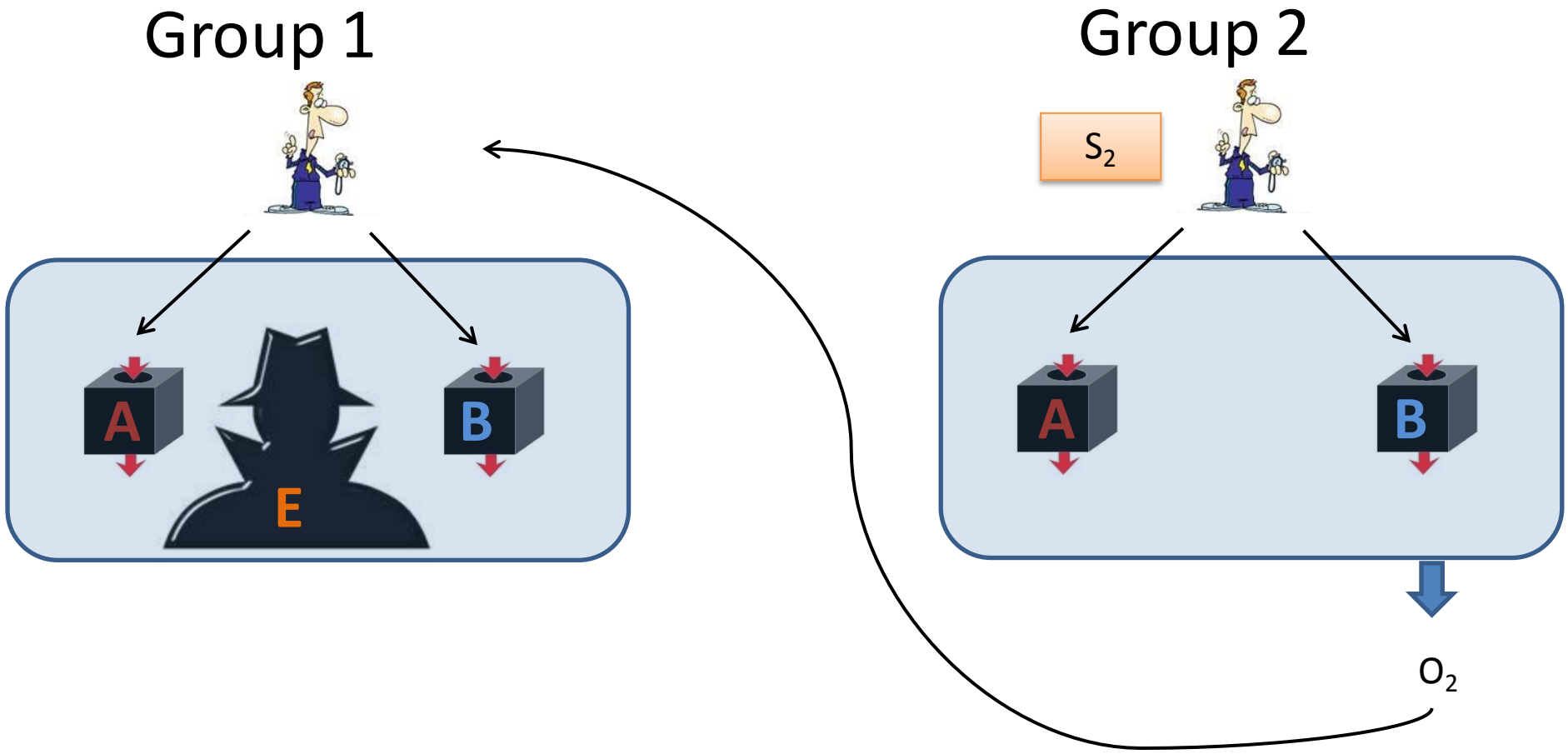

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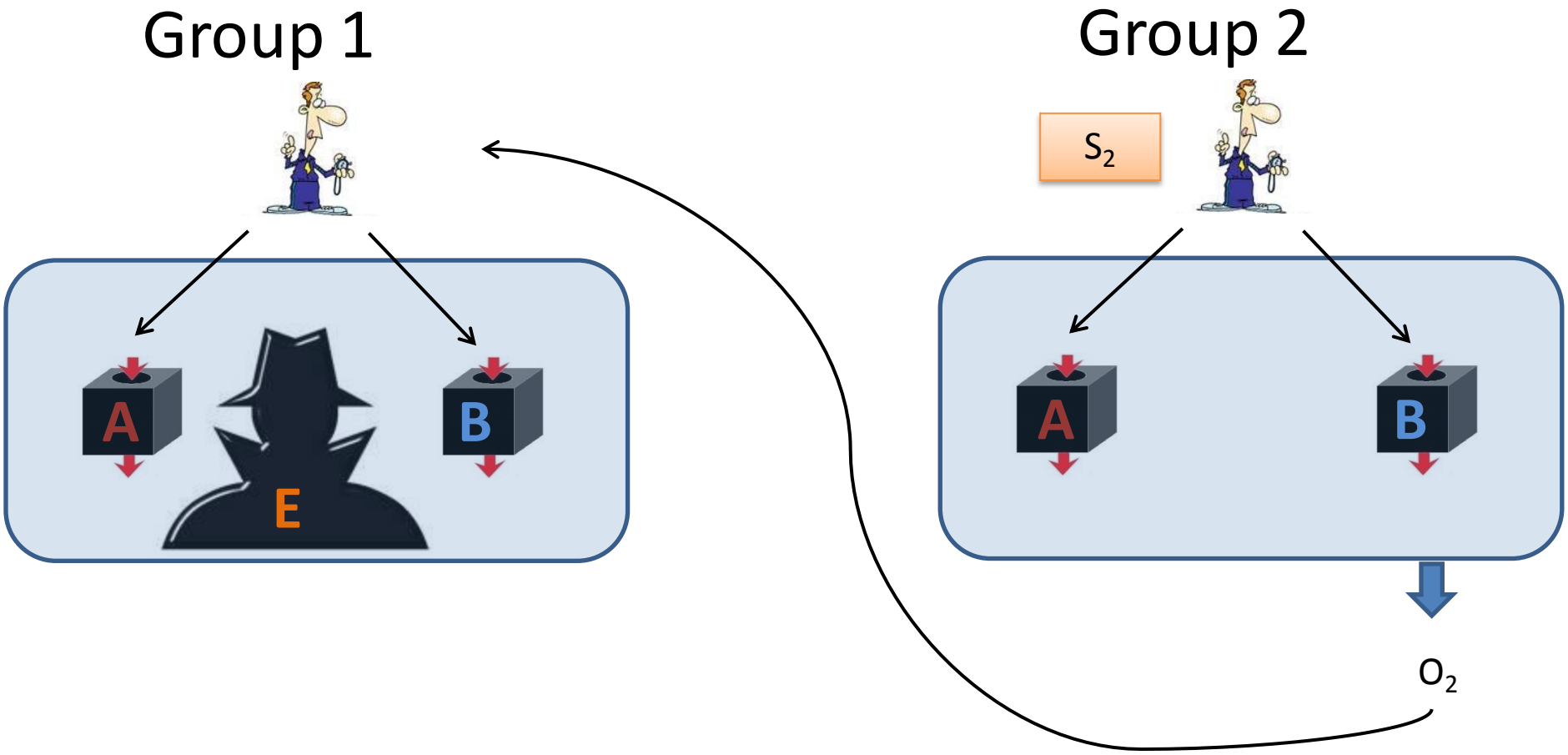

