The σ-resonance: dispersive treatment and its role in hadronic light-by-light scattering

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arXiv:1402.7081, arXiv:1309.6877, and work in progress

b UNIVERSITÄT BERN

AEC ALBERT EINSTEIN CENTER FOR FUNDAMENTAL PHYSICS

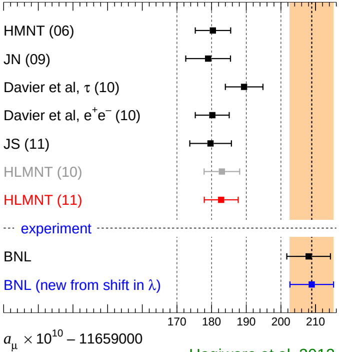
Benasque, July 25, 2014

Anomalous magnetic moment of the muon

- **Experimental precision** 0.5 ppm BNL E821 2006

$$a_\mu^{\text{exp}} = (116\,592\,089 \pm 63) \cdot 10^{-11}$$

- **Theory error** of **similar size**
- Deviation from SM prediction around 3σ
- New experiment at **FNAL** (E989) aiming at 0.14 ppm, beam in 2016/2017
- **J-PARC** aiming at 0.1 ppm, new approach with ultra-cold muons, R&D in progress



Hagiwara et al. 2012

$\Rightarrow$ **Need to improve theory by a factor of 4**

Overview of SM prediction

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
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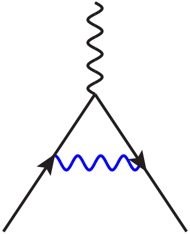

Schwinger 1948

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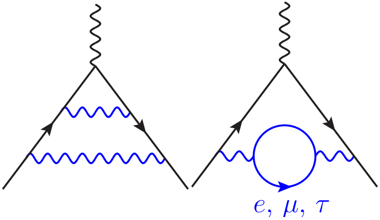

Sommerfield, Petermann 1957

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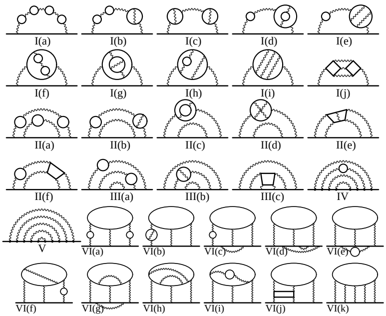

Kinoshita et al. 2012

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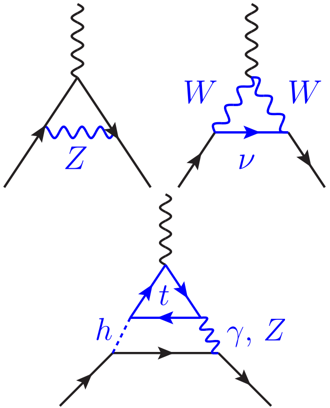

1-loop: Jackiw, Weinberg and others 1972

2-loop: Kukhto et al. 1992, Czarnecki, Krause, Marciano 1995, Degrassi, Giudice 1998, Knecht, Peris, Perrottet, de Rafael 2002, Vainshtein 2003, Heinemeyer, Stöckinger, Weiglein 2004, Gribouk, Czarnecki 2005

Update after Higgs discovery: Gnendiger et al. 2013

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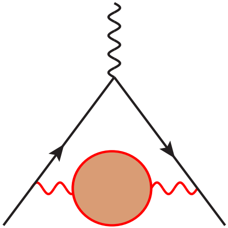

Hagiwara et al. 2011

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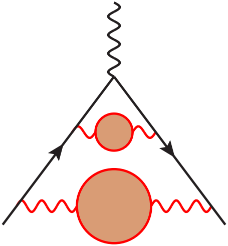

Calmet et al. 1976, Hagiwara et al. 2011

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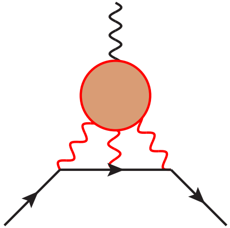

Hayakawa, Kinoshita, Sanda 1995

Bijnens, Pallante, Prades 1995

Knecht, Nyffeler 2001

Jegerlehner, Nyffeler 2009

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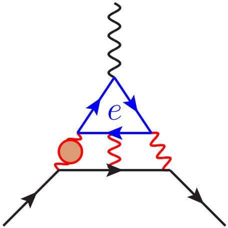

Kurz, Liu, Marquard, Steinhauser 2014

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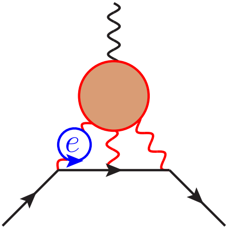

Colangelo, MH, Nyffeler, Passera, Stoffer 2014

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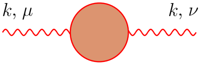
$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (234 \pm 86)\cdot 10^{-11} [2.7\sigma]$$

⇒**Theory error** comes almost exclusively from **hadronic part**

Hadronic vacuum polarization

- General principles yield **direct connection with experiment**
  - **Gauge invariance**



$$= -i\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\Pi(k^2)$$

  - **Analyticity**

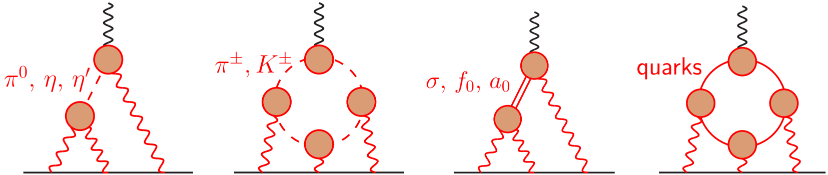
$$\Pi_{\text{ren}} = \Pi(k^2) - \Pi(0) = \frac{k^2}{\pi}\int_{4M_\pi^2}^{\infty} ds \frac{\operatorname{Im}\Pi(s)}{s(s-k^2)}$$

  - **Unitarity**

$$\operatorname{Im}\Pi(s) = \frac{s}{4\pi\alpha}\sigma_{\text{tot}}\left(e^+e^- \to \text{hadrons}\right) = \frac{\alpha}{3}R(s)$$

- 1 Lorentz structure, 1 kinematic variable, parameter-free
- **Dedicated e^+e^- program** under way: BaBar, Belle, BESIII, CMD3, KLOE2, SND

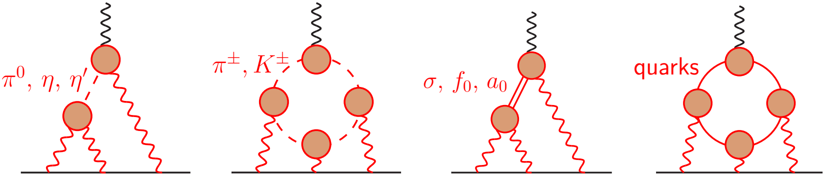
Anatomy of HLbL scattering



- Huge **model dependence**

  $\hookrightarrow$ can one find a **data-driven** approach also for **HLbL**?

Anatomy of HLbL scattering



- Huge **model dependence**

  $\hookrightarrow$ can one find a **data-driven** approach also for **HLbL**?

- **Dispersive point of view**
  - Analytic structure: poles and cuts

    $\hookrightarrow$ **residues** and **imaginary parts** $\Rightarrow$ by definition **on-shell** quantities

    $\hookrightarrow$ **form factors** and **scattering amplitudes** from experiment
  - Out of the above only **pion pole** model independent
  - Expansion: mass of intermediate states, partial waves

- This talk: focus on **σ-resonance**

  $\hookrightarrow$ pole approximation certainly not meaningful

Outline

1. The σ-resonance
   - Dispersion relations
   - Pole position
   - Two-photon coupling
   - Quark-mass dependence

2. Hadronic light-by-light scattering
   - Dispersive approach: one- and two-pion intermediate states
   - FsQED pion loop
   - $\pi\pi$ rescattering: the σ-contribution

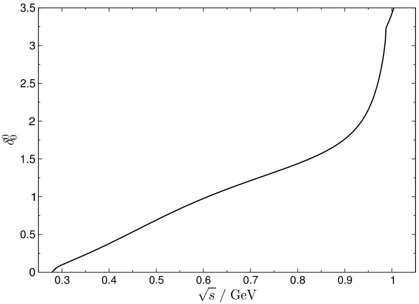
The σ-resonance

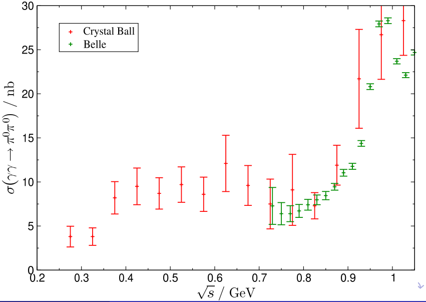
- σ seen as a **broad bump** in $\pi\pi$ or $\gamma\gamma\to\pi\pi$
- Vacuum quantum numbers $J^{PC}=0^{++}$
- Large width, not at all Breit–Wigner shape
- Pole deep in the complex plane

$f_0(600)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\ \mathrm{Im}(\sqrt{s_{\text{pole}}})$.

