Nucleon-Nucleon scattering from dispersion relations and chiral symmetry up to N^{2}LO

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Outline

Introduction

NN interactions are a basic building block

Application of ChPT to *NN* interactions
S. Weinberg, PLB **251** (1990) 288; NPB **363** (1991) 3; PLB **295** (1992) 114. **It is already a long story**

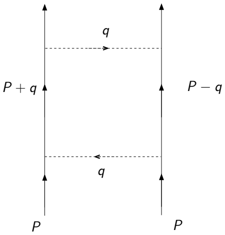
Weinberg's scheme: Calculate V_{NN} in ChPT and solve the LS equation:

$$T_{NN}(\mathbf{p}',\mathbf{p}) = V_{NN}(\mathbf{p}',\mathbf{p}) + \int d\mathbf{p}'' V_{NN}(\mathbf{p}',\mathbf{p}'') \frac{m}{\mathbf{p}^2 - \mathbf{p}''^2 + i\epsilon} T_{NN}(\mathbf{p}'',\mathbf{p})$$

C. Ordóñez, L. Ray and U. van Kolck, PRL **72** (1994) 1982; PRC **53** (1996) 2086.

- Typical three-momentum cut-offs $\Lambda \sim 600$ MeV are fined tuned to data.

- *NN* scattering is nonperturbative: (Anti)bound states, $m \gg M_\pi$



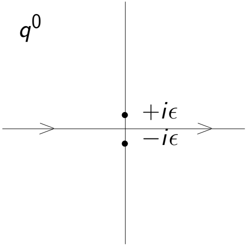
$$\int d^4q\,(q^0+i\epsilon)^{-1}(q^0-i\epsilon)^{-1}(q^2+M_\pi^2)^{-2}\,P(q)$$

Infrared enhancement

$1/|\mathbf{q}| \to 1/|\mathbf{q}| \times m/|\mathbf{q}|.$

Extreme non-relativistic propagator (or Heavy-Baryon propagator)

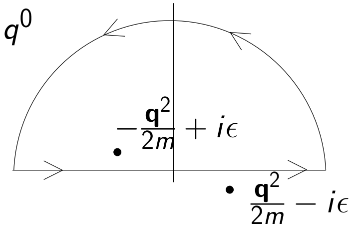
$$\frac{1}{q^0 + i\epsilon}$$



"Pinch" singularity
The integration contour cannot be deformed

Non-relativistic propagator with recoil correction:

$$\frac{1}{q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon}$$



$$\int dq^0 \, (q^0 - \frac{\mathbf{q}^2}{2m} + i\epsilon)^{-1}(q^0 + \frac{\mathbf{q}^2}{2m} - i\epsilon)^{-1} = -2\pi i \frac{m}{\mathbf{q}^2}$$

- V_{NN} is calculated up to next-to-next-to-next-to-leading order (N^{3}LO) and applied with great phenomenological success

Entem and Machleidt, PLB **254** (2002) 93; PRC **66** (2002) 014002; PRC **68** (2003) 041001
Epelbaum, Glöckle, Meißner, NPA **637** (1998) 107; **671** (2000) 195; **747** (2005) 362

- **Remaining cut-off dependence**

Chiral counterterms introduced in V_{NN} following naive chiral power counting are not enough to reabsorb the dependence on the cut-off

Nogga, Timmermans and van Kolck, PRC **72** (2005) 054006
Pavón Valderrama and Arriola, PRC **72** (2005) 054002; **74** (2006) 054001; **74** (2006) 064004
Kaplan, Savage, Wise NPB **478** (1996) 629
Birse, PRC **74** (2006) 014003 ; C.-J. Yang, Elster and Phillips, PRC **80** (2009) 034002; *idem* 044002.

▷ In Nogga *et al.* one counterterm is **promoted** from higher to lower orders in $^3P_0$, $^3P_2$ and $^3D_2$ and then stable results for $\Lambda < 4$ GeV are obtained.
▷ *Higher order contributions would be treated perturbatively*

Pavón Valderrama, PRC **83** (2011) 024003; **84** (2011) 064002
B. Long, C.-J. Yang, PRC **84** (2011) 057001; **85** (2011) 034002; **86** (2012) 024001

• Given an attractive/repulsive singular potential only one/none counterterm is effective.
Pavón Valderrama and Arriola, Phys.Rev.C72,054002 (2005)
Zeoli *et al.*, Few Body Sys. 54,2191 (2013)

• This procedure is **criticized** by Epelbaum and Gegelia, Eur.Phys. J.A41, 341 (2009).

It is not enough to obtain a finite T-matrix in the limit $\Lambda \to \infty$

One should absorb all divergences from loops in counterterms

To avoid renormalization scheme dependence and violation of low-energy theorems when $\Lambda \to \infty$

• **Covariant ChPT** Epelbaum and Gegelia, Phys.Lett.B716,338 (2012)
Avoid $1/m$ expansion in nucleon denominators + OPE
Ultraviolet divergences are absorbed by leading S-wave counterterms
Contrary to the HBChPT case Eiras,Soto, Eur.Phys.J.A17,89(2003)

• V_{NN} is calculated up to next-to-next-to-next-to-leading order (N^3LO) and applied with great phenomenological success

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▷ **The main goal of our study is to establish a sound framework that allows to study *NN* interactions in chiral EFT without any regulator dependence.**

It is an interesting problem

N/D Method

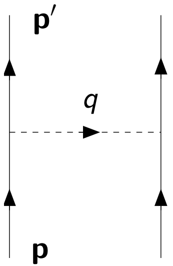
Chew and Mandelstam, Phys. Rev. **119** (1960) 467

A NN partial wave amplitude has two type of cuts:

Unitarity or Right Hand Cut (RHC)