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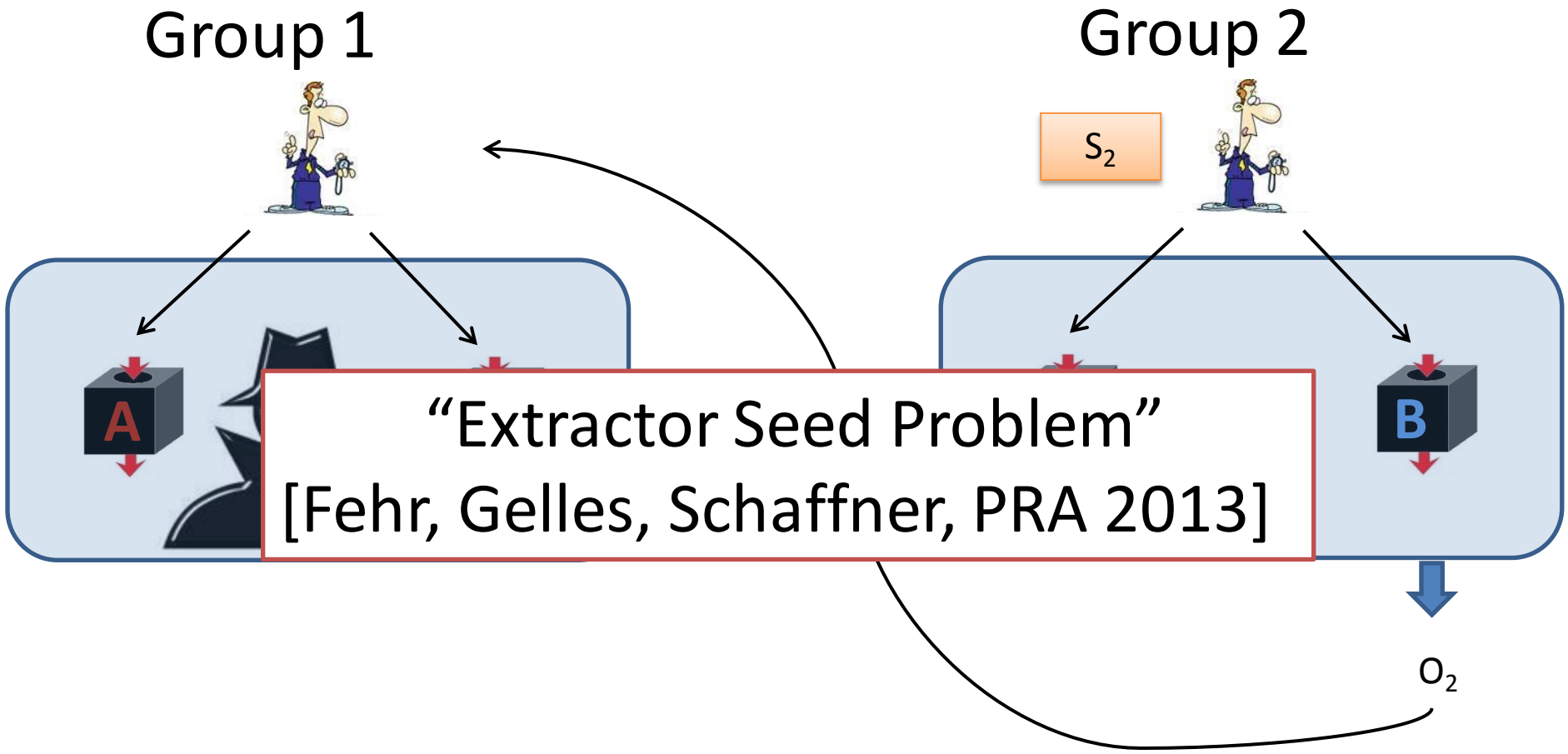

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Input Security

Secure Against Q. Eavesdropper

$$\rho_{SDE} \approx \rho_U \otimes \rho_{DE}$$

$$\rho_{OE} \approx \rho_U \otimes \rho_E$$

"Input Secure"

$$\rho_{SD} \approx \rho_U \otimes \rho_D$$

$$\rho_{OE} \approx \rho_U \otimes \rho_E$$

Input Security

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- Can we obtain input security in a randomness expansion protocol?

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- Provably input secure randomness expansion protocol ⟶ Infinite randomness expansion.
- Can we obtain input security in a randomness expansion protocol?
- Randomness Extractors are provably not input secure.

S = Input Seed

a = 1

b = 0

A

B

x	a	b	y
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1	0	1	0
1	0	1	1
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A New Tool

[Reichardt, Unger, Vazirani 2012]

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A New Tool

[Reichardt, Unger, Vazirani 2012]

"RUV" Protocol

- Device Independent protocol

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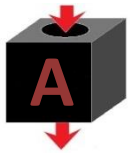

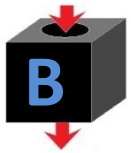

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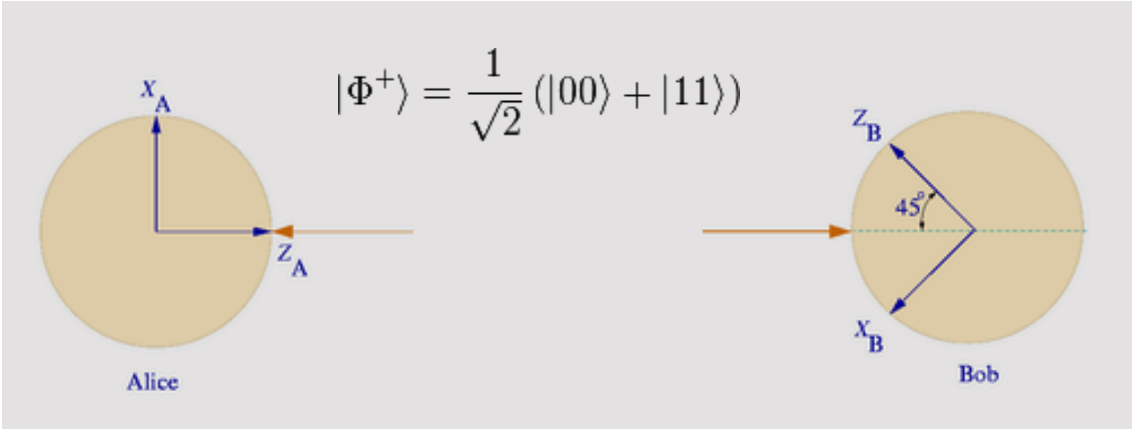
A New Tool

[Reichardt, Unger, Vazirani 2012]

"RUV" Protocol

- Device Independent protocol
- Certifies that devices are measuring an EPR pair in certain rounds.



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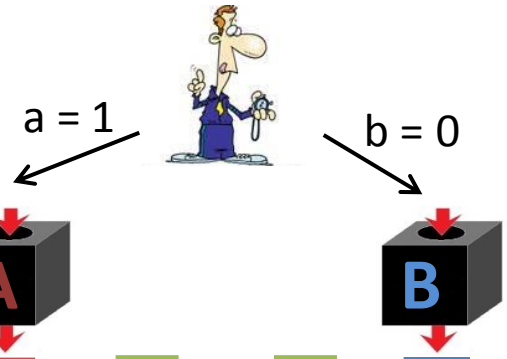

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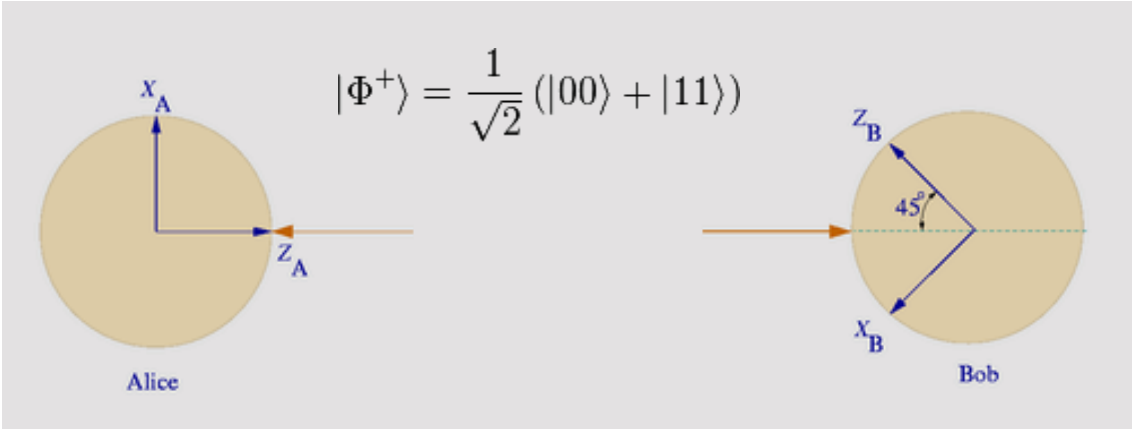
A New Tool

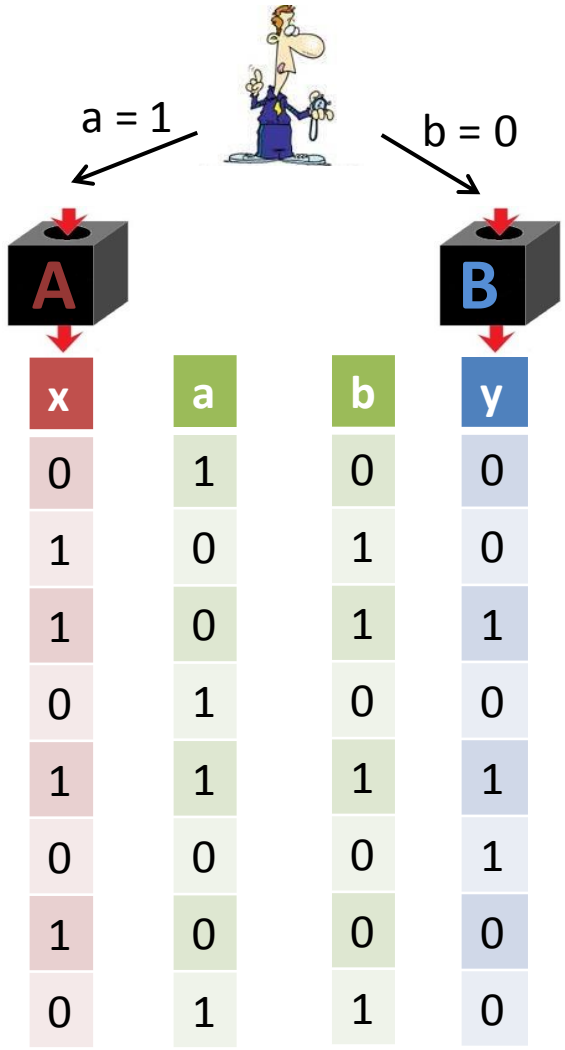
[Reichardt, Unger, Vazirani 2012]

"RUV" Protocol

- Device Independent protocol
- Certifies that devices are measuring an EPR pair in certain rounds.
- Employs CHSH Rigidity



Input Secure?