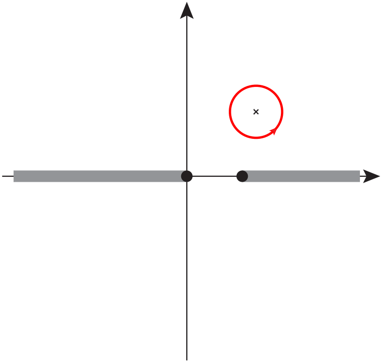
VALUE (MeV)	DOCUMENT ID	TECN	COMMENT
$(400\text{–}1200)-i(250\text{–}500)$ OUR ESTIMATE			
• • • We do not use the following data for averages, fits, limits, etc. • • •			
$(455\pm6{}^{+31}_{-13})-i(556\pm12{}^{+68}_{-86})$	[1] CAPRINI 08	RVUE	Compilation
$(463\pm6{}^{+31}_{-17})-i(518\pm12{}^{+66}_{-68})$	[2] CAPRINI 08	RVUE	Compilation
$(552{}^{+84}_{-106})-i(232{}^{+81}_{-72})$	[3] ABLIKIM 07A	BES2	$\psi(2S)\to\pi^+\pi^- J/\psi$
$(466\pm18)-i(223\pm28)$	[4] BONVICINI 07	CLEO	$D^+\to\pi^-\pi^+\pi^+$
$(472\pm30)-i(271\pm30)$	[5] BUGG 07A	RVUE	Compilation
$(484\pm17)-i(255\pm10)$	GARCIA-MAR..07	RVUE	$Ke4$
$(441{}^{+16}_{-8})-i(272{}^{+9}_{-12.5})$	[6] CAPRINI 06	RVUE	$\pi\pi\to\pi\pi$
$(470\pm50)-i(285\pm25)$	[7] ZHOU 05	RVUE	
$(541\pm39)-i(252\pm42)$	[8] ABLIKIM 04A	BES2	$J/\psi\to\omega\pi^+\pi^-$
$(528\pm32)-i(207\pm23)$	[9] GALLEGOS 04	RVUE	Compilation
$(440\pm8)-i(212\pm15)$	[10] PELAEZ 04A	RVUE	$\pi\pi\to\pi\pi$
$(533\pm25)-i(247\pm25)$	[11] BUGG 03	RVUE	
$532-i272$	BLACK 01	RVUE	$\pi^0\pi^0\to\pi^0\pi^0$
$(470\pm30)-i(295\pm20)$	[6] COLANGELO 01	RVUE	$\pi\pi\to\pi\pi$
$(535{}^{+48}_{-36})-i(155{}^{+76}_{-53})$	[12] ISHIDA 01		$\Upsilon(3S)\to\Upsilon\pi\pi$





- **Cauchy's theorem**

$$f(s) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{ds'\, f(s')}{s' - s}$$



From Cauchy's theorem to dispersion relations

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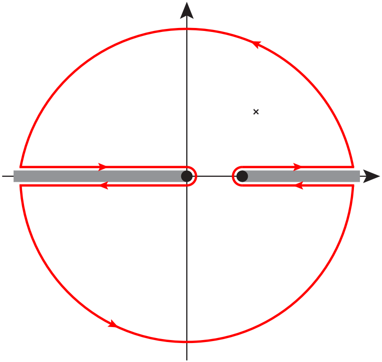

From Cauchy's theorem to dispersion relations

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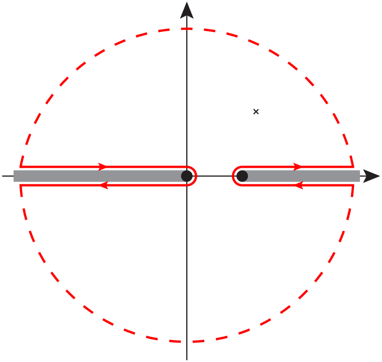
$$f(s) = \frac{1}{\pi}\int_{\text{cuts}} \frac{ds'\,\text{Im}\,f(s')}{s'-s}$$

$\hookrightarrow$ **analyticity**



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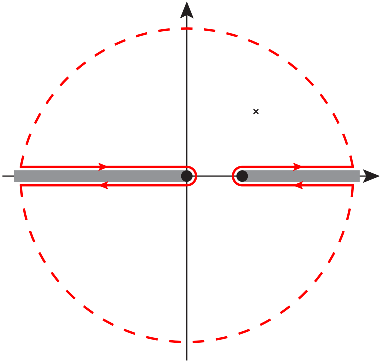
$$f(s) = f(0) + \frac{s}{\pi}\int_{\text{cuts}} \frac{ds'\,\text{Im}\,f(s')}{s'(s'-s)}$$



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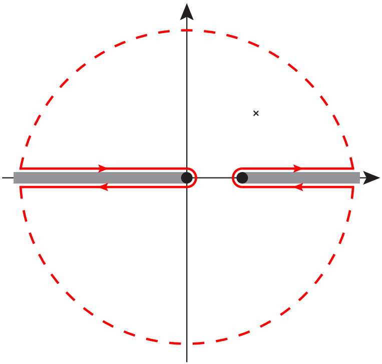

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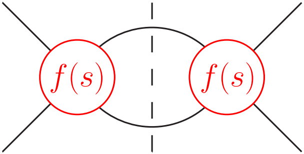
- Imaginary part from **Cutkosky rules**

$\hookrightarrow$ forward direction: **optical theorem**

- **Unitarity** for partial waves

$$\text{Im}\,f(s) = \rho(s)|f(s)|^2$$



$\pi\pi$ Roy equations

Roy equations = Dispersion relations + partial-wave expansion

+ crossing symmetry + unitarity

- **Coupled system of integral equations** for partial waves $t_J^I(s)$ Roy 1971

$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2} \sum_{J'=0}^{\infty} \int_{4M_\pi^2}^{\infty} \mathrm{d}s' K_{JJ'}^{II'}(s,s') \operatorname{Im} t_{J'}^{I'}(s')$$

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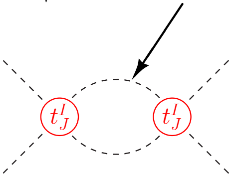
$$\underbrace{t_J^I(s)}_{\frac{e^{2i\delta_J^I(s)}-1}{2i\sigma(s)}} = k_J^I(s) + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int\limits_{4M_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s,s')\, \underbrace{\text{Im}\, t_{J'}^{I'}(s')}_{\frac{1}{\sigma(s)}\sin^2\delta_{J'}^{I'}(s')}$$

$$\sigma(s) = \sqrt{1-\frac{4M_\pi^2}{s}}$$



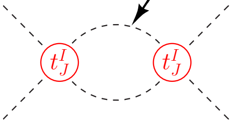
Roy equations $=$ **Dispersion relations** $+$ **partial-wave expansion**

$+$ **crossing symmetry** $+$ **unitarity**

- **Coupled system of integral equations** for partial waves $t^I_J(s)$ Roy 1971

$$\underbrace{t^I_J(s)}_{\frac{e^{2i\delta^I_J(s)}-1}{2i\sigma(s)}} = \underbrace{k^I_J(s)}_{\delta^I_0 a^I_0+\cdots} + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int_{4M_\pi^2}^{\infty} ds' K^{II'}_{JJ'}(s,s') \underbrace{\text{Im}\, t^{I'}_{J'}(s')}_{\frac{1}{\sigma(s)}\sin^2\delta^{I'}_{J'}(s')}$$

$$\sigma(s) = \sqrt{1-\frac{4M_\pi^2}{s}}$$



free parameters a^0_0, a^2_0

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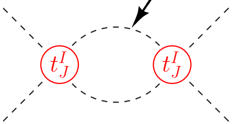
+ crossing symmetry + unitarity