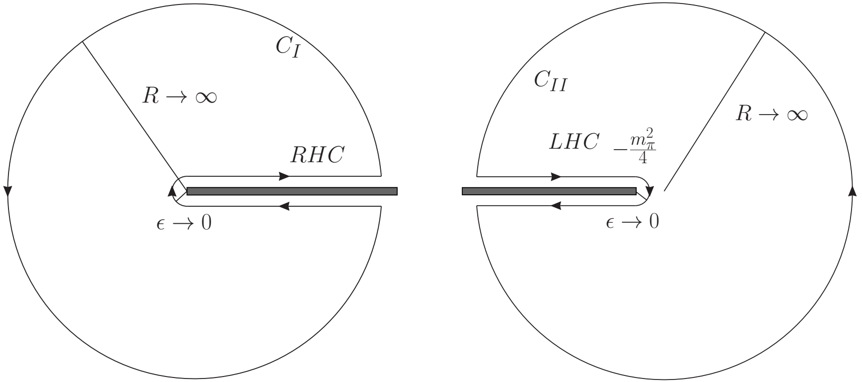
$$\Im T = \frac{m|\mathbf{p}|}{4\pi} T T^\dagger \quad , \; \mathbf{p}^2 > 0 \longrightarrow \Im T^{-1} = -\frac{m|\mathbf{p}|}{4\pi}\mathbb{I}$$

Left Hand Cut (LHC)



$$\frac{1}{(\mathbf{p}-\mathbf{p}')^2 + M_\pi^2}$$

$$\mathbf{p}^2 = -\frac{M_\pi^2/2}{1-\cos\theta} \rightarrow \mathbf{p}^2 \in]-\infty, -M_\pi^2/4]$$



$$T_{J\ell S}(A) = \frac{N_{J\ell S}(A)}{D_{J\ell S}(A)}$$

$N_{J\ell S}(A)$ has Only LHC

$D_{J\ell S}(A)$ has Only RHC

Uncoupled Partial Waves

$$T_{J\ell S}(A) = N_{J\ell S}(A)/D_{J\ell S}(A)$$

$$\Im \frac{1}{T_{J\ell S}(A)} = -\rho(A) \equiv \frac{m\sqrt{A}}{4\pi} \qquad , \; A > 0$$

$$\Im D_{J\ell S}(A) = -N_{J\ell S}(A)\rho(A) \qquad , \; A > 0$$

$$\Im N_{J\ell S}(A) = D_{J\ell S}(A)\, \Im T_{J\ell S}(A) \quad , \; A < -M_\pi^2/4$$

$$A \equiv |\mathbf{p}|^2 \quad , \quad \Delta(A) = \Im T_{J\ell S}(A) \quad , \; A < -M_\pi^2/4$$

Let us start with one subtraction in $D(A)$ and $N(A)$

$\boxed{\text{Coupled system of linear integral equations}}$

$$D_{J\ell S}(A) = 1 - \frac{A-D}{\pi}\int_0^\infty dq^2 \frac{\rho(q^2) N_{J\ell S}(q^2)}{(q^2-A)(q^2-D)}$$

$$N_{J\ell S}(A) = N_{J\ell S}(D) + \frac{A-D}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta_{J\ell S}(k^2) D_{J\ell S}(k^2)}{(k^2-A)(k^2-D)}$$

$$L \equiv -\frac{M_\pi^2}{4}$$

$$\Delta(A) = \Im T_{J\ell S}(A)\ , \ \ A < L$$

$$D_{J\ell S}(A) = 1 - A N_{J\ell S}(0)\mathbf{g}(\mathbf{A},\mathbf{0}) + \frac{A}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta_{J\ell S}(k^2) D_{J\ell S}(k^2)}{k^2}\mathbf{g}(\mathbf{A},\mathbf{k^2})$$

$$\mathbf{g}(\mathbf{A},\mathbf{k^2}) = \frac{1}{\pi}\int_0^{+\infty} dq^2 \frac{\rho(q^2)}{(q^2-A)(q^2-k^2)}$$

Convergent, $\rho(A) \propto \sqrt{A}$

CHANGE OF VARIABLE:

$$A = \frac{L}{x}\ , \ \ x \in [1,0]$$

$$D_{J\ell S}(x) = 1 - \frac{L}{x} N_{J\ell S}(0)\mathbf{g}(\mathbf{x},\mathbf{0}) + \frac{L}{\pi x}\int_0^1 dy \frac{\Delta(y)\mathbf{g}(\mathbf{x},\mathbf{y})}{y} D(y)$$

Fredholm Integral Equation of the Second Kind

$$D_{J\ell S}(x) = f_{J\ell S}(x) + \int_0^1 dy\, K(x,y) D(y)$$

$$K(x,y) = \frac{L}{\pi} \frac{\mathbf{g}(\mathbf{x},\mathbf{y})}{x\, y} \Delta(y)$$

- Not L_2 for $\Delta(A)$ at NLO and at higher orders in ChPT
- Not symmetric

The N/D method provides nonperturbative scattering equations that requires as input $\Delta(A)$ **that is calculated in perturbation theory**

Integrals of infinite extent are convergent by introducing enough number of subtractions

• In connection with ChPT this dispersive method was recently applied to NN scattering in LO: M. Albaladejo and J.A. Oller, Phys.Rev.C84, 054009 (2011); 86,034005 (2011) employing OPE

NLO: Z.-H.Guo, G. Ríos, J.A. Oller, Phys.Rev.C89,014002(2014) OPE+leading TPE

N^{2}LO: J.A. Oller, arXiv:1402.2449 OPE+leading+subleading TPE

High-Energy behavior

- Let $|D(A)| \leq A^n$ for $A \to \infty$

$$N(A) = T(A)D(A)$$
$$T(A) = \frac{S(A)-1}{2\rho(A)} \to A^{-1/2} \quad , \quad A \to +\infty$$
$$N(A) \leq A^{n-1/2}$$