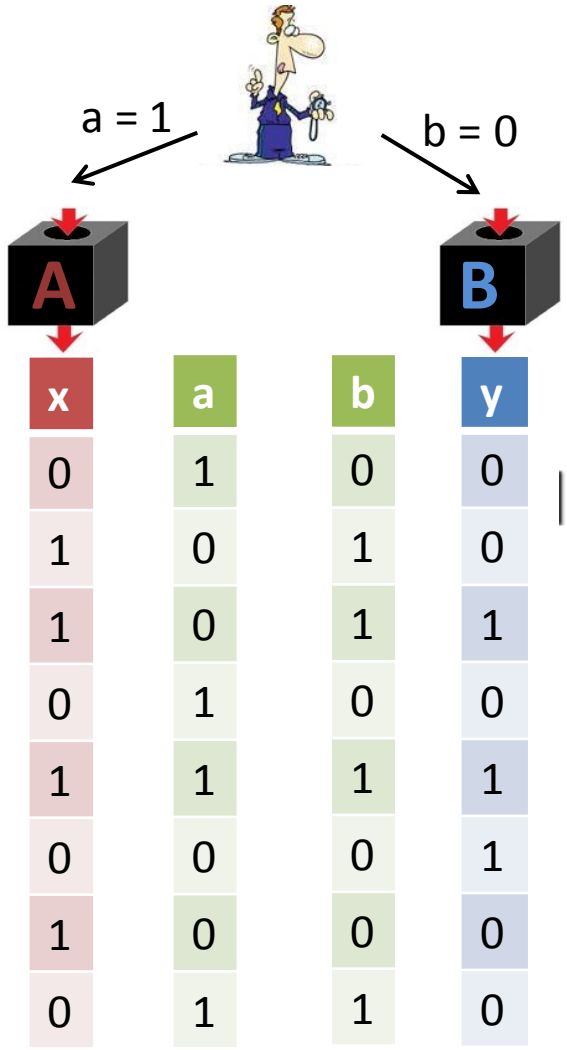


x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

The RUV protocol seems Input Secure!

.

Input Secure?



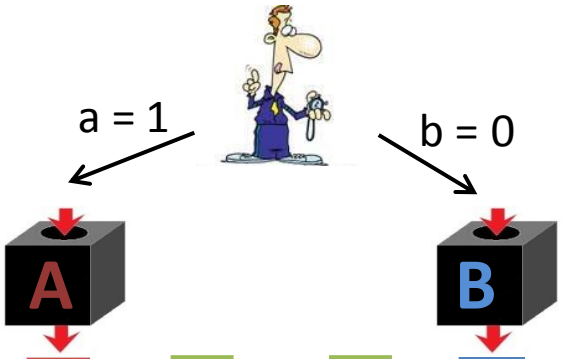
x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

The RUV protocol seems Input Secure!

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

.

Input Secure?



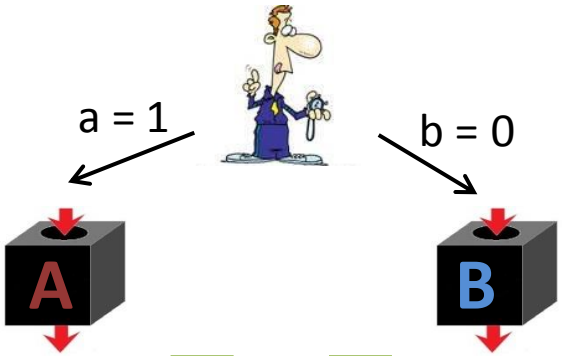
The RUV protocol seems Input Secure!

x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

.

Input Secure?



x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

The RUV protocol seems Input Secure!

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

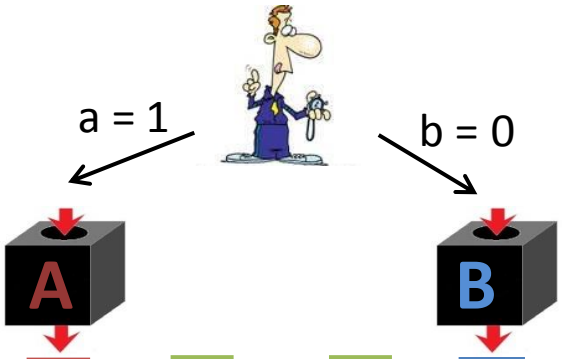
.

"Input Secure"

$$\rho_{SD} \approx \rho_U \otimes \rho_D$$

$$\rho_{OE} \approx \rho_U \otimes \rho_E$$

Input Secure?



x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

The RUV protocol seems Input Secure!

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

"Input Secure"

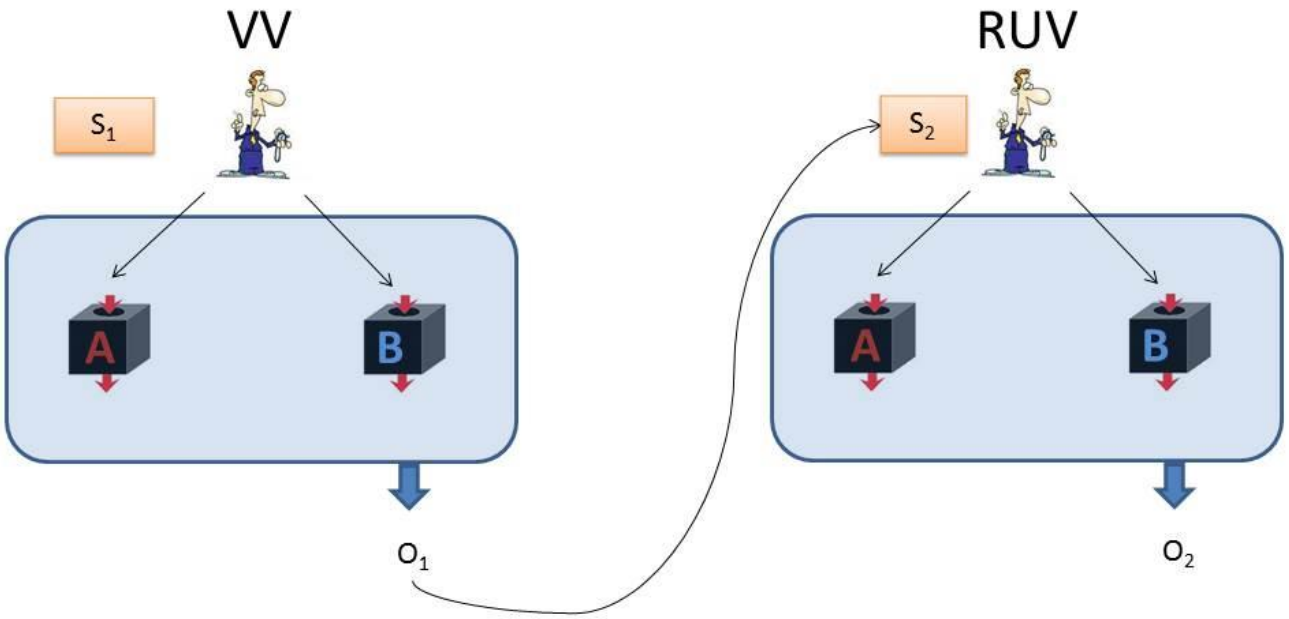
$$\rho_{SD} \approx \rho_U \otimes \rho_D$$

$$\rho_{OE} \approx \rho_U \otimes \rho_E$$

However:

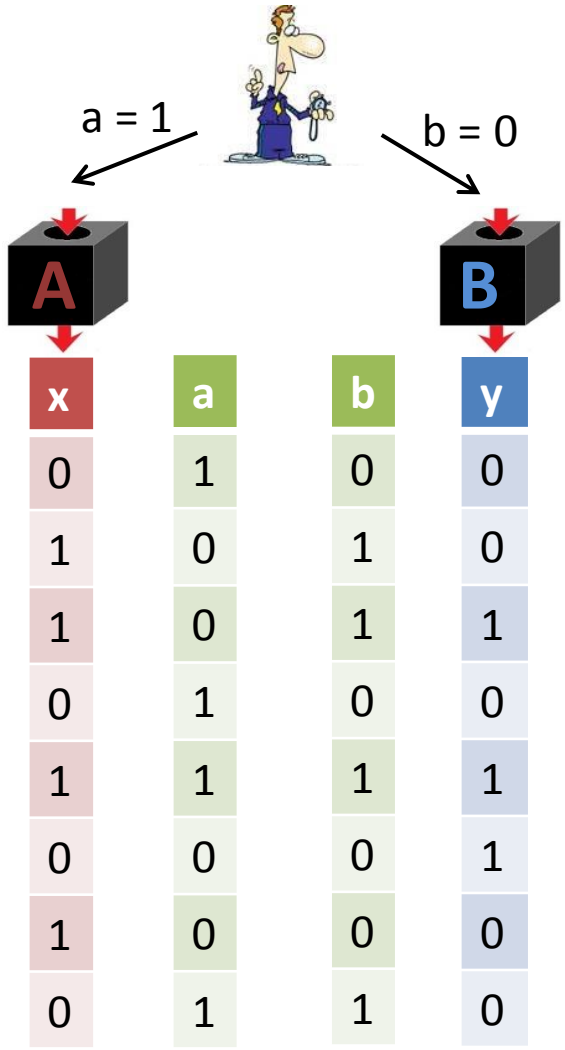
1) Not randomness *expanding*.

Obtaining Expansion



- VV
  - Exponential Expansion
  - Q. Secure
- RUV:
  - Polynomial Contraction
- Net:
  - Exponential Expansion

Input Secure?



This RUV protocol seems Input Secure!

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

"Input Secure"

$$\rho_{SD} \approx \rho_U \otimes \rho_D$$

However:

$$\rho_{OE} \approx \rho_U \otimes \rho_E$$

1) Not randomness *expanding.*
2) Not input secure *conditioned on passing*.

Input Security revisited

a = 1

b = 0

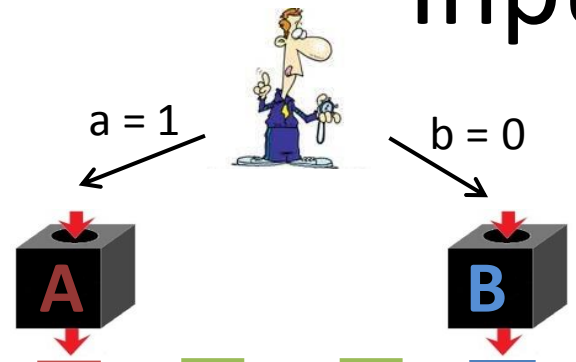
A

B

x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

- We only use the output of RUV in the event that the protocol passes

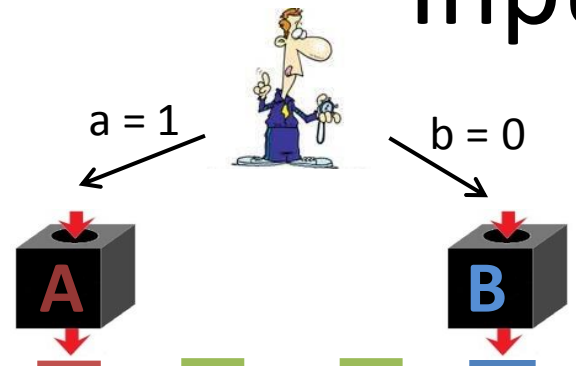
Input Security revisited



x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

- We only use the output of RUV in the event that the protocol passes
- In general conditioning on this event can reveal output information to the eavesdropper

Input Security revisited



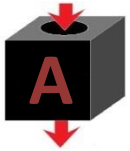
x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

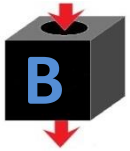
- We only use the output of RUV in the event that the protocol passes
- In general conditioning on this event can reveal output information to the eavesdropper
- This would invalidate the Input Security gained from RUV

Input Security: The Solution

a = 1

b = 0





x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

X_1

X_2

X_3

X

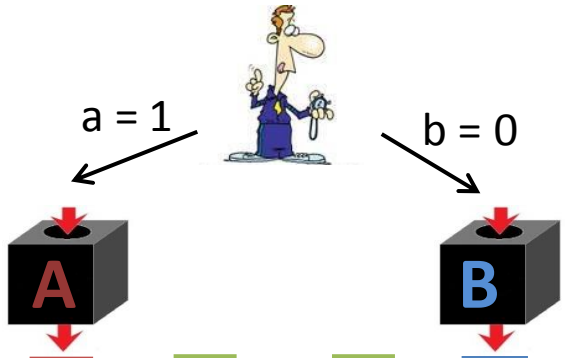
F

Pass	Fail
F = 1	F = 0

E

Input Security: The Solution

- Divide the output X into blocks



x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

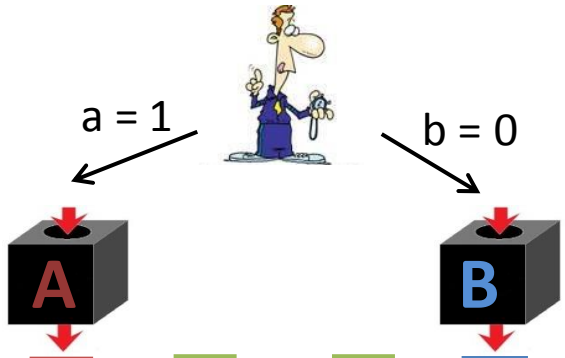
X

F

Pass	Fail
F = 1	F = 0

E

Input Security: The Solution



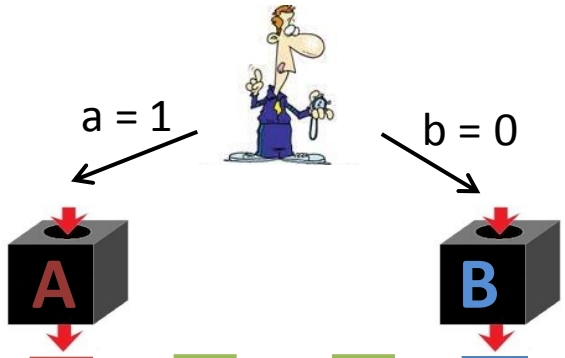
x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

- Divide the output X into blocks
- On average each block will be nearly unentangled with the combined system FE

X F E

Pass	Fail
F = 1	F = 0

Input Security: The Solution



x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

- Divide the output X into blocks
- On average each block will be nearly unentangled with the combined system FE
- Output a random block