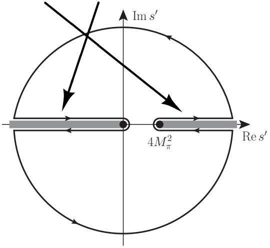
- **Coupled system of integral equations** for partial waves $t_J^I(s)$ Roy 1971

$$\underbrace{t_J^I(s)}_{\frac{e^{2i\delta_J^I(s)}-1}{2i\sigma(s)}} = \underbrace{k_J^I(s)}_{\delta_0^I a_0^I+\cdots} + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int\limits_{4M_\pi^2}^{\infty} ds' \quad \underbrace{K_{JJ'}^{II'}(s,s')}_{\frac{1}{\pi}\frac{\delta_{JJ'}\delta_{II'}}{s'-s-i\varepsilon}+\bar{K}_{JJ'}^{II'}(s,s')} \quad \underbrace{\text{Im}\, t_{J'}^{I'}(s')}_{\frac{1}{\sigma(s)}\sin^2\delta_{J'}^{I'}(s')}$$

$$\sigma(s) = \sqrt{1-\frac{4M_\pi^2}{s}}$$



free parameters a_0^0, a_0^2



$\hookrightarrow$ **Self-consistency condition** for phase shifts

Roy equations and ChPT

- Roy equations: $\pi\pi$ phase shifts in terms of a_0^0, a_0^2 Ananthanarayan et al. 2001
- Matching of two-loop ChPT and Roy equations Colangelo, Gasser, Leutwyler 2001
  - Match low-energy polynomials $\Rightarrow \bar{l}_1, \bar{l}_2$ as by-product
  - Scattering lengths in terms of quark-mass LECs $\bar{l}_3, \bar{l}_4$

$$a_0^0 = 0.198 \pm 0.001 + 0.0443\,\text{fm}^{-2}\langle r^2\rangle_\pi^S - 0.0017\bar{l}_3 = 0.220 \pm 0.005$$

$$a_0^2 = -0.0392 \pm 0.0003 - 0.0066\,\text{fm}^{-2}\langle r^2\rangle_\pi^S - 0.0004\bar{l}_3 = -0.0444 \pm 0.0010$$

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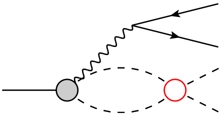
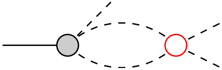
- Prediction tested in K_{e4} and $K \to 3\pi$ decays NA48/2 2010

$$a_0^0 = 0.2210 \pm 0.0047_{\text{stat}} \pm 0.0040_{\text{syst}}$$

$$a_0^2 = -0.0429 \pm 0.0044_{\text{stat}} \pm 0.0028_{\text{syst}}$$





The σ-resonance: pole position

- **Unitarity**

$$S^0_{0,\text{I}}(s+i\varepsilon)S^0_{0,\text{I}}(s+i\varepsilon)^* = 1$$

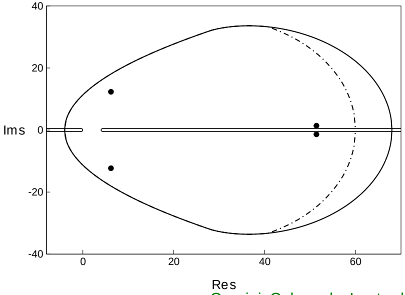
$$S^0_{0,\text{I}}(s+i\varepsilon) = 1/S^0_{0,\text{II}}(s+i\varepsilon)$$

$\hookrightarrow$ **pole** on the second sheet

$=$ **zero** on the first sheet

- Need a representation of $t^0_0(s)$ valid in the complex plane $\Rightarrow$ **Roy equations**

$\hookrightarrow$ established from axiomatic field theory



Caprini, Colangelo, Leutwyler 2006

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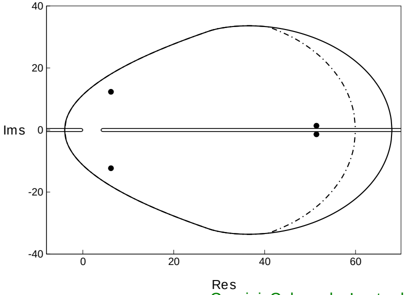
- Results for $s_\sigma = (M_\sigma - i\Gamma_\sigma/2)^2$

	M_σ [MeV]	Γ_σ [MeV]
Caprini et al. 2006	441^{+16}_{-8}	544^{+18}_{-25}
García-Martín et al. 2011	457^{+14}_{-13}	558^{+22}_{-14}
Moussallam 2011	442^{+5}_{-8}	548^{+12}_{-10}



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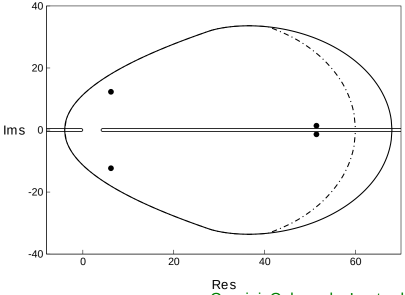


Caprini, Colangelo, Leutwyler 2006

$f_0(500)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\,\mathrm{Im}(\sqrt{s_{\mathrm{pole}}})$.

VALUE (MeV)	DOCUMENT ID	TECN	COMMENT
(400–550)−i(200–350) OUR ESTIMATE			
• • • We do not use the following data for averages, fits, limits, etc. • • •			
$(445 \pm 25) - i(278^{+22}_{-18})$	[1,2] GARCIA-MAR..11	RVUE	Compilation
$(457^{+14}_{-13}) - i(279^{+11}_{-7})$	[1,3] GARCIA-MAR..11	RVUE	Compilation
$(442^{+5}_{-8}) - i(274^{+6}_{-5})$	[4] MOUSSALLAM 11	RVUE	Compilation
$(452 \pm 13) - i(259 \pm 16)$	[5] MENNESSIER 10	RVUE	Compilation
$(448 \pm 43) - i(266 \pm 43)$	[6] MENNESSIER 10	RVUE	Compilation
$(455 \pm 6^{+31}_{-13}) - i(278 \pm 6^{+34}_{-43})$	[7] CAPRINI 08	RVUE	Compilation
$(463 \pm 6^{+31}_{-17}) - i(259 \pm 6^{+33}_{-34})$	[8] CAPRINI 08	RVUE	Compilation
$(552^{+84}_{-106}) - i(232^{+81}_{-72})$	[9] ABLIKIM 07A	BES2	$\psi(2S) \to \pi^+\pi^-$
$(466 \pm 18) - i(223 \pm 28)$	[10] BONVICINI 07	CLEO	$D^+ \to \pi^-\pi^+\pi$
$(472 \pm 30) - i(271 \pm 30)$	[11] BUGG 07A	RVUE	Compilation

The σ-resonance: couplings

- Coupling to $\pi\pi$ from derivative $\dot{t}^0_{0,\text{I}}(s)$ at s_σ

$$32\pi t^0_{0,\text{II}}(s) = \frac{g^2_{\sigma\pi\pi}}{s_\sigma - s}$$

- Coupling to $\gamma\gamma$ from $\gamma\gamma \to \pi\pi$

$$h^0_{0,++,\text{II}}(s) = \frac{g_{\sigma\pi\pi} g_{\sigma\gamma\gamma}}{s_\sigma - s} = \big(1 - 2i\sigma(s) t^0_{0,\text{II}}(s)\big) h^0_{0,++,\text{I}}(s) \qquad \sigma(s) = \sqrt{1 - \frac{4M_\pi^2}{s}}$$

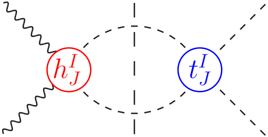
- “Two-photon width” $\Gamma_{\sigma\gamma\gamma}$

$$\frac{g^2_{\sigma\gamma\gamma}}{g^2_{\sigma\pi\pi}} = -\left(\frac{\sigma(s_\sigma)}{16\pi}\right)^2 \big(h^0_{0,++,\text{I}}(s_\sigma)\big)^2 \qquad \Gamma_{\sigma\gamma\gamma} = \frac{\pi\alpha^2 |g_{\sigma\gamma\gamma}|^2}{M_\sigma}$$

- Determine $h^0_{0,++,\text{I}}(s_\sigma)$ from a dispersive representation of $\gamma\gamma \to \pi\pi$

  Pennington 2006, Pennington et al. 2008, Oller, Roca, Schat 2008, Mao et al. 2009, MH, Phillips, Schat 2011, Moussallam 2011, Dai, Pennington 2014