We divide $N(A)$ and $D(A)$ by $(A-C)^m$ with $m > n$

$$\frac{D(A)}{A^m} \to 0 \ , \ \text{when } A \to \infty$$
$$L < C < 0$$

Dispersive integrals are convergent with $m > n$ subtractions

$$D(A) = \sum_{i=1}^{m} \delta_i (A-C)^{m-i} - \frac{(A-C)^m}{\pi} \int_0^{\infty} dq^2 \frac{\rho(q^2) N(q^2)}{(q^2-A)(q^2-C)^m}$$
$$N(A) = \sum_{i=1}^{m} \nu_i (A-C)^{m-i} + \frac{(A-C)^m}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2-A)(k^2-C)^m}$$

$m = 1$ IS THE MINIMUM

Once-subtracted DRs for $N(A)$ and $D(A)$

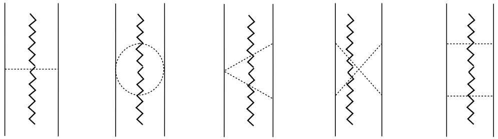
Unnatural size of S-wave scattering lengths

- C could be taken different in $D(A)$ and $N(A)$
  - $N(A)$: $C = 0$
  - $D(A)$: One subtraction at $C = 0$ and the rest at $C = -M_\pi^2$.
  - Normalization: $D(0) = 1$

In our study $\Delta(A)$ is given by $\frac{-i}{2}$ the discontinuity across the LHC:

- LO: OPE
- NLO: Leading TPE (irreducible) + Once-iterated OPE
- N^{2}LO: Subleading TPE

Kaiser, Brockmann and Weise, NPA625(1997)758



$\Delta(A)$ is finite

$$\lim_{A\to\infty} \Delta(A) \to A^{3/2} \qquad \text{N}^2\text{LO, at most}$$
$$\to A \qquad \text{NLO, at most}$$
$$\to A^{-1} \qquad \text{LO, at most}$$

Existence of solution of the IEs: $\Delta(A) = \lambda(-A)^\gamma$

Change of Variable: $x = L/k^2, \;\; y = L/A$

$$D(y) = 1 + \nu_1 \frac{m(-L)^{\frac{1}{2}}}{4\pi y^{\frac{1}{2}}} + \frac{\lambda m}{4\pi^2}(-L)^{\gamma+\frac{1}{2}} \int_0^1 \frac{dx}{x^{\gamma+\frac{1}{2}} y^{\frac{1}{2}}} \frac{D(x)}{\sqrt{x}+\sqrt{y}}$$

Symmetryzing the kernel. Change of function:

$$\widetilde{D}(y) = y^{-\frac{\gamma}{2}} D(y)$$

$$\widetilde{D}(y) = y^{-\gamma/2} + y^{-\frac{\gamma+1}{2}} \nu_1 \frac{m(-L)^{\frac{1}{2}}}{4\pi} + \frac{\lambda m}{4\pi^2}(-L)^{\gamma+\frac{1}{2}} \int_0^1 dx \frac{\widetilde{D}(x)}{(xy)^{\frac{\gamma+1}{2}}(\sqrt{x}+\sqrt{y})}$$

Kernel:

$$K(y,x) = \frac{1}{(xy)^{\frac{\gamma+1}{2}}(\sqrt{x}+\sqrt{y})}$$

- It is quadratically integrable for $\gamma < -1/2$

$$\int_0^1 \int_0^1 dxdy\ K(x,y)^2 < \infty$$

- The inhomogeneous term is also quadratically integrable
- Because of Fredholm Theorem $\rightarrow$ There is a unique solution
- The eigenvalues have no accumulation point in the finite domain. Just change infinitesimally g_A, c_i, etc.

For **OPE** $\gamma = -1$ or -2

There is a unique solution when $\Delta(A)$ is given at LO with the N/D method

In the Lippmann-Schwinger + OPE potential this is not the case.
Singular nature of the OPE potential ($1/r^3$ for $r \to 0$) in the triplet waves. $\rightarrow$**Introduction**

- Adding more subtractions does not modify the symmetric kernel

$$\frac{A^2}{\pi} \int dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2)^2} \ldots$$

One extra $\frac{L}{y} \cdot \frac{x}{L} \longrightarrow \widetilde{D}(y) = y^{-\frac{\gamma}{2}+1} D(y)$

The degree of divergence does not increase in the inhomogeneous term $\leftrightarrow$ We multiply by the extra y

NLO: $\gamma \geq 1$

Integration Interval: $x \in [\varepsilon, 1] \quad \varepsilon \to 0^+$
To recover $[0,1]$ define $t = (x-\varepsilon)/(1-\varepsilon)$, $u = (y-\varepsilon)/(1-\varepsilon)$

$$\widetilde{D}_\varepsilon(u) = (1-\varepsilon)^{-\frac{\gamma}{2}}(u+\frac{\varepsilon}{1-\varepsilon})^{-\frac{\gamma}{2}}\left(1+(1-\varepsilon)^{-\frac{1}{2}}(u+\frac{\varepsilon}{1-\varepsilon})^{-\frac{1}{2}}\nu_1\frac{m(-L)^{\frac{1}{2}}}{4\pi}\right)$$
$$+\frac{\lambda m}{4\pi^2}(-L)^{\gamma+\frac{1}{2}}\int_0^1 dt\frac{\widetilde{D}_\varepsilon(t)\,(1-\varepsilon)^{-\gamma-\frac{1}{2}}}{\left[(t+\frac{\varepsilon}{1-\varepsilon})(u+\frac{\varepsilon}{1-\varepsilon})\right]^{\frac{\gamma+1}{2}}\left(\sqrt{t+\frac{\varepsilon}{1-\varepsilon}}+\sqrt{u+\frac{\varepsilon}{1-\varepsilon}}\right)}$$