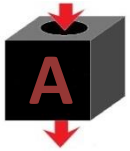
X F E

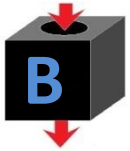
Pass	Fail
F = 1	F = 0

Input Security: The Solution

a = 1

b = 0





x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

X_1

X_2

X_3

X

F

Pass	Fail
F = 1	F = 0

E

Input Security: The Solution

$$I(X:E) \approx 0$$

a = 1

b = 0

A

B

x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

X

F

Pass	Fail
F = 1	F = 0

E

Input Security: The Solution

$$I(X:E) \approx 0$$

$$I(X:FE) \leq 2\,H(F) \leq 2$$

a = 1

b = 0

A

B

x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

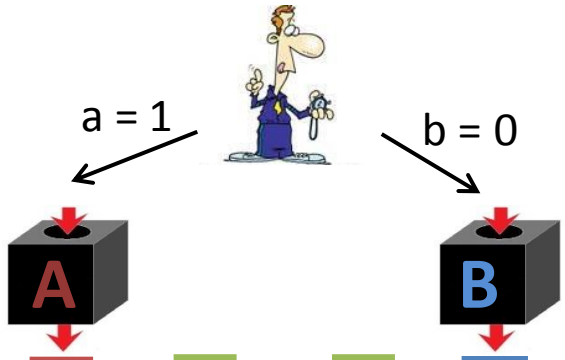
X

F

Pass	Fail
F = 1	F = 0

E

Input Security: The Solution



x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

X

$$I(X:E) \approx 0$$

$$I(X:FE) \leq 2\,H(F) \leq 2$$

$$2 \geq I(X:FE) = \sum_i I(X_i:FE|X_{<i})$$

$$\geq \sum_i I(X_i:FE)$$

F

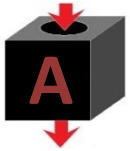
Pass	Fail
F = 1	F = 0

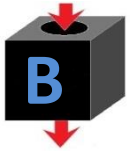
E

Input Security: The Solution

a = 1

b = 0





x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

X_1

X_2

X_3

X

F

Pass	Fail
F = 1	F = 0

E

Input Security: The Solution

$$\frac{1}{N}\sum_i I(X_i : FE) \le \frac{1}{N} I(X : FE) \le \frac{2}{N}$$

a = 1

b = 0

A

B

x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

X

F

Pass	Fail
F = 1	F = 0

E

Input Security: The Solution

$$\frac{1}{N}\sum_i I(X_i : FE) \le \frac{1}{N} I(X : FE) \le \frac{2}{N}$$

Pinsker's Inequality

a = 1

b = 0

A

B

x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

X_1

X_2

X_3

X

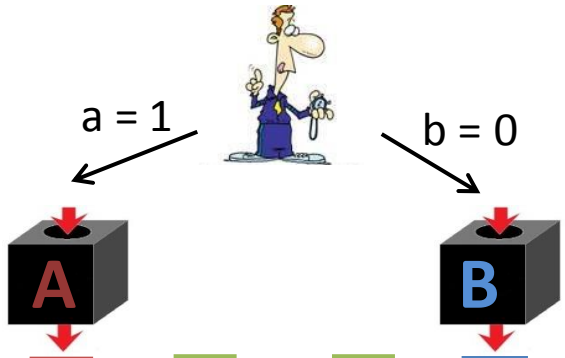
F

Pass	Fail
F = 1	F = 0

E

Input Security: The Solution

$$\frac{1}{N}\sum_i I(X_i : FE) \le \frac{1}{N} I(X : FE) \le \frac{2}{N}$$



x	a	b	y
0	1	0	0
1	0	1	0
1	0	1	1
0	1	0	0
1	1	1	1
0	0	0	1
1	0	0	0
0	1	1	0

X_1 X_2 X_3 X

Pinsker's Inequality

$$\frac{1}{N}\sum_i || \rho_{X_i FE} - \rho_{X_i} \otimes \rho_{FE} ||^2$$

$$\le \frac{1}{N}\sum_i I(X_i : FE) \le \frac{2}{N}$$

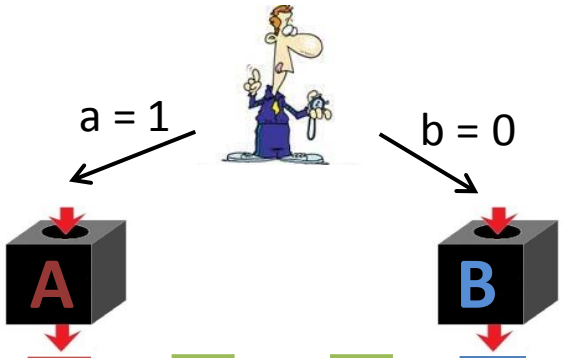
F

Pass	Fail
F = 1	F = 0

E

Selecting Random Blocks

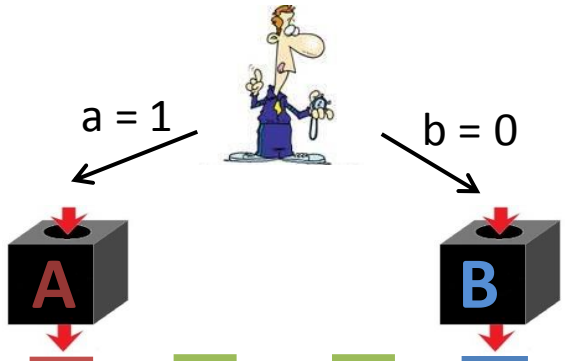
- Our solution selects output blocks at random….



x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

X

Selecting Random Blocks



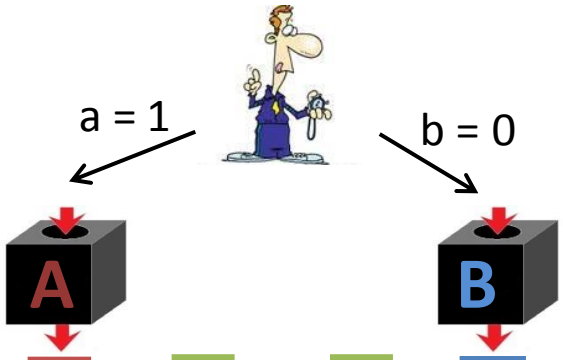
x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

X

- Our solution selects output blocks at random….

  ….using an input seed unknown to the devices

Selecting Random Blocks



x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

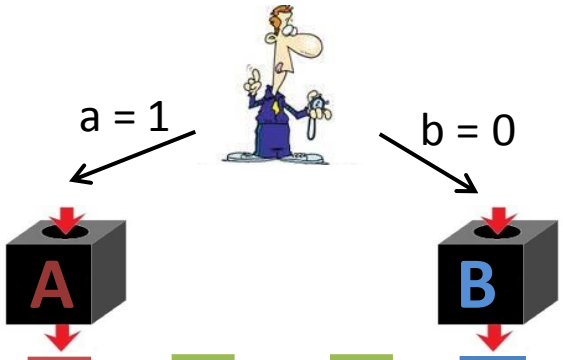
X

- Our solution selects output blocks at random….

  ….using an input seed unknown to the devices

- But could the seed be correlated with the position of "bad" blocks?

Selecting Random Blocks

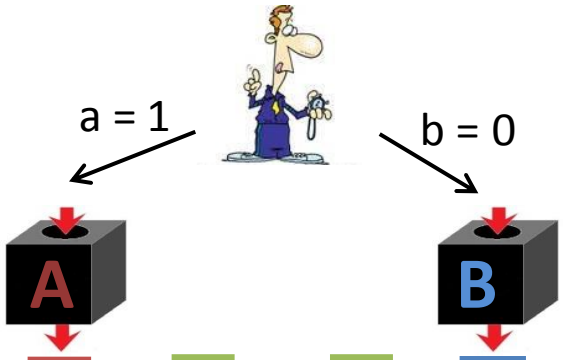


- Such adversarial correlations can be ruled out using a purification and simulation argument.

x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

X

Selecting Random Blocks



x	a	b	y	
0	1	0	0	X_1
1	0	1	0	
1	0	1	1	
0	1	0	0	X_2
1	1	1	1	
0	0	0	1	
1	0	0	0	X_3
0	1	1	0	

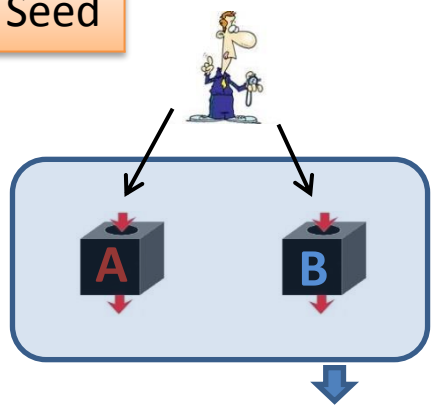
X

- Such adversarial correlations can be ruled out using a purification and simulation argument.
- This implies full input security for this composition of VV and RUV.