  $\hookrightarrow$ differences in treatment of LHC

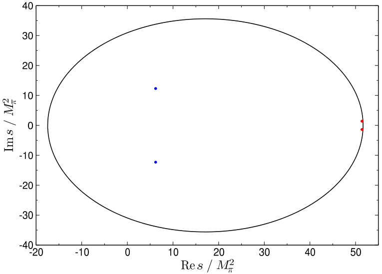


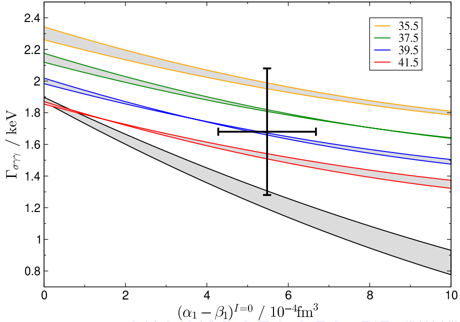
The σ-resonance: two-photon coupling

- **Roy–Steiner equations** MH, Phillips, Schat 2011
- Suppress LHC integral with subtractions

  $\hookrightarrow$ **pion polarizabilities**
- With ChPT for pion polarizabilities
  Gasser, Ivanov, Sainio 2005, 2006

$$\Gamma_{\sigma\gamma\gamma} = (1.7 \pm 0.4)\text{keV}$$





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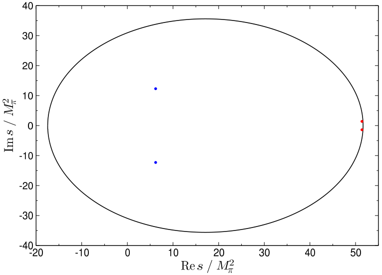

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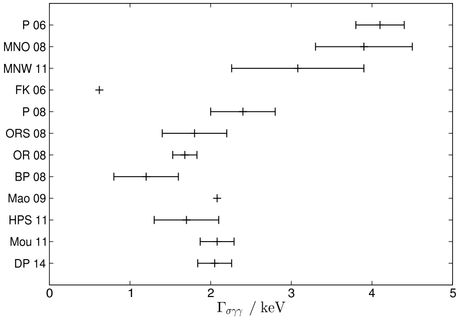

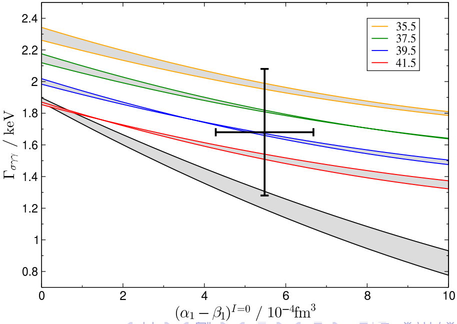

The σ-resonance: quark-mass dependence

- Cannot get **quark-mass dependence** in dispersion theory $\Rightarrow$ ChPT
- **Inverse amplitude method** Truong, Dobado, Herrero, Peláez, Guerrero, Oller, Gómez Nicola, Oset, ...

$$\mathrm{Im}\, t(s) = \sigma(s)|t(s)|^2 \quad \Rightarrow \quad \mathrm{Im}\, t(s)^{-1} = -\sigma(s) \quad \Rightarrow \quad t(s) = \frac{1}{\mathrm{Re}\, t(s)^{-1} - i\sigma(s)}$$

- Single-channel IAM justified from DRs, approximation: ChPT for LHC

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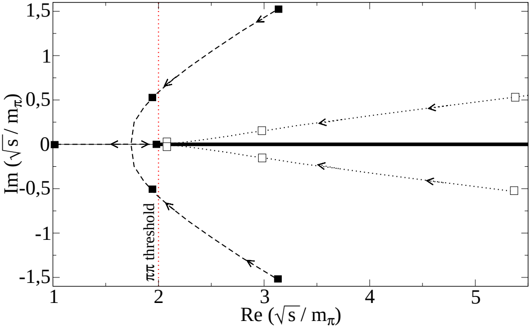
- Trajectory of σ- and ρ-pole Hanhart, Peláez, Ríos 2008, Peláez, Ríos 2010



dashed: σ

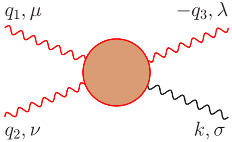
dotted: ρ

squares: $M_\pi = 1/2/3\, M_\pi^{\text{phys}}$

- **HLbL tensor**

  $\gamma^*(q_1,\mu)\gamma^*(q_2,\nu) \to \gamma^*(-q_3,\lambda)\gamma(k,\sigma)$

  $i\Pi^{\mu\nu\lambda\sigma} =$



  - **Gauge invariance**: 29 independent gauge-invariant structures cf. Bijnens et al. 1995

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{29} A_i^{\mu\nu\lambda\sigma} \Pi_i$$

  $\hookrightarrow$ in practice use 45 (redundant) structures

  - 5 kinematic variables: $s = (q_1+q_2)^2$, $t = (q_1+q_3)^2$, q_1^2, q_2^2, q_3^2

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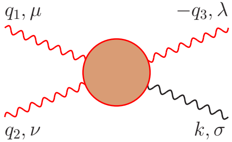

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  - 5 kinematic variables: $s=(q_1+q_2)^2$, $t=(q_1+q_3)^2$, q_1^2, q_2^2, q_3^2

- Decompose the tensor according to

$$\Pi_{\mu\nu\lambda\sigma}=\Pi_{\mu\nu\lambda\sigma}^{\pi^0\text{-pole}}+\Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}}+\bar{\Pi}_{\mu\nu\lambda\sigma}+\cdots$$

  $\hookrightarrow$ accounts for **one-** and **two-pion** intermediate states

- Generalizes immediately to η, η', $K\bar{K}$, but e.g. 3π more difficult

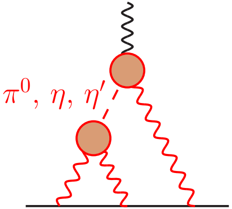
Pion pole: pion transition form factor

Knecht, Nyffeler 2001

Master formula for pion-pole contribution

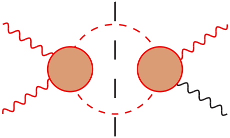
$$a_\mu^{\pi^0\text{-pole}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s\big((p+q_1)^2-m^2\big)\big((p-q_2)^2-m^2\big)}$$
$$\times \left\{ \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi^0\gamma^*\gamma^*}(s,0)}{s-M_\pi^2} T_1(q_1,q_2;p) + \frac{F_{\pi^0\gamma^*\gamma^*}(s,q_1^2)F_{\pi^0\gamma^*\gamma^*}(q_2^2,0)}{q_2^2-M_\pi^2} T_2(q_1,q_2;p) \right\}$$

- Crucial ingredient: **pion transition form factor** $F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)$
- **Wick rotation**: only **space-like** s, q_1^2, q_2^2 contribute
- **Dispersive approach**
  - **On-shell** form factor
  - Fix parameters wherever data are available
  - Use **analyticity** to go to the space-like region



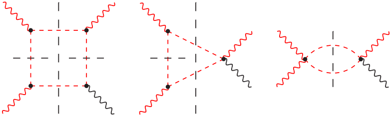
$\pi\pi$ intermediate states

- σ-contribution corresponds to **$I = 0$ $\pi\pi$ rescattering**



- First step: separate terms with **simultaneous cuts**

$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)F^V_\pi(q_2^2)F^V_\pi(q_3^2) \times \Bigg[\Bigg]$$



- Multiplication of sQED diagrams with F^V_π gives correct q^2-dependence

  $\hookrightarrow$ **not an approximation**

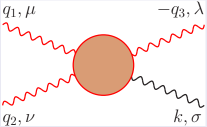
- Remaining $\pi\pi$ contribution included in $\bar{\Pi}_{\mu\nu\lambda\sigma}$ has cuts only in one channel

  $\hookrightarrow$ partial-wave expansion, dispersion relations for this part

Colangelo, MH, Procura, Stoffer arXiv:1402.7081

Master formula for $\pi\pi$ intermediate states

$$a_\mu^{\pi\pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_i T_i(q_1,q_2;p) I_i(s,q_1^2,q_2^2)}{q_1^2 q_2^2 s\big((p+q_1)^2-m^2\big)\big((p-q_2)^2-m^2\big)}$$

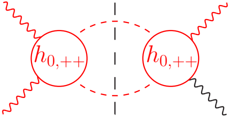


- **Partial waves** up to $L=2$, manifest **crossing symmetry**
- $I_i(s,q_1^2,q_2^2)$: dispersive integrals over $\gamma^*\gamma^* \to \pi\pi$ **helicity partial waves**, e.g.