With the modified kernel $K_\varepsilon(u,t)$ given by

$$K_\varepsilon(u,t) = \frac{(1-\varepsilon)^{-\gamma-\frac{1}{2}}}{\left[(t+\frac{\varepsilon}{1-\varepsilon})(u+\frac{\varepsilon}{1-\varepsilon})\right]^{\frac{\gamma+1}{2}}\left(\sqrt{t+\frac{\varepsilon}{1-\varepsilon}}+\sqrt{u+\frac{\varepsilon}{1-\varepsilon}}\right)} > 0\ .$$

$$\widetilde{D}_\varepsilon(u) = f(u) + \frac{\lambda m}{4\pi^2}(-L)^{\gamma+\frac{1}{2}} \int_0^1 dt\, H_\varepsilon(u,t) f(t)$$

$$H_\varepsilon(u,t) = \sum_{n=1} \beta^{n-1} K_{\varepsilon;n}(u,t) \ ,$$

$$K_{\varepsilon;n+1}(u,t) = \int_0^1 dv\, K_\varepsilon(u,v)\, K_{\varepsilon;n}(v,t) \quad , \quad (n \geq 1) \ ,$$

$$K_{\varepsilon;1}(u,t) \equiv K_\varepsilon(u,t) \ .$$

$$H_\varepsilon > 0 \ \textbf{if} \ \lambda > 0$$

- $\gamma \geq \mathbf{1/2}$: For having a cancellation between both terms it is necessary that $\lambda < 0$

We can get rid of this limitation by adding more subtractions

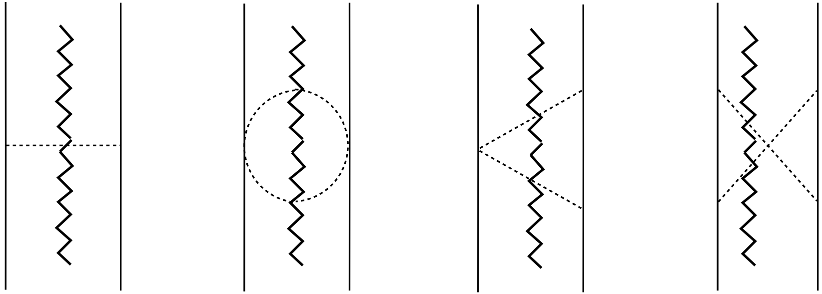
The subtraction constants have no sign defined.

The factor A^n also changes sign according to whether n is even or odd.

Adding more subtractions increases also the sensitivity to lower energies

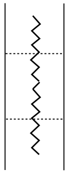
Perturbative calculation of $\Delta(A)$.

- Irreducible diagrams contributing to $\Delta(A)$

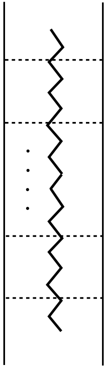


Amenable to a chiral expansion, much like V_{NN}

- Two-nucleon reducible diagrams



Similar size to the other NLO irreducible diagrams



- All pion lines must be put on-shell $\longrightarrow A \leq -n^2 M_\pi^2/4$.
- As n increases their physical contribution fades away.
- This only occurs for the imaginary part!

Chiral scaling of subtraction constants.

The change of the subtraction point makes the subtraction constants change

$$
\begin{aligned}
N(A) &= \nu_1 + \nu_2 A + \frac{A^2}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2} \\
&= \nu_1' + \nu_2' A + \frac{(A-C)^2}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2-C)^2}
\end{aligned}
$$

$$
\begin{aligned}
\nu_1' &= \nu_1 - \frac{C^2}{\pi}\int_{-\infty}^{L} dk^2 \Delta(k^2)D(k^2)\frac{1}{(k^2-C)^2 k^2} \\
\nu_2' &= \nu_2 + \frac{C}{\pi}\int_{-\infty}^{L} dk^2 \Delta(k^2)D(k^2)\frac{2k^2-C}{(k^2-C)^2(k^2)^2}
\end{aligned}
$$

$$\nu_1' = \nu_1 - \frac{C^2}{\pi}\int_{-\infty}^{L} dk^2 \Delta(k^2) D(k^2) \frac{1}{(k^2-C)^2 k^2} \sim \mathcal{O}(p^n)$$
$$\nu_2' = \nu_2 + \frac{C}{\pi}\int_{-\infty}^{L} dk^2 \Delta(k^2) D(k^2) \frac{2k^2-C}{(k^2-C)^2 (k^2)^2} \sim \mathcal{O}(p^{n-1})$$

Coefficient	$\Delta(A) =$	$\mathcal{O}(p^0)$	$\mathcal{O}(p^2)$
ν_1		p^0	p^2
ν_2		p^{-2}	p^0
ν_3		p^{-4}	p^{-2}

- $C \sim M_\pi^2$
- $\Delta(k^2) \sim \mathcal{O}(p^n)$
- $D(k^2) \sim \mathcal{O}(p^0)$
  $D(0) = 1$

This chiral power counting coincides with Weinberg power counting

$$D(A) = 1 + \delta_2 A - \frac{A(A-E)}{\pi}\int_0^\infty dq^2 \frac{\rho(q^2)N(q^2)}{q^2(q^2-E)(q^2-A)}$$

$$E \longrightarrow C$$

$$\delta_2 \to \delta_2 + \frac{E-C}{\pi}\int_0^\infty dq^2 \frac{\rho(q^2)N(q^2)}{q^2(q^2-E)(q^2-C)}$$

Coefficient	$\Delta(A) =$	$\mathcal{O}(p^0)$	$\mathcal{O}(p^2)$
δ_2		p^{-2}	p^0
δ_3		p^{-4}	p^{-2}

$$\rho(A) = m\sqrt{A}/4\pi \sim \mathcal{O}(p^0)$$

⋆ $D(A)$ is attached to two-nucleon reducible diagrams

$$\boxed{\nu_n,\ \delta_n \sim \mathcal{O}(p^{-2(n-1)+m})}$$

for $\Delta(A) \sim \mathcal{O}(p^m)$

How many subtraction to include for a given m?

$$n \leq [\tfrac{m}{2}] \quad \text{, such that } -2(n-1)+m \geq 0$$

However, more relevant that the counting is to have a meaningful IE.