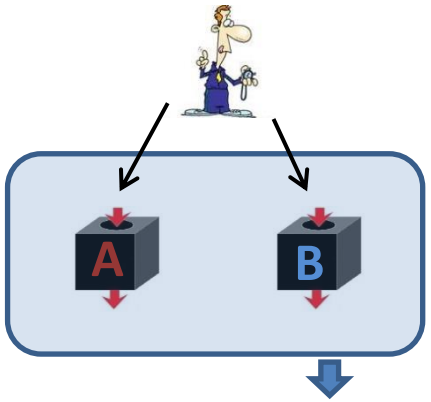
Our Protocol

S = Input Seed

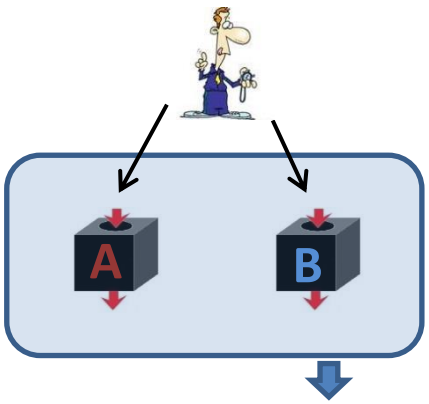
VV



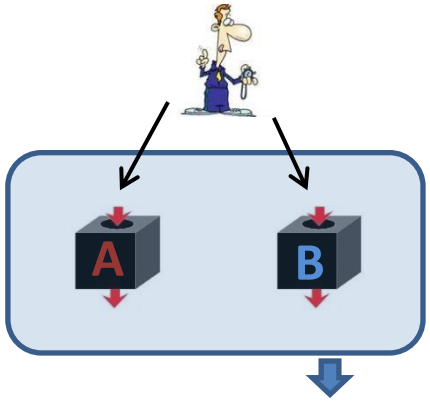
VV



RUV

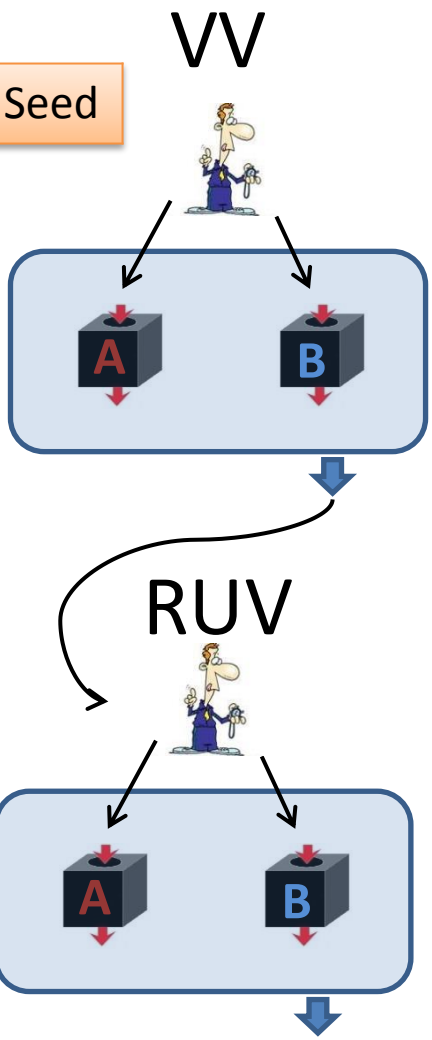


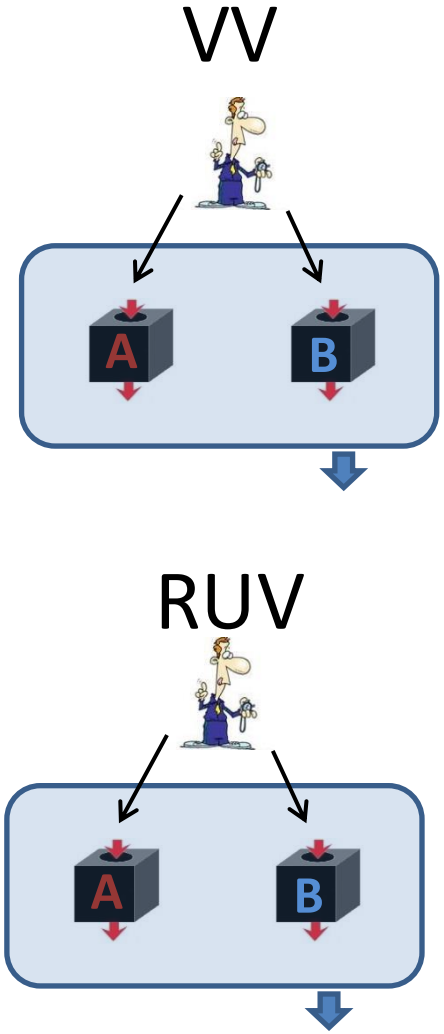
RUV



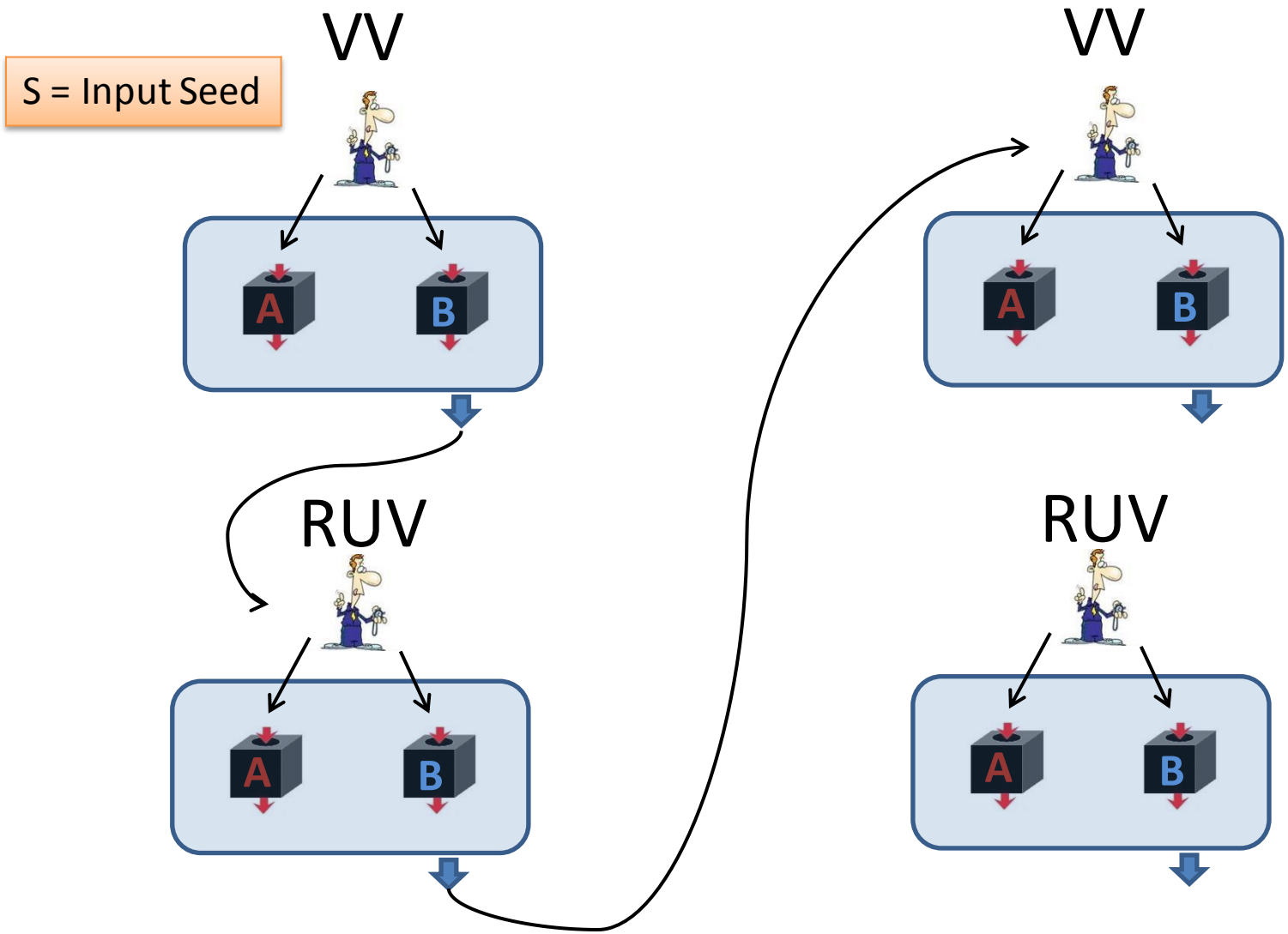
Our Protocol

S = Input Seed