$$I_1(s,q_1^2,q_2^2) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'-s}\bigg[\bigg(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda(s',q_1^2,q_2^2)}\bigg)\text{Im}\, h^0_{++,++}(s';q_1^2,q_2^2;s,0)$$

$$+\frac{2\xi_1\xi_2}{\lambda(s',q_1^2,q_2^2)}\text{Im}\, h^0_{00,++}(s';q_1^2,q_2^2;s,0)\bigg]$$

$$\text{Im}\, h^0_{++,++}(s';q_1^2,q_2^2;s,0) = \frac{\sigma(s')}{16\pi} h_{0,++}(s';q_1^2,q_2^2) h_{0,++}(s';s,0)$$



$$\lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz+yz)$$

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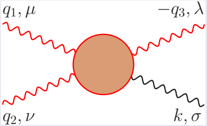
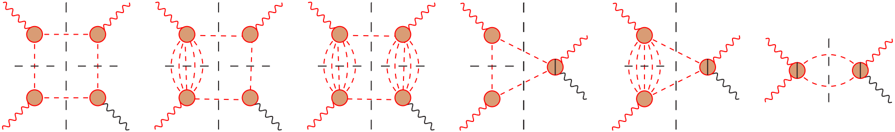
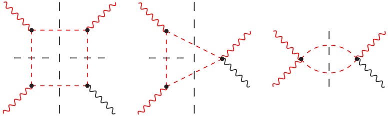


- **Partial waves** up to $L = 2$, manifest **crossing symmetry**
- What is included? How?



$\hookrightarrow$ sorted by analytic structure in the crossed channel

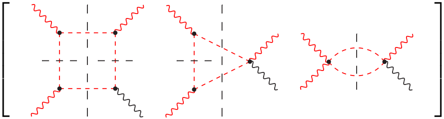
$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2) F^V_\pi(q_2^2) F^V_\pi(q_3^2) \times \Bigg[$$  $$\Bigg]$$

- Input for $F^V_\pi(q^2)$: Omnès factor $F^V_\pi(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}$
- Results for a_μ^{FsQED} in units of 10^{-11}

phase shift δ_1^1	loop integrals	disp 1	disp 2
CCL	-13.77 ± 0.01	-15.87 ± 0.01	-14.57 ± 0.01
CCL + ρ', ρ''	-14.65 ± 0.01	-16.90 ± 0.02	-15.53 ± 0.01

Warm-up: numbers for FsQED

$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)F^V_\pi(q_2^2)F^V_\pi(q_3^2) \times$



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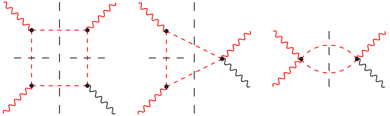
- Dependence on $F^V_\pi(s)$: analytic continuation can be stabilized using space-like data

Warm-up: numbers for FsQED

$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)F^V_\pi(q_2^2)F^V_\pi(q_3^2) \times \Bigg[$  $\Bigg]$

- Input for $F^V_\pi(q^2)$: Omnès factor $F^V_\pi(s) = \exp\left\{\frac{s}{\pi}\int\limits_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}$
- Results for a_μ^{FsQED} in units of 10^{-11}

phase shift δ_1^1	loop integrals	disp 1	disp 2
CCL	-13.77 ± 0.01	-15.87 ± 0.01	-14.57 ± 0.01
CCL + ρ',ρ''	-14.65 ± 0.01	-16.90 ± 0.02	-15.53 ± 0.01

- Dependence on $F^V_\pi(s)$: analytic continuation can be stabilized using space-like data
- Basis $A_i^{\mu\nu\lambda\sigma}$ not unique, Π_i need to be free of **kinematic singularities**

  $\hookrightarrow$ disp 1/2 equivalent for suitable high-energy behavior $\Rightarrow$ theoretical uncertainty
- **Why does this work so well?** it shouldn't: double-spectral regions, only S-waves!

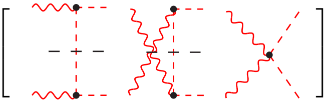
Simplified input for $\gamma^*\gamma^* \to \pi\pi$

Omnès representation for S-wave

$$h^I_{0,++}(s) = N^I_{0,++}(s) + \frac{\Omega^I_0(s)}{\pi}\int\limits_{4M_\pi^2}^{s_\text{m}} ds' \frac{\sin\delta^I_0(s')}{|\Omega^I_0(s')|}\bigg[\bigg(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda(s',q_1^2,q_2^2)}\bigg)N^I_{0,++}(s') + \frac{2\xi_1\xi_2}{\lambda(s',q_1^2,q_2^2)}N^I_{0,00}(s')\bigg]$$

- Starting point: **Roy–Steiner** equations for $\gamma^*\gamma^* \to \pi\pi$
- Omnès factors $\Omega^I_0(s) = \exp\bigg\{\frac{s}{\pi}\int\limits_{4M_\pi^2}^{s_\text{m}} ds' \frac{\delta^I_0(s')}{s'(s'-s)}\bigg\}$
- LHC approximated by **pion pole** $N^I_{0,\lambda_1\lambda_2}$ only

$$F^V_\pi(q_1^2)F^V_\pi(q_2^2) \times$$



- **Finite matching point**: $h^I_{0,++}(s) = 0$ above s_m
- Take $\sqrt{s_\text{m}} = 0.98\,\text{GeV}$, sanity check: $\Gamma_{\sigma\gamma\gamma} = 1.66\,\text{keV}$ ✓

  $\hookrightarrow$ no $f_0(980)$ or coupling to $K\bar{K}$ $\Rightarrow$ "σ-contribution"

$\pi\pi$ rescattering: some preliminary numbers for S-waves

- $a_\mu^{\pi\pi}$ in units of 10^{-11}

phase shift δ_1^1	$I = 0$ disp 1	$I = 0$ disp 2	$I = 2$ disp 1	$I = 2$ disp 2
CCL	-7.13 ± 0.03	-6.75 ± 0.06	1.82 ± 0.01	1.68 ± 0.01
CCL + ρ', ρ''	-7.79 ± 0.03	-7.38 ± 0.06	2.00 ± 0.01	1.84 ± 0.01

$\pi\pi$ rescattering: some preliminary numbers for S-waves

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- Adding the FsQED contribution

phase shift δ_1^1	FsQED	sum disp 1	sum disp 2
CCL	-13.77 ± 0.01	-19.08 ± 0.03	-18.84 ± 0.06
CCL + ρ',ρ''	-14.65 ± 0.01	-20.44 ± 0.03	-20.19 ± 0.06

$\pi\pi$ rescattering: some preliminary numbers for S-waves

- $a_\mu^{\pi\pi}$ in units of 10^{-11}

phase shift δ_1^1	$I=0$ disp 1	$I=0$ disp 2	$I=2$ disp 1	$I=2$ disp 2
CCL	-7.13 ± 0.03	-6.75 ± 0.06	1.82 ± 0.01	1.68 ± 0.01
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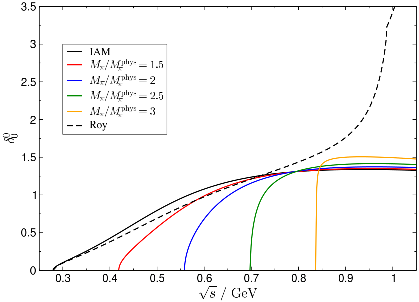
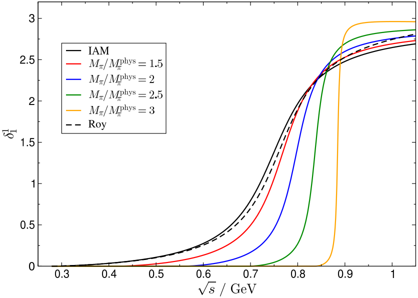
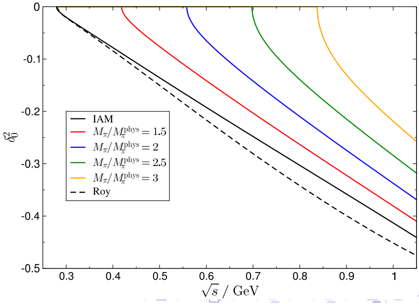
- Comparing to the literature Jegerlehner, Nyffeler 2009

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	–	–	–	$2.3\pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