This could eventually require including more subtraction constants.

Analogy with ChPT, e.g. meson-meson sector $\mathcal{O}(p^4)$:

$$f_4 = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \frac{s^3}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{Im f_4(s')}{(s')^3 (s'-s)}$$

Doing the same

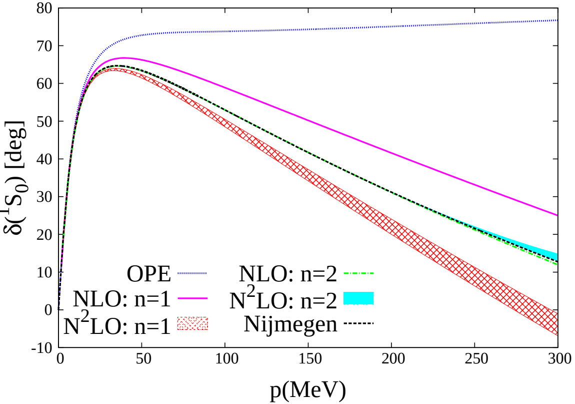
$$\alpha_0 = \mathcal{O}(p^4)\ ,\ \alpha_1 = \mathcal{O}(p^2)\ ,\ \alpha_2 = \mathcal{O}(p^0)$$

and no more subtractions are included because they would scale with negative powers (low-energy propagation)

The α_i are combination of the $L_i = \mathcal{O}(A^0)$

Once-subtracted DR

$\nu_1 = -4\pi a_s/m \sim 31\ M_\pi^{-2}$



$$r_s = \frac{m}{2\pi^2 a_s}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2}\left\{\sqrt{-k^2} - \frac{1}{a_s}\right\}$$

Correlation between a_s and r_s

$$r_s = 2.92(6)\ \text{fm}$$

Exp: 2.75 ± 0.05 fm
NijmII: 2.67 fm Arriola, Pavón, nucl-th/0407113

$$-\frac{A}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{k^2} g(A,k^2) + D(A) = 1 + A\frac{4\pi a_s}{m} g(A,0)$$

$D(A) = D_0(A) + a_s D_1(A)$ with $D_{0,1}(A)$ independent of a_s

Low-energy correlation:

$$r_s = \alpha_0 + \frac{\alpha_{-1}}{a_s} + \frac{\alpha_{-2}}{a_s^2}\ ,$$

$$\alpha_0 = 2.61 \sim 2.73\ \mathrm{fm}\ ,$$

$$\alpha_{-1} = -5.93 \sim -5.65\ \mathrm{fm}^2\ ,$$

$$\alpha_{-2} = 5.92 \sim 6.12\ \mathrm{fm}^3\ .$$

$$\alpha_0 = \frac{m}{2\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D_1(k^2)}{(k^2)^2}\sqrt{-k^2}$$

$$\alpha_{-1} = \frac{m}{2\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)}{(k^2)^2}\left[D_0(k^2)\sqrt{-k^2} - D_1(k^2)\right]$$

$$\alpha_{-2} = -\frac{m}{2\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D_0(k^2)}{(k^2)^2}$$

Pavón Valderrama, Ruiz Arriola PRC74(2006)054001: solving a Lippmann-Schwinger equation with V_{NN} that includes OPE+TPE + boundary conditions + orthogonality of wave functions

Twice-subtracted DR: a_s $[\nu_1]$ is fixed — ν_2 and δ_2 are fitted

$$\delta_2 = -8.0(3)\ M_\pi^{-2}\ ,$$
$$\nu_2 = -23(1)\ M_\pi^{-4}\ .$$

From the once-subtracted DR:

$$\nu_2^{\rm pred} = \frac{1}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2}$$

At N^{2}LO: $\nu_2^{\rm pred} \sim -7.5\ M_\pi^{-4}$

$$p\cot\delta = -\frac{1}{a_s} + \frac{1}{2} r_s p^2 + \sum_{i=2} v_i p^{2i}$$

	r_s	v_2	v_3	v_4	v_5	v_6
NLO	2.32	-1.08	6.3	-36.2	225	-1463
NNLO-I	2.92(6)	$-0.32(8)$	4.9(1)	$-27.7(8)$	177(4)	$-1167(30)$
NNLO-II	2.699(4)	$-0.657(3)$	5.20(2)	$-30.39(9)$	191.9(6)	$-1263(3)$
[A]	2.68	-0.61	5.1	-30.0		
[B]	$2.62 \sim 2.67$	$-0.52 \sim -0.48$	$4.0 \sim 4.2$	$-20.5 \sim -19.9$		
[C]	2.68	-0.48	4.0	-20.0		

	$v_7 \times 10^{-1}$	$v_8 \times 10^{-2}$	$v_9 \times 10^{-3}$	$v_{10} \times 10^{-4}$
NLO	985	-681	480	$-344(3)$
NNLO-I	795(18)	$-554(12)$	393(8)	$-284(6)$
NNLO-II	857.1(1.9)	$-595.7(1.3)$	421.7(9)	$-304(3)$