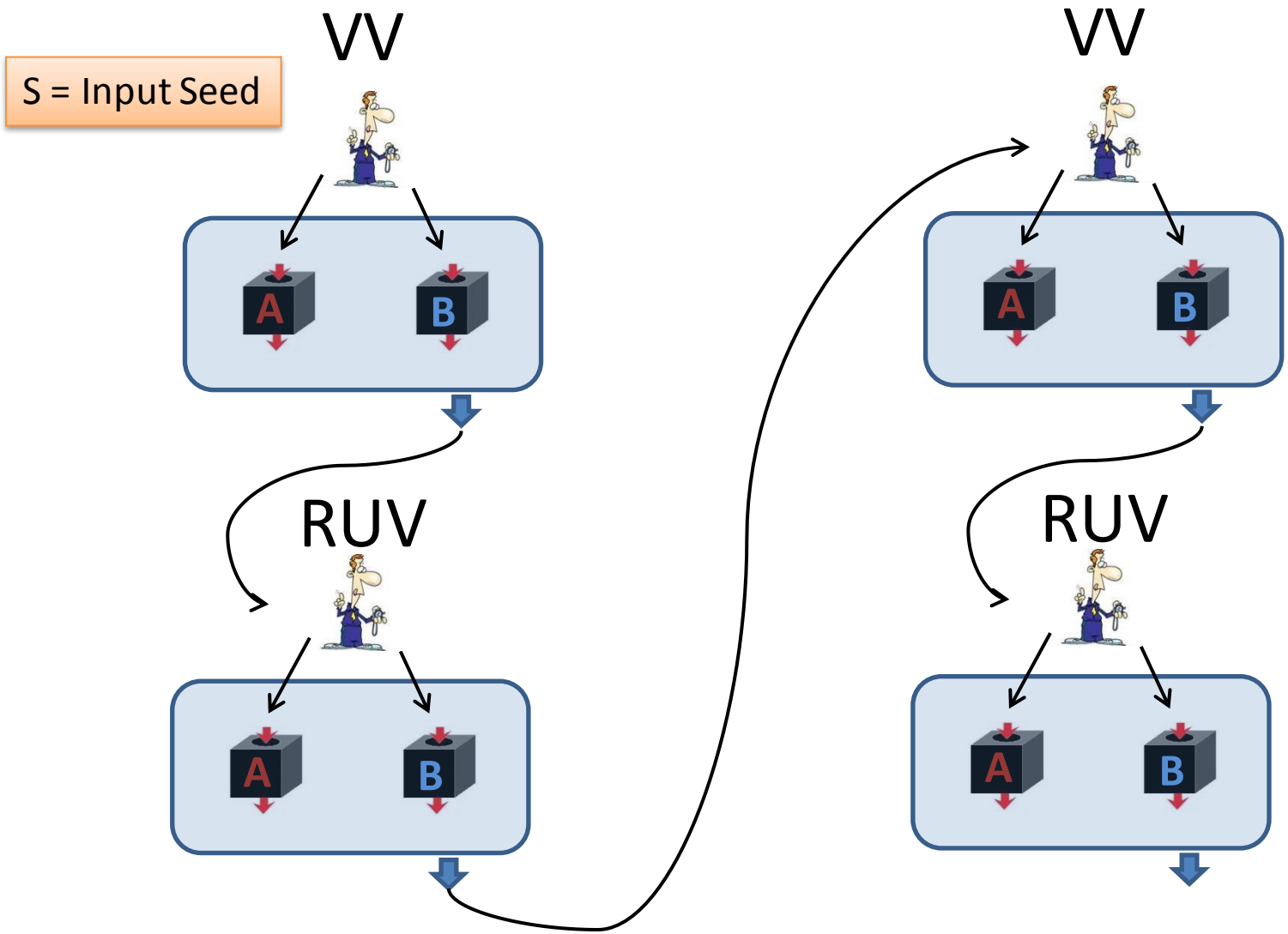




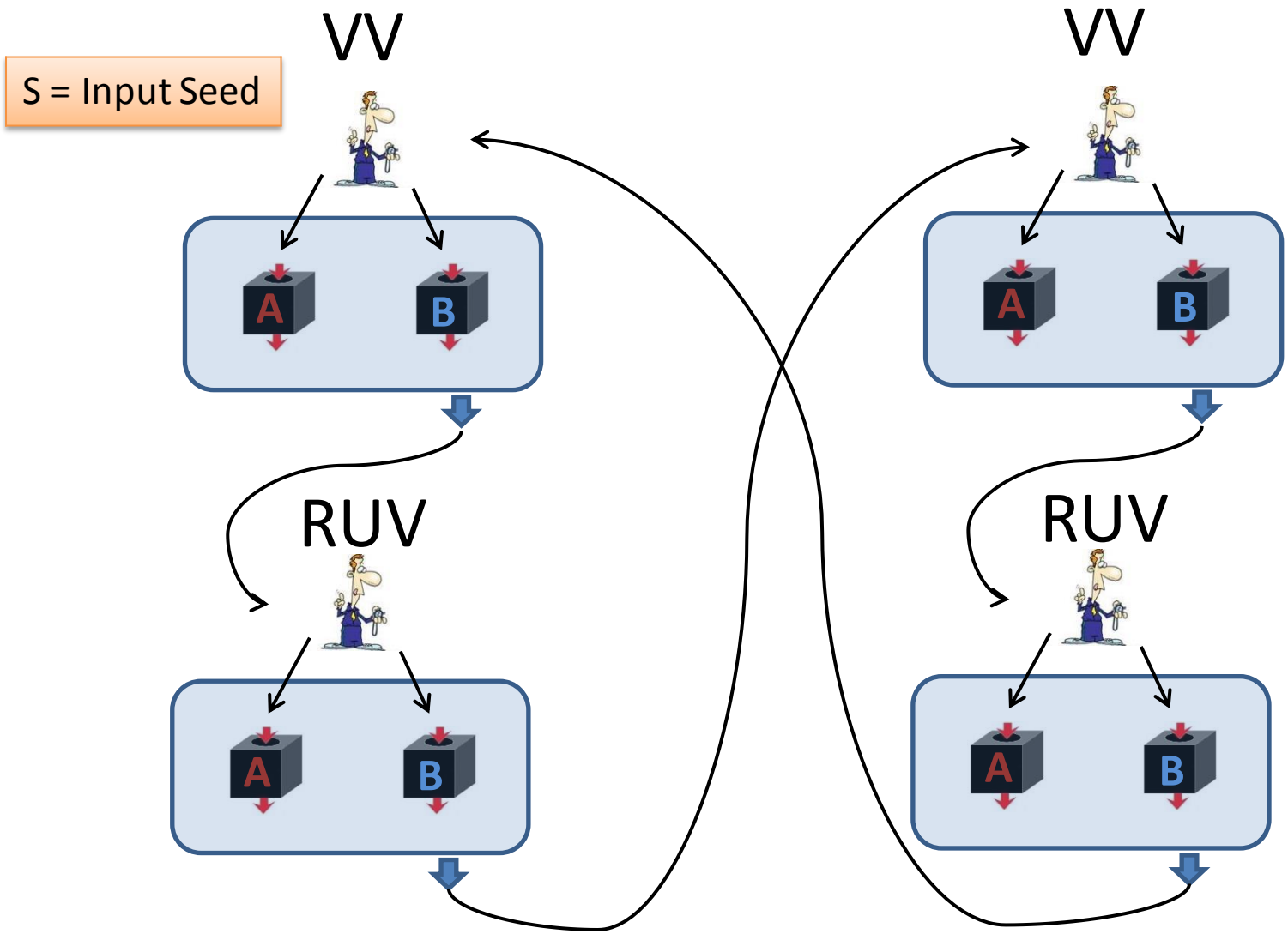
Our Protocol



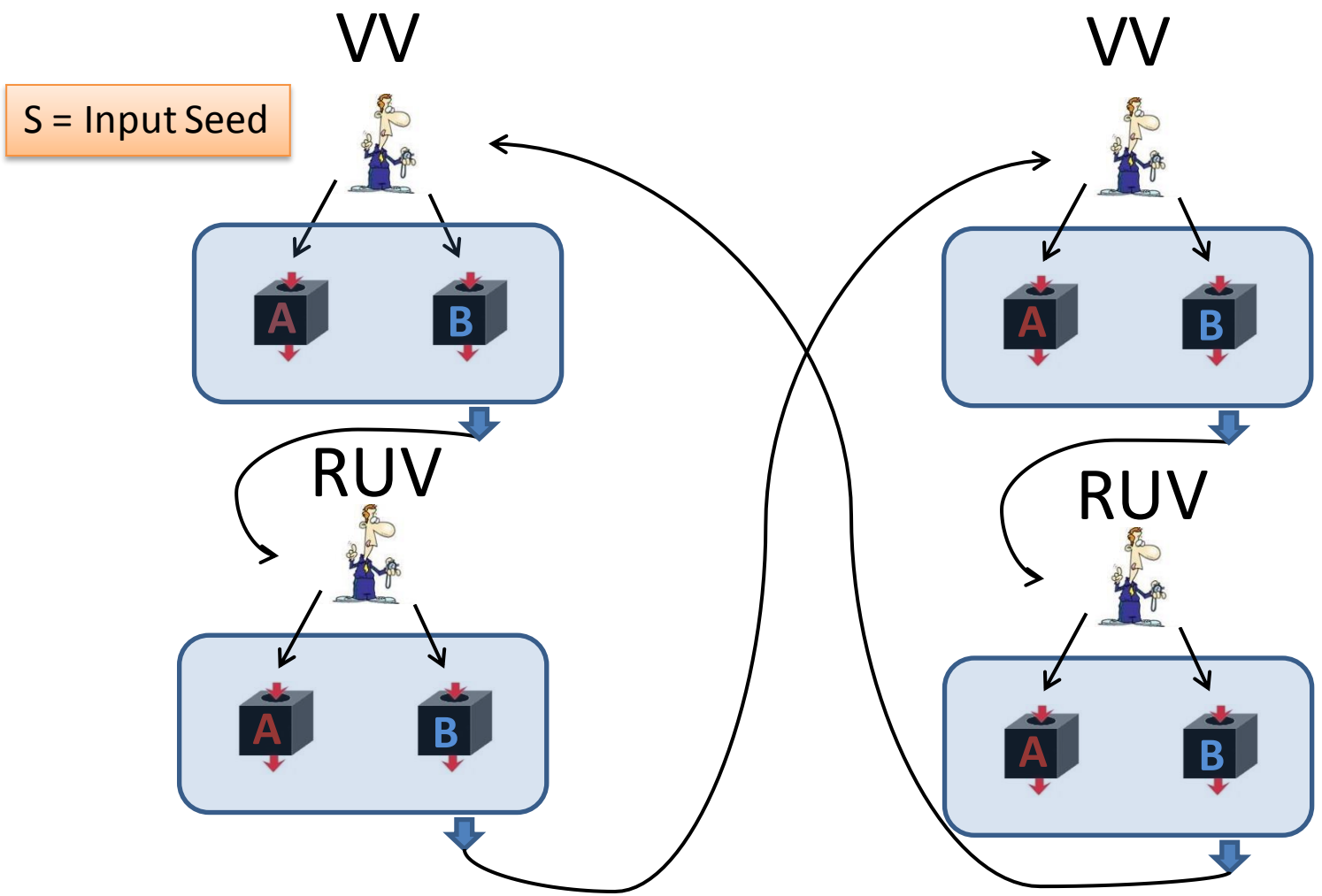
Our Protocol



Our Protocol

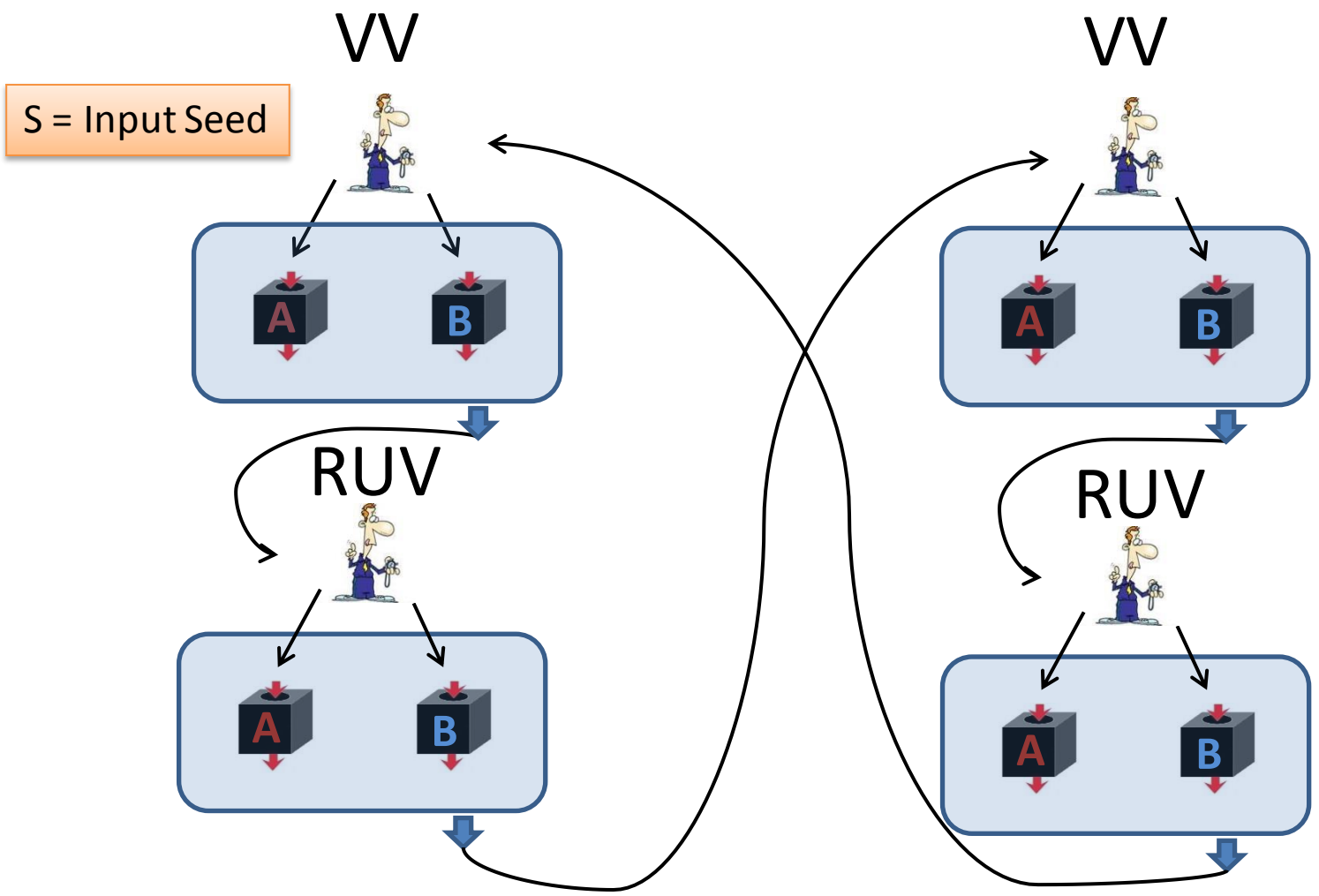


Infinite Randomness Expansion

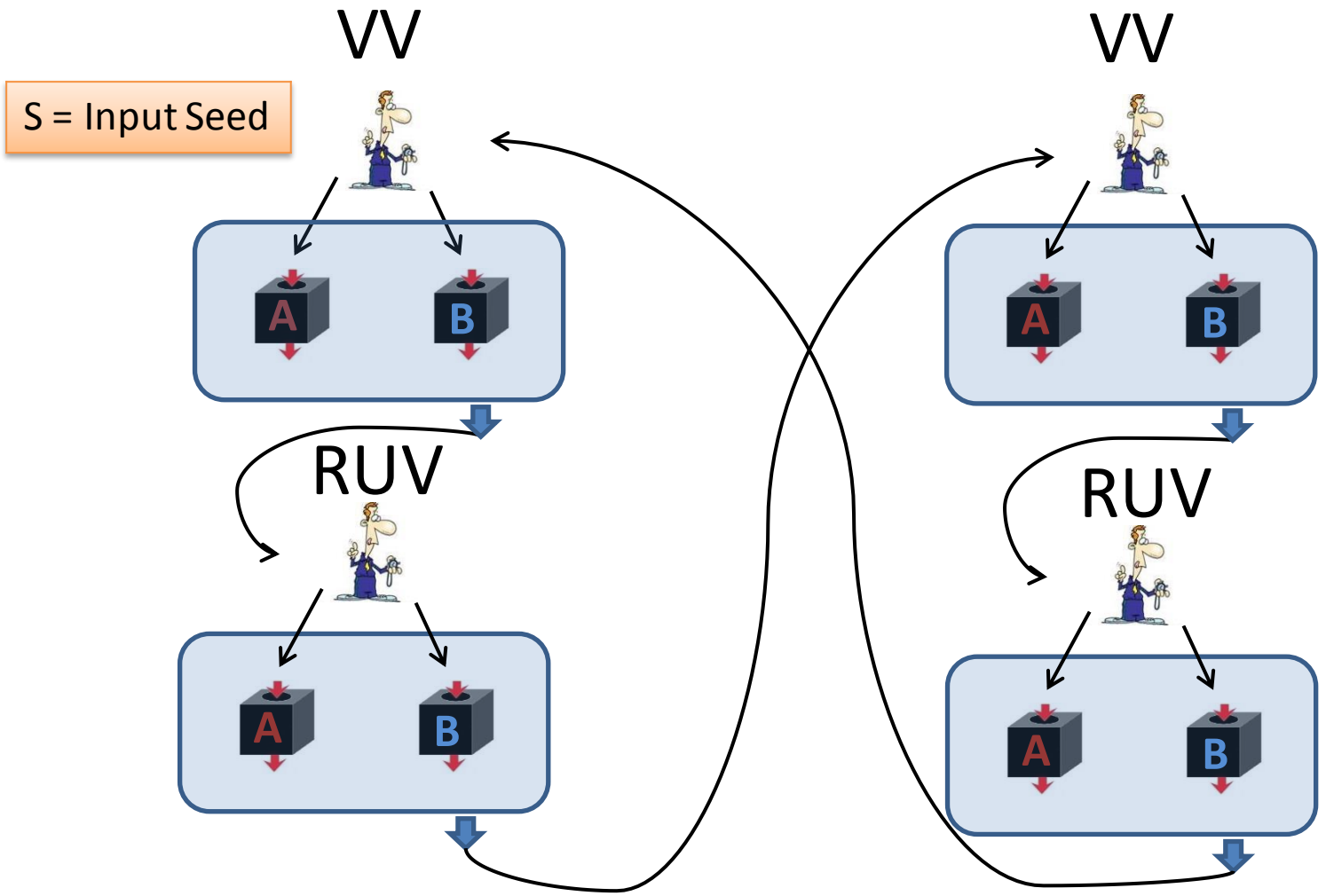


- Approximately Input Secure steps (composable)
- Exponential Expansion at each step

Infinite Randomness Expansion



- Accumulated error converges
- Output is $\frac{1}{exp(|S|)}$ -close to uniform and secure against quantum eavesdropper.



Open Questions

- Robust protocols [Miller, Shi], [Chung, Shi, Wu]
- Optimal Parameters?
- Protocols other than Randomness Expansion [Reichardt, Unger, Vazirani 2012]