- **1-loop IAM** with low-energy constants from Hanhart, Peláez, Ríos 2008
- Quark-mass dependence of the **phase shifts**



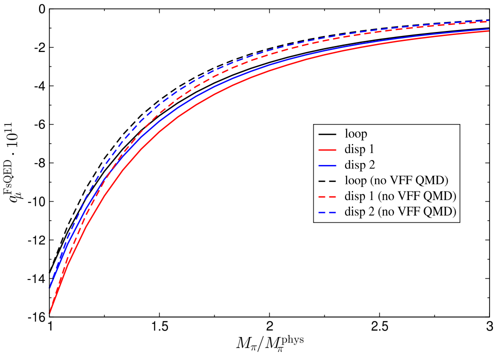




Quark-mass dependence: FsQED

- Check accuracy of IAM

phase shift δ_1^1	loop integrals	disp 1	disp 2
CCL	-13.77 ± 0.01	-15.87 ± 0.01	-14.57 ± 0.01
CCL + ρ', ρ''	-14.65 ± 0.01	-16.90 ± 0.02	-15.53 ± 0.01
IAM	-13.72 ± 0.01	-15.82 ± 0.01	-14.51 ± 0.01



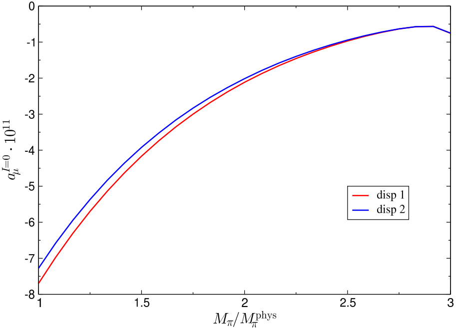
- Quark-mass dependence of $F_\pi^V(s)$ via phase shift in Omnès representation
  Guo, Hanhart, Llanes-Estrada, Meißner 2009
  $\hookrightarrow$ relevant for large M_π
- Quark-mass dependence of $a_\mu^{\rm FsQED}$ roughly $\propto M_\pi^{-2}$

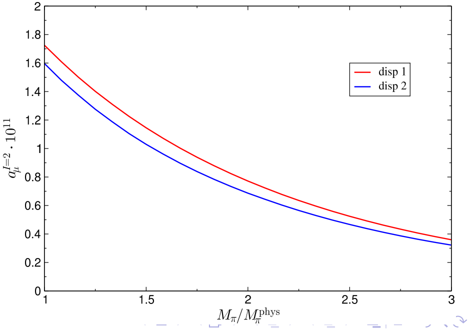
Quark-mass dependence: S-waves

	disp 1	disp 1 + IAM	disp 2	disp 2 + IAM
$I=0$	-7.13 ± 0.03	-7.70 ± 0.04	-6.75 ± 0.06	-7.28 ± 0.06
$I=2$	1.82 ± 0.01	1.73 ± 0.01	1.68 ± 0.01	1.60 ± 0.01

- Choose $\sqrt{s_\text{m}}=2M_K=2\sqrt{\big(M_K^\text{phys}\big)^2+\big(M_\pi^2-\big(M_\pi^\text{phys}\big)^2\big)/2}$, compare to CCL δ_1^1
- Beyond $3M_\pi$: σ and ρ become **bound states**

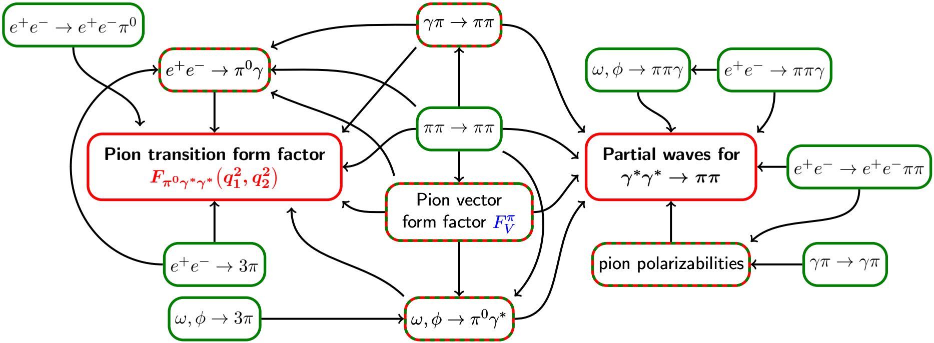
$\hookrightarrow$ highly non-trivial quark-mass dependence





Summary

- Dispersive treatment of **broad resonances**
  - Reliable continuation into the complex plane $\Rightarrow$ pole position, couplings
  - Quark-mass dependence: combination with EFT
- **Hadronic light-by-light scattering**
  - Goal: data-driven analysis similar to HVP
  - Preliminary numbers for $\pi\pi$, including **σ-physics** for $I = 0$
  - Quark-mass dependence
- Next steps
  - Refinement of $\gamma^*\gamma^* \to \pi\pi$ input
  - Comprehensive treatment of D-waves
  - Error analysis: which input quantity has the biggest impact on a_μ?



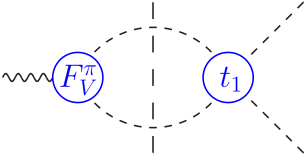
- Major omission: pion transition form factor
- Reconstruction of $\gamma^*\gamma^* \to \pi\pi, \pi^0$: combine experiment and theory constraints

  $\hookrightarrow$ simplified version for $\pi\pi$ in this talk
- Beyond: η, η', $K\bar{K}$, multi-pion channels (resonances), pQCD constraints, ...

Pion vector form factor

- **Dispersive approach**: resum $\pi\pi$ rescattering F_V^π as example
- **Unitarity** for **pion vector form factor**

$$\operatorname{Im} F_V^\pi(s) = \theta\big(s - 4M_\pi^2\big) F_V^\pi(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

Pion vector form factor

- **Dispersive approach**: resum $\pi\pi$ rescattering F_V^π as example
- **Unitarity** for **pion vector form factor**

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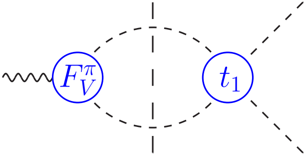


$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

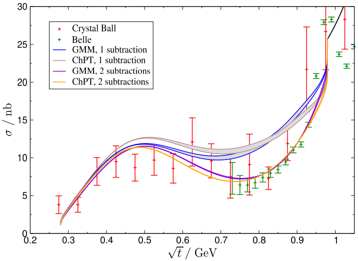
- Solution in terms of **Omnès function** Omnès 1958

$$F_V^\pi(s) = P(s)\Omega_1(s) \qquad \Omega_1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\frac{\delta_1(s')}{s'(s'-s)}\right\}$$

- Asymptotics + normalization $\Rightarrow P(s) = 1$

Roy(–Steiner) equations $=$ Dispersion relations $+$ partial-wave expansion $+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- **On-shell case** $\gamma\gamma \to \pi\pi$ Moussallam 2010, MH, Phillips, Schat 2011 $\hookrightarrow$ precision determination of $\sigma \to \gamma\gamma$ coupling
- **Singly-virtual** $\gamma^*\gamma \to \pi\pi$ Moussallam 2013
- **Doubly-virtual** $\gamma^*\gamma^* \to \pi\pi$: **anomalous thresholds**
  Colangelo, MH, Procura, Stoffer arXiv:1309.6877



$\gamma^*\gamma^* \to \pi\pi$ partial waves

Roy(–Steiner) equations $=$ Dispersion relations $+$ partial-wave expansion $+$ crossing symmetry $+$ unitarity $+$ gauge invariance

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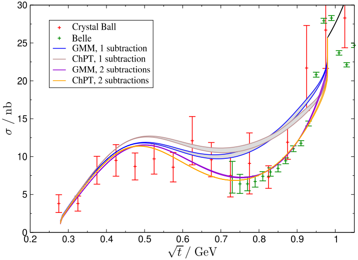
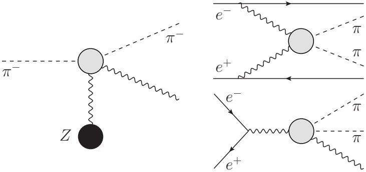
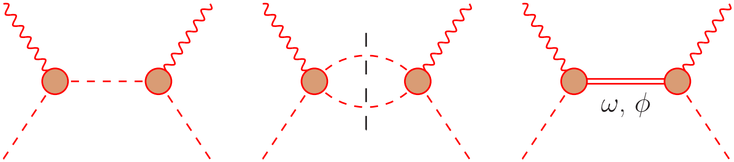
- **Doubly-virtual** $\gamma^*\gamma^* \to \pi\pi$: **anomalous thresholds** Colangelo, MH, Procura, Stoffer arXiv:1309.6877
- Constraints
  - **Low energies**: pion polarizabilities, ChPT
  - **Primakoff**: $\gamma\pi \to \gamma\pi$ (COMPASS), $\gamma\gamma \to \pi\pi$ (JLab)
  - **Scattering**: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - **(Transition) Form factors**: F_V^π, $\omega, \phi \to \pi^0\gamma^*$