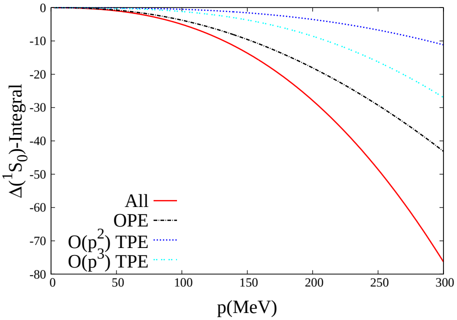
[A] Epelbaum *et al.*, NPA671,295(2000);
[B] Epelbaum *et al.*, EPJA19,401(2004);
[C] Stoks *et al.*, PRC48,792(1993)

Quantifying contributions to $\Delta(A)$

A typical integral from twice-subtracted DR:

$$\frac{A(A+M_\pi^2)}{\pi^2}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^2}\int_0^\infty dq^2 \frac{q^2\rho(q^2)}{(q^2-A)(q^2-k^2)(q^2+M_\pi^2)}$$

The integral displays the dominant role played by the nearest region in the LHC



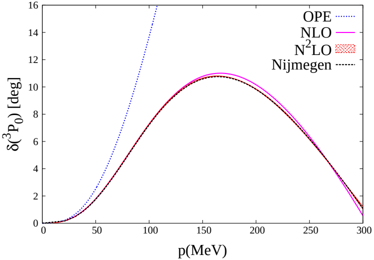
Uncoupled P-waves

$$\lambda_P = \lim_{A\to -\infty} \frac{\Delta(A)}{(-A)^{(3/2)}} > 0 \ ,$$

Once-subtracted DRs are not meaningful.

Three-time subtracted DRs are needed for $^3P_0$ and $^3P_1$

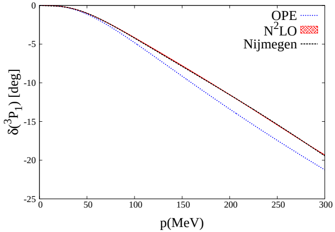
$$\nu_2 = 4\pi \, a_V / m \quad , \quad \nu_3 = 0^*$$



^{3}P$_0$

$\delta_2 = 2.82(5) \ M_\pi^{-2}$

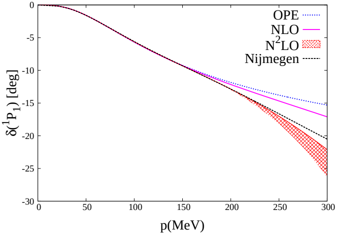
$\delta_3 = 0.18(6) \ M_\pi^{-4}$



$^3P_1$

$\delta_2 = 2.7(1)\ M_\pi^{-2}$

$\delta_3 = 0.47(3)\ M_\pi^{-4}$



$^3P_0$

Twice-subtracted DRs are enough

$\delta_2 = 0.4(1)\ M_\pi^{-2}$

A partial wave should vanish as A^ℓ in the limit $A \to 0^+$ (threshold)

$$\boxed{\textbf{Method: } \ell\text{-TIME-SUBTRACTED DR}}$$

$$N(A) = \frac{A^\ell}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D_{J\ell S}(k^2)}{(k^2)^\ell (k^2 - A)}$$

$$\nu_1, \ldots, \nu_{\ell-1} = 0, \qquad \lim_{A \to 0} N(A) \longrightarrow A^\ell$$

$$D(A) = 1 + \sum_{i=2}^{\ell} \delta_i A^{i-1} + \frac{A^\ell}{\pi} \int_{-\infty}^{L} dk^2 \frac{\Delta(k^2) D(k^2)}{(k^2)^\ell} g(A, k^2)$$

$$\lim_{A \to 0} D(A) \longrightarrow 1 + \mathcal{O}(A)$$

$$\lim_{A \to 0} \frac{N(A)}{D(A)} \longrightarrow A^\ell$$

$\ell - 1$ free parameters: $\delta_i \quad (i = 2, \ldots, \ell)$

Principle of maximal smoothness:

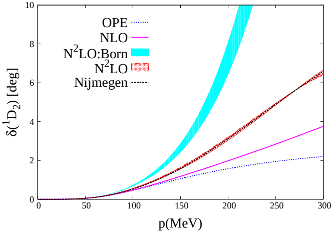
$\delta_p = 0^*$, $2 \leq p \leq \ell - 1$

δ_ℓ is the only free parameter

Uncoupled D-waves

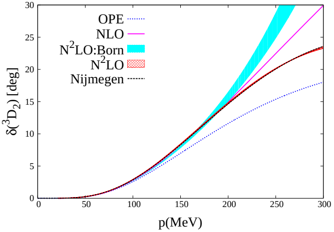
Twice-subtracted DRs

$$\lambda_D = \lim_{A \to -\infty} \frac{\Delta(A)}{(-A)^{3/2}} < 0$$



$^1D_2$

$$\delta_2 = -0.07(1)\ M_\pi^{-2}$$



$^3D_2$

$$\delta_2 = -0.017(3)\ M_\pi^{-2}$$

Born approximation

High-ℓ partial waves are expected to be perturbative

$$N(A) = \frac{A^\ell}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^\ell(k^2-A)}$$
$$N_B(A) = \frac{A^\ell}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta_B(k^2)}{(k^2)^\ell(k^2-A)}$$

$\Delta_B(A)$ only includes irreducible contributions

$$\boxed{\text{For } \ell \geq 2 \quad N_B(A) = V_{NN}(A)}$$

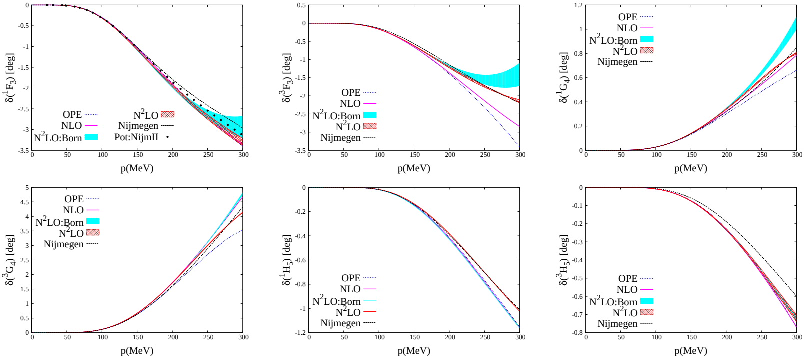
Perturbative phase shifts: $\delta_B(A) = \rho(A) N_B(A)$