$\hookrightarrow$ discuss these constraints in the following





Left-hand cut



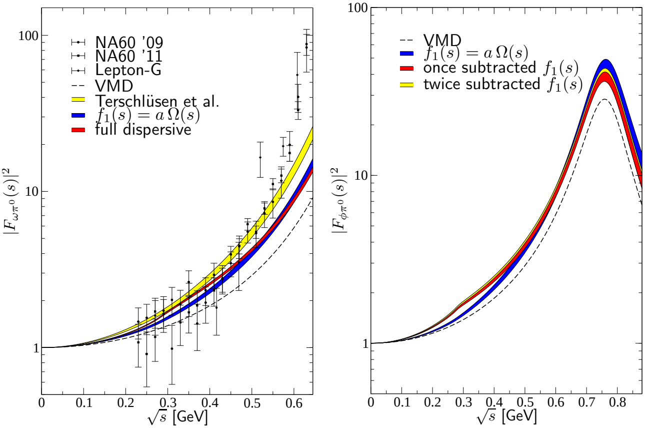
- **Pion pole**: coupling determined by F_V^π as before
- **Multi-pion intermediate states**: approximate in terms of **resonances**
  - $2\pi \sim \rho$: can even be done **exactly** using $\gamma^* \to 3\pi$ amplitude

    $\hookrightarrow$ see pion transition form factor
  - $3\pi \sim \omega, \phi$: narrow-width approximation

    $\hookrightarrow$ **transition form factors** for $\omega, \phi \to \pi^0\gamma^*$
  - Higher intermediate states also potentially relevant: **axials**, **tensors**

    $\hookrightarrow$ **sum rules** to constrain their transition form factors Pauk, Vanderhaeghen 2014

$\omega, \phi \to \pi^0\gamma^*$ transition form factor



Schneider, Kubis, Niecknig 2012

- Puzzle of steep rise in $F_{\omega\pi^0}$

  $\hookrightarrow$ measurement of $F_{\phi\pi^0}$ would be extremely valuable

- Clarification important for pion transition form factor, but also $\gamma^*\gamma^* \to \pi\pi$

Subtraction functions

Omnès representation for S-wave

$$h_{0,++}(s) = \Delta_{0,++}(s) + \Omega_0(s)\bigg[\frac{1}{2}(s-s_+)a_+(q_1^2,q_2^2) + \frac{1}{2}(s-s_-)a_-(q_1^2,q_2^2) + q_1^2q_2^2 b(q_1^2,q_2^2)$$

$$+\frac{s(s-s_+)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_+)(s'-s)|\Omega_0(s')|} + \frac{s(s-s_-)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_-)(s'-s)|\Omega_0(s')|}$$

$$+\frac{2q_1^2q_2^2 s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,00}(s')}{s'(s'-s_+)(s'-s_-)|\Omega_0(s')|}\bigg] \qquad s_\pm = q_1^2+q_2^2 \pm 2\sqrt{q_1^2q_2^2}$$

- Inhomogeneities $\Delta_{0,++}(s), \Delta_{0,00}(s)$ include left-hand cut
- **Subtraction functions**
  - $b(q_1^2,q_2^2)$ and $a_+(q_1^2,q_2^2) - a_-(q_1^2,q_2^2)$ multiply $q_1^2q_2^2$ and $\sqrt{q_1^2q_2^2}$
    $\hookrightarrow$ inherently doubly-virtual observables $\Rightarrow$ need ChPT (or lattice)
  - However: $a(q_1^2,q_2^2) = (a_+(q_1^2,q_2^2) + a_-(q_1^2,q_2^2))/2$ fixed by singly-virtual measurements
    $\hookrightarrow$ compare with chiral prediction, uncertainty estimates for the other functions

- 1-loop result for arbitrary q_i^2, e.g.

$$a^{\pi^0}(q_1^2,q_2^2) = -\frac{M_\pi^2}{8\pi^2F_\pi^2(q_1^2-q_2^2)^2}\bigg\{q_1^2+q_2^2+2\Big(M_\pi^2(q_1^2+q_2^2)+q_1^2q_2^2\Big)C_0(q_1^2,q_2^2) + q_1^2\bigg(1+\frac{6q_2^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_1^2)+q_2^2\bigg(1-\frac{6q_1^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_2^2)\bigg\}$$

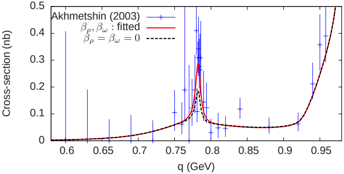
- Special case: $q_1^2=q_2^2=0$

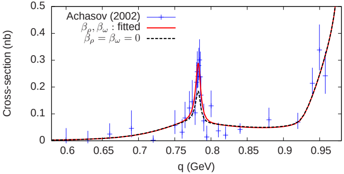
$$a^{\pi^\pm}(0,0) = \frac{\bar{l}_6-\bar{l}_5}{48\pi^2F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^\pm} \qquad b^{\pi^\pm}(0,0)=0$$

$$a^{\pi^0}(0,0) = -\frac{1}{96\pi^2F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^0} \qquad b^{\pi^0}(0,0) = -\frac{1}{1440\pi^2F_\pi^2M_\pi^2}+\cdots$$

$\hookrightarrow$ resum higher chiral orders into **pion polarizabilities**

Subtraction functions: dispersive representation





Moussallam 2013

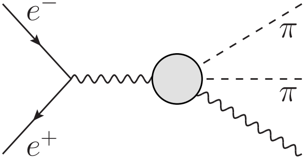
- Singly-virtual case: phenomenological representation with chiral constraints
  $\hookrightarrow$ parameters fixed from $e^+e^- \to \pi^0\pi^0\gamma$ (CMD2 and SND) Moussallam 2013
- **Dispersive representation**: imaginary part from $2\pi, 3\pi, \ldots$
  $\hookrightarrow$ analytic continuation from time-like to space-like kinematics
- Example: $I = 2 \Rightarrow$ isovector photons $\Rightarrow 2\pi \sim \rho$

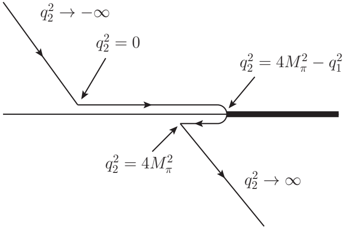
$$a^2(q_1^2,q_2^2) = \alpha_0\Big[\alpha^2 + \alpha\Big(q_1^2\mathscr{F}^\rho(q_1^2) + q_2^2\mathscr{F}^\rho(q_2^2)\Big) + q_1^2 q_2^2\mathscr{F}^\rho(q_1^2)\mathscr{F}^\rho(q_2^2)\Big]$$

$$\mathscr{F}^\rho(q^2) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds\, \frac{q_{\pi\pi}^3(s)\big(F_\pi^V(s)\big)^*\Omega_1(s)}{s^{3/2}(s-q^2)} \qquad q_{\pi\pi}(s) = \sqrt{\frac{s}{4} - M_\pi^2}$$

$\hookrightarrow$ α_0 and α can be determined from $a^2(q^2,0)$ alone!