- Connection between the subtraction constants ν_i and ChPT counterterms

Uncoupled $F-$, $G-$ and H-waves

Standard treatment:

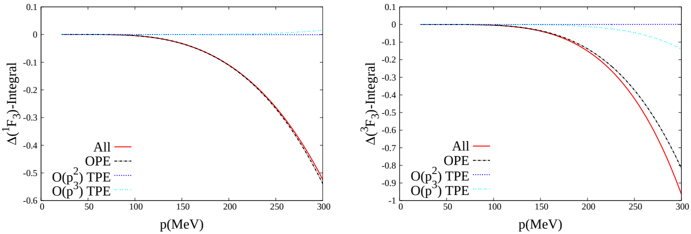
$\ell-$time subtracted DRs
1 free parameter per wave



Quantifying contributions to $\Delta(A)$

The perturbative character for $\ell \geq 3$ can also be seen here:

$$\frac{A^\ell}{\pi}\int_{-\infty}^{L} dk^2 \frac{\Delta(k^2)D(k^2)}{(k^2)^\ell} g(A,k^2)$$



Coupled Waves

$$S_{JIS} = I + i\frac{|\mathbf{p}|m}{4\pi}T$$

Along the RHC $A \geq 0$

$$S_{JIS} \cdot S_{JIS}^{\dagger} = S_{JIS}^{\dagger} \cdot S_{JIS} = I$$

$$S_{JIS} = \begin{pmatrix} \cos 2\varepsilon\, e^{i2\delta_1} & i\sin 2\varepsilon\, e^{i(\delta_1+\delta_2)} \\ i\sin 2\varepsilon\, e^{i(\delta_1+\delta_2)} & \cos 2\varepsilon\, e^{i2\delta_2} \end{pmatrix} \quad , \quad |\mathbf{p}|^2 \geq 0$$

ε is the mixing angle: $i = 1$ $(\ell = J-1)$, $i = 2$ $(\ell = J+1)$

$$\operatorname{Im}\frac{1}{T_{ii}(A)} = -\rho(A)\Big[1 + \frac{\frac{1}{2}\sin^2 2\varepsilon}{1-\cos 2\varepsilon\,\cos 2\delta_i}\Big]^{-1} \equiv -\nu_{ii}(A)$$

$$\operatorname{Im}\frac{1}{T_{12}(A)} = -2\rho(A)\frac{\sin(\delta_1+\delta_2)}{\sin 2\varepsilon} \equiv -\nu_{12}(A)$$

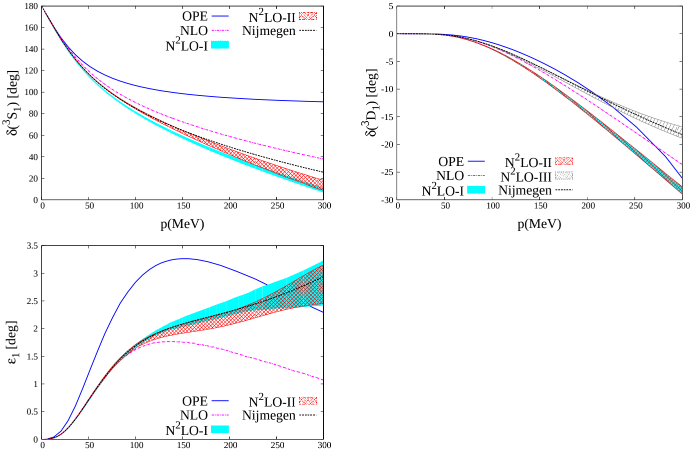
$$T_{ij}(A) = \frac{N_{ij}(A)}{D_{ij}(A)} \quad , \quad (ij = 11,\ 12,\ 22)$$

One proceeds in a coupled-iterative way:

1. We take an input.
2. Solve the integral equations and get new $\nu_{ij}(A)$.
3. Repeat the process until convergence is obtained.

$^3S_1 - {}^3D_1$

- Minimum number of subtractions in the DRs: 1 free parameter, $E_d = 2.225$ MeV



Correlation between r_t and a_t

$$r_t = -\frac{m}{2\pi^2 a_t}\int_{-\infty}^{L} dk^2 \frac{\Delta_{11}(k^2) D_{11}(k^2)}{(k^2)^2}\left\{\frac{1}{a_t} + \frac{4\pi k^2}{m} g_{11}(0,k^2)\right\} - \frac{8}{m}\int_0^{\infty} dq^2 \frac{\nu_{11}(q^2) - \rho(q^2)}{(q^2)^2}$$

$$S = \mathcal{O}\begin{pmatrix} S_0 & 0 \\ 0 & S_2 \end{pmatrix}\mathcal{O}^T \quad , \quad N_p^2 = \lim_{A\to k_d^2}\left(\sqrt{-k_d^2} + i\sqrt{A}\right) S_0$$

$$\eta = -\tan\epsilon_1 = [D/S \text{ ratio}]$$

	a_t [fm]	r_t [fm]	η	N_p^2 [fm^{-1}]	v_2	v_3
NLO	5.22	1.47	0.0295	0.714	-0.10572(12)	0.8818(11)
NNLO-I	5.52(3)	1.89(3)	0.0242(3)	0.818(10)	0.157(22)	0.645(9)
NNLO-II	5.5424*	1.759*	0.02535(13)	0.78173(2)	0.0848(4)	0.762(7)
[A]	5.4194(20)	1.7536(25)	0.0253(2)	0.7830(15)	0.040(7)	0.673(2)
[B]	5.424	1.753	0.0245		0.046	0.67

	v_7	$v_8 \times 10^{-1}$	$v_9 \times 10^{-2}$	$v_{10} \times 10^{-3}$
NLO	1867(11)	$-1375(11)$	1008(11)	$-760(12)$
NNLO-I	1161(41)	$-840(30)$	625(22)	$-463(17)$
NNLO-II	1426(13)	$-1015(15)$	764(17)	$-545(20)$

[A] de Swart *et al.*, *Proceedings of 3rd International Symposium on Dubna Deuteron 95*, Dubna, Moscow, July 4–7, 1995, arXiv: nucl-th/9509032

[B] Epelbaum *et al.*, NPA671, 295 (2000).

The differences between NNLO-I and NNLO-II are much smaller than in the $^1S_0$ wave

πN physics is more dominant in $^3S_1 - {}^3D_1$

Twice-subtracted DRs:
NNLO-II Results

E_d, r_t, a_t
ν_2^{12} is fitted

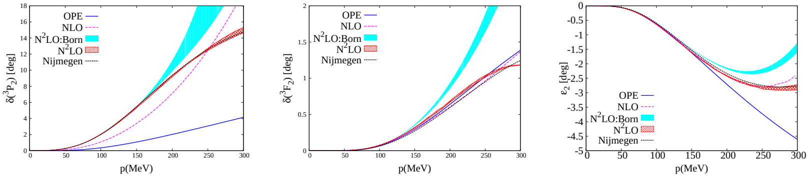
- $^3D_1$ is not accurately reproduced
- Three-time subtracted DRs for this wave: **NNLO-III**
- ν_3^{22} is around a 20% larger than predicted from NNLO-II

$$\frac{\delta\nu_3}{\nu_3^{\mathrm{pred}}} = \mathcal{O}(p) = 0.23 \sim \frac{M_\pi}{\Lambda}$$
$$\Lambda \sim 4M_\pi \sim 500\ MeV$$

$^3P_2 - {}^3F_2$

$$\lambda_{11} = \lim_{A\to-\infty} \frac{\Delta_{11}(A)}{(-A)^{3/2}} > 0 \ ,$$

$^3P_2$ requires at least three subtractions $\quad$ $^3P_2$ [2] ; $^3F_2$ [1] ; ϵ_2 [0]

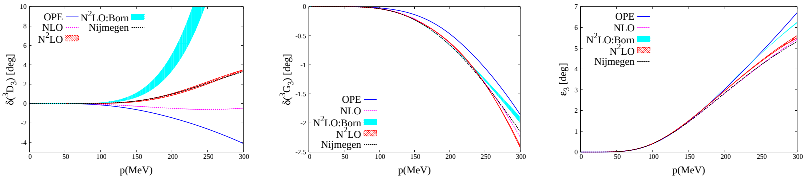


Clear improvement compared to the Born approximation in $^3F_2$ and ϵ_2 without modifying the values of the c_i's

The improvement does not come by modifying the potential

$^3D_3 - {}^3G_3$

Standard formalism

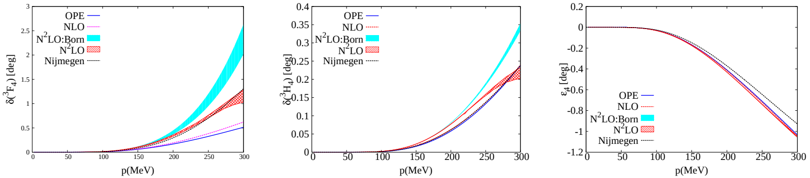


Clear improvement compared to the Born approximation in $^3D_3$ without modifying the values of the c_i's

$^3D_3$ [1] ; $^3G_3$ [1] ; ϵ_3 [0]

${}^3F_4 - {}^3H_4$

Standard formalism

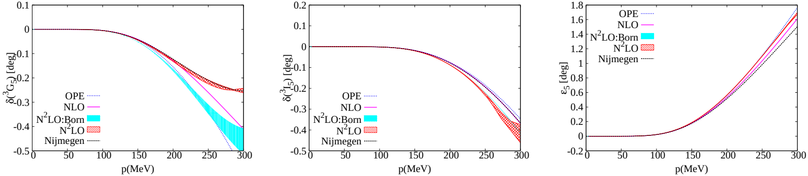


Clear improvement compared to the Born approximation in ${}^3F_4$ without modifying the values of the c_i's

${}^3F_4$ [1] ; ${}^3H_4$ [0] ; ϵ_4 [0]

$^3G_5 - {}^3I_5$

Standard formalism



Note the improvement in $^3G_5$.

This was not accomplished before.

Neither perturbatively (even following the spectral-function regularization) nor by iterating V_{NN}.

A genuine effect of NN rescattering.

$^3G_5$ [1] ; $^3I_5$ [1] ; ϵ_5 [0]

Conclusions

- $\Delta(A)$ is calculated perturbatively in ChPT up to N^{2}LO
  OPE, leading and subleading TPE, once-iterated OPE.

- Accurate reproduction of Nijmegen phase shifts.

- No need to modify $V_{NN}(A)$ in order to achieve such accurate reproduction.

- Dispersion relations provide a sound framework where NN rescattering can be studied in a well-defined way and independent of regulator.

- Born approximation is much more dependent on the c_i's than the full results.

- Correlation between scattering length and effective range in S-waves $\sim 10\%$

- Chiral power counting for the subtraction constants.

- $\Lambda_{NN} \sim 0.4$–0.5 GeV.

- Can one connect the δ_i with the chiral Lagrangians?