Anomalous thresholds





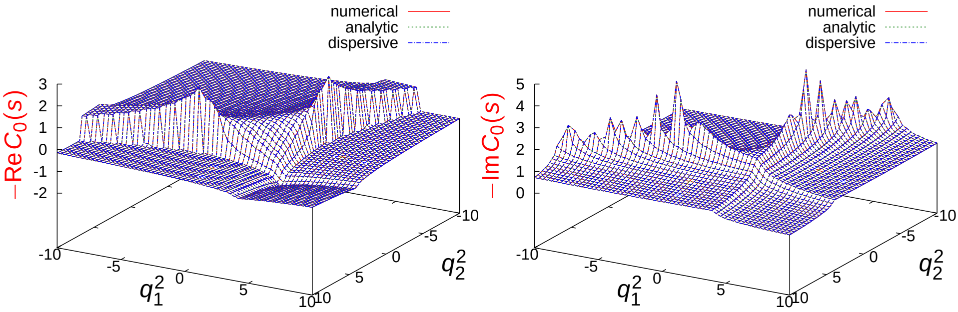
- **Analytic continuation** in q_i^2 in time-like region non-trivial in doubly-virtual case
- Singularities from second sheet move onto first one

  $\hookrightarrow$ need to **deform** the **integration contour**
- Problem already occurs for a simple triangle loop function $C_0(s)$

  $\hookrightarrow$ extra factor $t_\ell(s)/\Omega_\ell(s)$ is well defined in the whole complex plane

  $\hookrightarrow$ remedy in case of $C_0(s)$ can be taken over to full Omnès solution
- Becomes relevant for $e^+e^+ \to e^+e^-\pi\pi$ in time-like kinematics

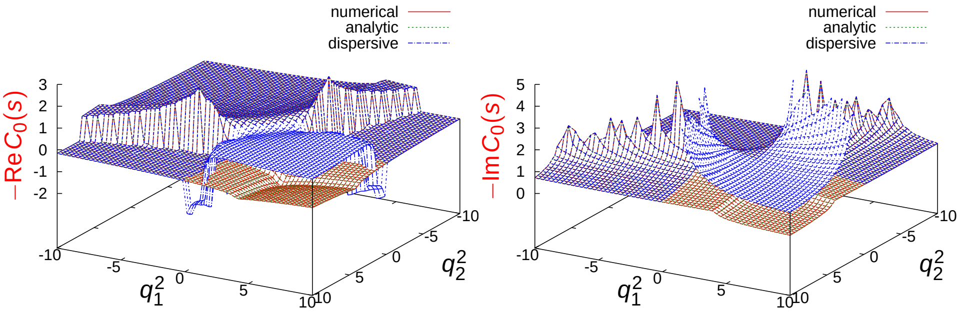
Numerical check of anomalous thresholds



- Comparison for $s = 5$, $M_\pi = 1$

  $\hookrightarrow$ **dispersive reconstruction** of $C_0(s)$ works!

Numerical check of anomalous thresholds



- Ignore anomalous piece

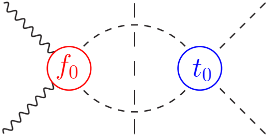
  $\hookrightarrow$ substantial deviations for **large virtualities**!

$\gamma^*\gamma^* \to \pi\pi$

- **Left-hand cut** approximated by **pion pole** + **resonances**
- **Unitarity** for $\gamma^*\gamma^* \to \pi\pi$ system: Watson's theorem

$$\text{disc}\, f_0(s;q_1^2,q_2^2) = 2i\sigma_s f_0(s;q_1^2,q_2^2)\, t_0^*(s)$$

$$t_0(s) = \frac{1}{\sigma_s} e^{i\delta_0(s)} \sin\delta_0(s) \qquad \sigma_s = \sqrt{1 - \frac{4M_\pi^2}{s}}$$



$\hookrightarrow$ solution in terms of **Omnès function**, e.g. for pion pole only

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|} \qquad N_0(s;q_1^2,q_2^2) = \frac{2L}{\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}$$

$$L = \log \frac{s - q_1^2 - q_2^2 + \sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}{s - q_1^2 - q_2^2 - \sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}} \qquad \lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$$

- **Analytic continuation** in q_i^2?

$\gamma^*\gamma^* \to \pi\pi$: analytic continuation

$$L = \log \frac{s - q_1^2 - q_2^2 + \sigma_s \sqrt{\lambda(s,q_1^2,q_2^2)}}{s - q_1^2 - q_2^2 - \sigma_s \sqrt{\lambda(s,q_1^2,q_2^2)}} \overset{q_2^2 \to 0}{\longrightarrow} \pm \log \frac{1+\sigma_s}{1-\sigma_s}$$

- Singularities of the log: **anomalous thresholds**

$$s_\pm = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} \pm \frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2 - 4M_\pi^2) q_2^2 (q_2^2 - 4M_\pi^2)}$$

$\hookrightarrow$ usual Omnès derivation breaks down

- Idea: consider first the **scalar loop function**

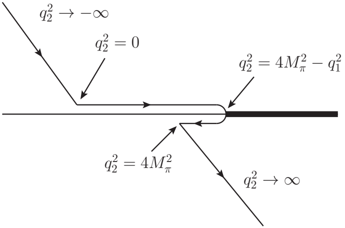
$$C_0(s) \equiv C_0\big((q_1+q_2)^2; q_1^2, q_2^2\big) = \frac{1}{i\pi^2}\int \frac{d^4k}{(k^2 - M_\pi^2)\big((k+q_1)^2 - M_\pi^2\big)\big((k-q_2)^2 - M_\pi^2\big)}$$

$$\operatorname{disc} C_0(s) = -\frac{2\pi i}{\sqrt{\lambda\big(s, q_1^2, q_2^2\big)}} L = -\pi i \sigma_s N_0\big(s; q_1^2, q_2^2\big)$$

$$s_+ = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} + \frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2 - 4M_\pi^2)q_2^2(q_2^2 - 4M_\pi^2)}$$

- **Anomalous threshold** usually on the **second sheet**
- Trajectory of $s_+(q_2^2)$ for $0 \leq q_1^2 \leq 4M_\pi^2$

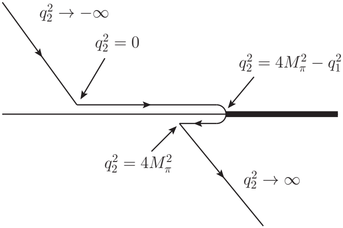
  $\hookrightarrow$ moves through unitarity cut onto first sheet



$\gamma^*\gamma^* \to \pi\pi$: anomalous thresholds

$$s_+ = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} + \frac{1}{2M_\pi^2}\sqrt{q_1^2\big(q_1^2 - 4M_\pi^2\big)q_2^2\big(q_2^2 - 4M_\pi^2\big)}$$

- **Anomalous threshold** usually on the **second sheet**
- Trajectory of $s_+(q_2^2)$ for $0 \leq q_1^2 \leq 4M_\pi^2$
  $\hookrightarrow$ moves through unitarity cut onto first sheet
- Need to deform the contour



$$C_0(s) = \frac{1}{2\pi i}\int\limits_{4M_\pi^2}^{\infty} ds' \frac{\text{disc}\, C_0(s')}{s' - s}$$

$$+\,\theta\big(q_1^2 + q_2^2 - 4M_\pi^2\big)\frac{1}{2\pi i}\int\limits_0^1 dx\, \frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}}\, C_0(s_x)}{s_x - s}$$

$$s_x = x 4M_\pi^2 + (1-x)s_+ \qquad \text{disc}_{\text{an}}\, C_0(s) = \frac{4\pi^2}{\sqrt{\lambda\big(s, q_1^2, q_2^2\big)}}$$

$\gamma^*\gamma^* \to \pi\pi$: back to the Omnès representation

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi}\int\limits_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

- Integrand similar to the scalar-loop example

$$\frac{N_0(s;q_1^2,q_2^2)\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i\,\sigma_s}\frac{\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i}\frac{t_0(s)}{\Omega_0(s)}$$

$\hookrightarrow$ additional factor **independent of q_i^2** and **well-defined in the whole s-plane**

$\gamma^*\gamma^* \to \pi\pi$: back to the Omnès representation

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

- Integrand similar to the scalar-loop example

$$\frac{N_0(s;q_1^2,q_2^2)\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i\,\sigma_s}\frac{\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i}\frac{t_0(s)}{\Omega_0(s)}$$

$\hookrightarrow$ additional factor **independent of q_i^2** and **well-defined in the whole s-plane**

Omnès representation for $\gamma^*\gamma^* \to \pi\pi$

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

$$+\theta\left(q_1^2+q_2^2-4M_\pi^2\right)\frac{\Omega_0(s)}{2\pi i}\int_0^1 dx\,\frac{\partial s_x}{\partial x}\,\frac{\text{disc}_{\text{an}}\, f_0(s_x;q_1^2,q_2^2)}{s_x-s}$$

$$s_x = x4M_\pi^2+(1-x)s_+ \qquad \text{disc}_{\text{an}}\, f_0(s;q_1^2,q_2^2) = -\frac{8\pi}{\sqrt{\lambda(s,q_1^2,q_2^2)}}\frac{t_0(s)}{\Omega_0(s)}$$